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Examination for FY3451 Astrophysics II
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Allowed tools: officially approved pocket calculator

1. Knowledge and basic understanding.

• Name the main source(s) of pressure in a
a.) main-sequence star,
b.) a white dwarf,
c.) a neutron star.

a.) Ideal gas pressure of nuclei (i.e. essentially hydrogen and helium) and free electrons contributes roughly equally, while radiation pressure is subdominant.
b.) Degeneracy pressure of electrons.
c.) Degeneracy pressure of neutrons.

• Name the main source of energy generation in a
a.) proto-star,
b.) main-sequence star,
c.) red giant,
d.) active galactic nucleus.

a.) Release of gravitational potential energy during contraction.
b.) Conversion of mass into energy via hydrogen burning, $4p \to {}^4\text{He} + 2e^+ + 2\nu_e$.
c.) Conversion of mass into energy, in addition burning of heavier elements (He, CNO, ...).
d.) Release of gravitational potential energy during the accretion of matter.

• Name the three main sources of energy transport in stars. Give for two an example of a star for which this transport mechanism is important.

i) radiative energy transfer, i.e. energy transport by photons, ii) conduction, i.e. the transport of heat by the Brownian motion of electrons or atoms, and iii) convection, i.e. macroscopic matter flows.

Conduction plays a prominent role as energy transport only for dense systems, and is therefore only relevant in the dense, final stages of stellar evolution as, e.g., white dwarfs. Convection dominates in (the outer part of) low-mass stars and the center of massive stars. Otherwise radiative energy transfer dominates.

• Explain briefly (without explicit calculation) the basic idea how one derives the Schwarzschild criterion for convection.

One considers how the density of a fluid element (characterised by ρ, T, P) changes when it is displaced by a small distance upwards (to a position characterised by $\rho + \mathrm{d}\rho, T + \mathrm{d}T, P + \mathrm{d}P$). If the density $\rho + \delta\rho$ of the fluid element decreases faster than the unperturbed density $\rho + \mathrm{d}\rho$ of the star, it will continue to move upwards, thus convection sets in. Using $P = K\rho^\gamma$ for an adiabatic change, one can relate density and pressure gradients, obtaining thereby a condition for the temperature gradient.
Alternative: Convection starts when the luminosity at the radius r exceeds the Eddington luminosity, i.e. for $L(r) > L_{\rm Edd}$, .

2. Scaling relations for main-sequence stars.
a.) Derive scaling relations (e.g. $P \propto M^\alpha R^\beta$) for the central temperature T_c and pressure P_c of a main-sequence star.
b.) Use the equation of radiative transfer to derive an approximate mass-luminosity relation.

a.) Replacing $\rho \propto M/R^3$ and $\mathrm{d}P/\mathrm{d}r \sim -P_c/R$, the hydrostatic equilibrium equation becomes

$$P_c/R \propto GM\rho/R^2 \tag{1}$$

or $P_c \propto M^2/R^4$. Using the ideal gas law as E.o.S., $P = nkT$, it is

$$P_c \propto \rho T_c$$

or $T_c \propto M/R$.

[Alternatives: integrate the hydrostatic equilibrium equation for constant density; use virial theorem for temperature.]

b.) Inserting the results for T_c and P_c gives

$$\frac{T}{R} \propto \frac{L}{R^2}\,\frac{M}{R^3T^3}, \tag{2}$$

or $L \propto M^3$.

3. Eddington luminosity.
Derive the formula for the Eddington luminosity.
[Hint: Compare two pressure gradients; all formulae needed are in the appendix.]

We first rewrite

$$aT^3\,\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{L(r)}{4\pi r^2}\frac{3\langle\kappa\rangle\rho}{4c},$$

as an equation for the radiation pressure $P_{\rm rad} = aT^4/3$,

$$\frac{3}{4}\,\frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}r} = -\frac{3\kappa\rho}{4c}\,\frac{L(r)}{4\pi r^2}.$$

Dividing by the equation for hydrostatic equilibrium,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho(r)}{r^2}$$

gives

$$\frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}P} = \frac{\kappa L(r)}{4\pi c GM(r)}\,.$$

Since the total pressure P is the sum of radiation and gas pressure, $P = P_{\rm rad} + P_{\rm gas}$, and both decrease outwards, it follows that $\mathrm{d}P_{\rm rad}$ and $\mathrm{d}P_{\rm gas}$ have the same sign. Hence, $\mathrm{d}P_{\rm rad} < \mathrm{d}P$ and

$$L(r) < \frac{4\pi c GM(r)}{\kappa}\,.$$

Setting $r = R$, one obtains the Eddington luminosity as a bound for maximal luminosity of a stable star or galaxy.

4. HII regions and the Strömgren sphere.

An OB star which emits per time the number $\dot{N}_\gamma$ of photons with energy above the ionisation threshold of hydrogen is surrounded by interstellar medium consisting of hydrogen atoms with the density n. Find the radius of the sphere containing ionized hydrogen, assuming spherical geometry and a sharp, stationary boundary as function of n and $\dot{N}_\gamma$.

In steady-state, the number of recombinations inside the Strömgren sphere with radius R_* must equal the number of ionizations, $\mathcal{R}_{\rm ion} = R_{\rm rec}$. The former is in turn equal to the rate $\dot{N}_\gamma$ of emitted ionizing photons,

$$\dot{N}_\gamma = \frac{4\pi}{3} R_*^3 \mathcal{R}_{\rm rec}, \tag{3}$$

with

$$\mathcal{R}_{\rm rec} = \frac{\mathrm{d}N_{\rm rec}}{\mathrm{d}V\mathrm{d}t} = n_e n_p \langle \sigma_{\rm rec} v\rangle. \tag{4}$$

The recombination rate $\mathcal{R}_{\rm rec}$ per volume and tim is given by the product of the proton and electron densities, $n_p n_e$, and the thermally averaged recombination cross section $\langle \sigma_{\rm rec} v\rangle$. For a pure hydrogen cloud, the number of free electrons equals the one of free protons, $n_e = n_p$. If we denote with $n = n_H + n_p$ the total number density of hydrogen atoms and free protons, then it is $n_e = n_p = xn$ with x as the fraction of ionized hydrogen.
Solving for the radius gives

$$R_* = \left(\frac{3\dot{N}_\gamma}{4\pi\langle \sigma_{\rm rec} v\rangle n^2}\right)^{1/3} \propto \dot{N}_\gamma^{1/3} n^{-2/3}, \tag{5}$$

where we could set $x = 1$ because of the sharp boundary.

5. Neutrinos from type II supernova.

The proto-neutron star formed during the core collapse of a massive star emits copiously neutrinos, transforming 99% of all the energy emitted into neutrinos. The mass of the proto-neutron stare is $\approx 1.4 M_\odot$, and its radius $\approx 15\,\mathrm{km}$.

a.) Estimate the total (gravitational potential) energy E_b released in the collapse.
b.) Apply the virial theorem to a nucleon N at the surface of the proto-neutron star and estimate its kinetic energy E_N.
c.) Estimate the number N_ν of neutrinos emitted and the duration of the neutrino signal applying a random walk picture. For the numerical estimate, you may use $E_\nu = E_N/2$, $E_b = N_\nu E_\nu$ and $\sigma_\nu = 10^{-43}\text{cm}^2(E_\nu/\text{MeV})^2$.

a.) The released gravitational potential energy is (assuming uniform density)

$$\Delta E \simeq \left[-\frac{3GM^2}{5R}\right]_{\text{NS}} \sim 3 \times 10^{53}\text{erg}\left(\frac{15\text{km}}{\text{R}}\right)\left(\frac{\text{M}_{\text{NS}}}{1.4\text{M}_\odot}\right)^2. \tag{6}$$

b.) Applying the virial theorem, the mean kinetic energy E_N of a nucleon is $E_N = E_{\text{pot}}/2 \approx Gm_N M/(2R) \approx 1 \times 10^{-4}$ erg or 64 MeV. The number of emitted neutrinos follows as $N_\nu = E_N/E_b \approx 2 \times 10^{57}$.

c.) The number of steps in a random walk with step size ℓ_{int} needed to reach the distance R is $N = R^2/\ell_{\text{int}}^2$. Hence the duration τ of the neutrino signal is $\tau \approx N\ell_{\text{int}}/c = R^2/(c\ell_{\text{int}})$. The step size ℓ_{int} is found as $\ell_{\text{int}} = 1/(n\sigma) \approx 1$ m for $E_\nu = 30$ MeV and $n \approx 10^{-38}/\text{cm}^3$. Thus $\tau \approx 1$ s.

Good luck!

1 Some formulas

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r), \qquad \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho(r)}{r^2}, \tag{7a}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3L(r)\langle\kappa\rangle\rho}{16\pi r^2\, acT^3}, \qquad \frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \varepsilon\rho\,. \tag{7b}$$

$$P = aT^4/3, \qquad P = nkT \qquad P = K\rho^\gamma. \tag{8}$$

2 Physical constants and measurements

Gravitational constant	$G = 6.674 \times 10^{-11}\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2} = 6.674 \times 10^{-8}\text{cm}^3\,\text{g}^{-1}\,\text{s}^{-2}$
Planck's constant	$\hbar = h/(2\pi) = 1.055 \times 10^{-27}\text{erg}\,\text{s}$
velocity of light	$c = 2.998 \times 10^8$ m/s= 2.998×10^{10} cm/s
Boltzmann constant	$k = 1.381 \times 10^{-23}$ J/K $= 1.38 \times 10^{-16}$ erg/K
electron mass	$m_e = 9.109 \times 10^{-28}\,\text{g} = 0.5110\,\text{MeV}/c^2$
proton mass	$m_p = 1.673 \times 10^{-24}\,\text{g} = 938.3\,\text{MeV}/c^2$
Fine-structure constant	$\alpha = e^2/(\hbar c) \approx 1/137.0$
Fermi's constant	$G_F/(\hbar c)^3 = 1.166 \times 10^{-5}$ GeV^{-2}
Stefan-Boltzmann constant	$\sigma = (2\pi^5 k^4)/(15c^2h^3) \approx 5.670 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4}
Radiation constant	$a = 4\sigma/c \approx 7.566 \times 10^{-15}$ erg cm^{-3} K^{-4}
Rydberg constant	$R_\infty = 1.10 \times 10^5$ cm^{-1}
Thomson cross-section	$\sigma_T = 8\pi\alpha_{\text{em}}^2/(3m_e^2) = 6.652 \times 10^{-25}$ cm^2

3 Astronomical constants and measurements

Astronomical Unit	AU $= 1.496 \times 10^{13}$ cm
Parsec	pc $= 3.086 \times 10^{18}$ cm $= 3.261$ ly
Tropical year	yr $= 31\,556\,925.2\,\text{s} \approx \pi \times 10^7\,\text{s}$
Solar radius	$R_\odot = 6.960 \times 10^{10}$ cm
Solar mass	$M_\odot = 1.998 \times 10^{33}$ g
Solar luminosity	$L_\odot = 3.84 \times 10^{33}$ erg/s
Solar apparent visual magnitude	$m = -26.76$
Earth equatorial radius	$R_\oplus = 6.378 \times 10^5$ cm
Earth mass	$M_\oplus = 5.972 \times 10^{27}$ g
Age of the universe	$t_0 = (13.7 \pm 0.2)$ Gyr
present Hubble parameter	$H_0 = 73\,\text{km}/(\text{s}\,\text{Mpc}) = 100\,h\,\text{km}/(\text{s}\,\text{Mpc})$
present CMB temperature	$T = 2.725$ K
present baryon density	$n_\text{b} = (2.5 \pm 0.1) \times 10^{-7}$ cm^3
	$\Omega_b = \rho_b/\rho_\text{cr} = 0.0223/h^2 \approx 0.0425$
dark matter abundance	$\Omega_\text{DM} = \Omega_m - \Omega_b = 0.105/h^2 \approx 0.20$

4 Other useful quantities

cross section	1 mbarn $= 10^{-27}$ cm^2
flux conversion	$L = 3.02 \times 10^{28}\text{W} \times 10^{-0.4M}$
energy conversion	erg $= 624$ GeV