NTNU Trondheim, Institutt for fysikk

Examination for FY3452 Gravitation and Cosmology
Contact: J.-O. Andersen
Allowed tools: –

1. Gravitational waves.

a.) Write down the polarisation tensor $\varepsilon_{\mu\nu}$ for a gravitational wave $h_{\mu\nu} \propto \varepsilon_{\mu\nu} \mathrm{e}^{-\mathrm{i}kx}$ in the TT gauge propagating in the x direction. (4 pts)
b.) Explain why gravitational waves do not exist in $d \leq 3$ spacetime dimensions. (2 pts)

a.) The polarisation tensor is always symmetric, $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$. In the TT (=transverse-traceless) gauge, the tensor is—not surprisingly—transverse, $\varepsilon_{0\nu} = \varepsilon_{1\nu} = 0$, and traceless, $\varepsilon_{yy} = -\varepsilon_{zz}$. It has therefore only two independent components,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{yy} & \varepsilon_{yz} \\ 0 & 0 & \varepsilon_{yz} & -\varepsilon_{yy} \end{pmatrix} .$$

They correspond to linear polarisation states.

b.) In $d = 3$, we have using the transverse condition,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{yy} \end{pmatrix} .$$

Imposing the traceless condition implies $\varepsilon_{yy} = 0$ and the polarisation tensor is identically zero. In $d = 2$, we can not even satisfy the transverse condition.

2. Empty universe.

Show that the Ricci tensor of an empty universe ($T_{\mu\nu} = 0$ and $\Lambda = 0$) vanishes, $R_{\mu\nu} = 0$. [Hint: Trace-reverse the Einstein equation.] (6 pts)

Taking the trace of the Einstein equation, it follows

$$R_\mu^{\ \mu} - \frac{1}{2} g_\mu^{\ \mu}(R + 2\Lambda) = R - 2(R + 2\Lambda) = -R - 4\Lambda = \kappa T_\mu^{\ \mu} \tag{1}$$

Defining $T \equiv T^\mu_\mu$, we can perform the replacement $R = -4\Lambda - \kappa T$ in the Einstein equation and obtain

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) - g_{\mu\nu}\Lambda \,. \tag{2}$$

[This is a special case of the rule $\overline{\bar{A}_{\mu\nu}} = A_{\mu\nu}$.] Thus an empty universe has a vanishing Ricci tensor, $R_{\mu\nu} = 0$.

3. Black holes.
The metric outside a spherically symmetric mass distribution with mass M is given in Schwarzschild coordinates by

$$ds^2 = dt^2\left(1-\frac{2M}{r}\right) - \frac{dr^2}{1-\frac{2M}{r}} - r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2)$$

a.) Use the "advanced time parameter"

$$p = t + r + 2M \ln|r/2M - 1|$$

to eliminate t in the line-element (i.e. introduce Eddington-Finkelstein coordinates) and show that in the new coordinates the singularity at $R = 2M$ is absent. (3 pts)
b.) Consider radial light-rays in these coordinates. (6 pts)
c.) Introducing a new time-like coordinate $t' \equiv p - r$,one ontains

$$\begin{aligned} t' &= -r + \text{const.} \\ t' &= r + 4M \ln|r/2M - 1|) + \text{const.} \end{aligned} \tag{3}$$

Draw a space-time diagram showing the light-cone structure and explain why $r = 2M$ is an event horizon. (6 pts)

a.) Forming the differential,

$$dp = dt + dr + \left(\frac{r}{2M} - 1\right)^{-1} dr = dt + \left(1 - \frac{2M}{r}\right)^{-1} dr\,,$$

we can eliminate dt from the Schwarzschild metric and find

$$ds^2 = \left(1 - \frac{2M}{r}\right) dp^2 - 2dpdr - r^2 d\Omega\,.$$

This metric is regular at $2M$ and valid for all $r > 0$.

b.) For radial light-rays, $ds = d\phi = d\vartheta = 0$, it follows

$$0 = \left(1 - \frac{2M}{r}\right) dp^2 - 2dpdr\,.$$

There exist three types of solutions: i) light-rays with d$p = 0$ and thus $p =$ const.; ii) dividing by dp,

$$0 = \left(1 - \frac{2M}{r}\right) dp + 2dr,$$

we separate variables and integrate,

$$p - 2(r + 2M \ln|r/2M - 1|) = \text{const.}$$

iii) additionally, light-rays for $r = 2M$ satisfy $\mathrm{d}s^2 = 0$.
c.) The light-rays of type i) are ingoing: as t increases, r has to increase to keep p constant. The light-rays of type ii) are ingoing for $r < 2M$ and outgoing for $r > 2M$. Thus for $r < 2M$ both radial light-rays moves towards $r = 0$; all wordlines of observers are inside such light-cones and have to move towards $r = 0$ too. Hence $r = 2M$ is an event horizon.

4. Cosmology.

Consider a flat universe dominated by one matter component with E.o.S. $w = P/\rho$ =const.
a.) Use that the universe expands adiabatically to find the connection $\rho = \rho(R, w)$ between the density ρ, the scale factor R and the state parameter w. (6 pts)
b.) Find the age t_0 of the universe as function of w and the current value of the Hubble parameter, H_0. (4 pts)
c.) Comment on the value of t_0 in the case of a positive cosmological constant, $w = -1$. (2 pts)
d.) Find the relative energy loss per time, $E^{-1}\,\mathrm{d}E/\mathrm{d}t$, of relativistic particles due to the expansion of the universe for $H_0 = 70$km/s/Mpc. (6 pt)

a.) For adiabatic expansion, the first law of thermodynmaics becomes $\mathrm{d}U = -P\mathrm{d}V$ or

$$\mathrm{d}(\rho R^3) = -3PR^2\mathrm{d}R$$

Eliminating P with $P = P(\rho) = w\rho$,

$$\frac{\mathrm{d}\rho}{\mathrm{d}R}R^3 + 3\rho R^2 = -3w\rho R^2\,.$$

Separating the variables,

$$-3(1+w)\frac{\mathrm{d}R}{R} = \frac{\mathrm{d}\rho}{\rho}\,,$$

we can integrate and obtain $\rho \propto R^{-3(1+w)}$.

b.) For a flat universe, $k = 0$, with one dominating energy component with $w = P/\rho =$ const. and $\rho = \rho_{\mathrm{cr}}\,(R/R_0)^{-3(1+w)}$, the Friedmann equation becomes

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 = H_0^2 R_0^{3+3w}\;R^{-(1+3w)}\,, \tag{4}$$

where we inserted the definition of $\rho_{\mathrm{cr}} = 3H_0^2/(8\pi G)$. Separating variables we obtain

$$R_0^{-(3+3w)/2}\int_0^{R_0}\mathrm{d}R\,R^{(1+3w)/2} = H_0\int_0^{t_0}\mathrm{d}t = t_0 H_0 \tag{5}$$

and hence the age of the Universe follows as

$$t_0 H_0 = \frac{2}{3+3w}\,.$$

c.) Models with $w > -1$ need a finite time to expand from the initial singularity $R(t=0)=0$ to the current value of the scale factor R_0, while a Universe with only a Λ has no "beginning", $t_0 H_0 \to \infty$.

d.) The connection between the energy E_0 today and the energy at redshift z is

$$E(z) = (1+z)E_0$$

and thus $\mathrm{d}E = \mathrm{d}zE_0$. Differentiating $1+z = R_0/R(t)$, we obtain with $H = \dot{R}/R$

$$\mathrm{d}z = -\frac{R_0}{R^2}\,\mathrm{d}R = -\frac{R_0}{R^2}\,\frac{\mathrm{d}R}{\mathrm{d}t}\,\mathrm{d}t = -(1+z)H\mathrm{d}t\,.$$

Combining the two equations, we find $\mathrm{d}E = -(1+z)H\mathrm{d}tE_0 = -H\mathrm{d}tE$ or

$$\frac{1}{E}\,\frac{\mathrm{d}E}{\mathrm{d}t} = -H(z)\,.$$

Numerically, we find for the current epoch

$$\frac{1}{E}\,\frac{\mathrm{d}E}{\mathrm{d}t} = -H_0 \simeq \frac{7.1\times 10^6\mathrm{cm}}{\mathrm{s}\;3.1\times 10^{24}\mathrm{cm}} \simeq 5.2\times 10^{-36}\mathrm{s}^{-1}\,.$$

5. Symmetries.

Consider in Minkowski space a complex scalar field ϕ with Lagrange density

$$\mathscr{L} = \partial_a\phi^\dagger\partial^a\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2\,.$$

a.) Name the symmetries of the Langrangian. [No calculation needed.] (3 pts)

b.) Derive Noether's theorem in the form

$$0 = \delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a - K^\mu\right)\,.$$

for a Lagrangian $\mathscr{L}(\phi_a, \partial_\mu\phi_a)$. (6 pts)

c.) Derive one conserved current of your choice for the given complex scalar field. (4 pts)

a. space-time symmetries: Translation, Lorentz, (scale invariance). internal: global SO(2) / U(1) invariance.

b.) We assume that the collection of fields ϕ_a has a continuous symmetry group. Thus we can consider an infinitesimal change $\delta\phi_a$ that keeps $\mathscr{L}(\phi_a, \partial_\mu\phi_a)$ invariant,

$$0 = \delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\,\delta\partial_\mu\phi_a\,. \tag{6}$$

Now we exchange $\delta\partial_\mu$ against $\partial_\mu\delta$ in the second term and use then the Lagrange equations, $\delta\mathscr{L}/\delta\phi_a = \partial_\mu(\delta\mathscr{L}/\delta\partial_\mu\phi_a)$, in the first term. Then we can combine the two terms using the Leibniz rule,

$$0 = \delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\,\partial_\mu\delta\phi_a = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a\right). \tag{7}$$

Hence the invariance of $\mathscr{L}$ under the change $\delta\phi_a$ implies the existence of a conserved current, $\partial_\mu J^\mu = 0$, with

$$J_1^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a\,. \tag{8}$$

If the transformation $\delta\phi_a$ leads to change in $\mathscr{L}$ that is a total four-divergence, $\delta\mathscr{L} = \partial_\mu K^\mu$, and boundary terms can be dropped, then the equation of motions are still invariant. The conserved current is changed to

$$J^\mu = \delta\mathscr{L}/\delta\partial_\mu\phi_a\,\delta\phi_a - K^\mu\,.$$

c. i) Translations: From $\phi_a(x) \to \phi_a(x-\varepsilon) \approx \phi_a(x) - \varepsilon^\mu\partial_\mu\phi(x)$ we find the change $\delta\phi_a(x) = -\varepsilon^\mu\partial_\mu\phi(x)$. The Lagrange density changes similiarly, $\mathscr{L}(x) \to \mathscr{L}(x-\varepsilon)$ or $\delta\mathscr{L}(x) = -\varepsilon^\mu\partial_\mu\mathscr{L}(x) = -\partial_\mu(\varepsilon^\mu\mathscr{L}(x))$. Thus $K^\mu = -\varepsilon^\mu\mathscr{L}(x)$ and inserting both in the Noether current gives

$$J^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\left[-\varepsilon^\nu\partial_\nu\phi(x)\right] + \varepsilon^\mu\mathscr{L}(x) = \varepsilon_\nu T^{\mu\nu}$$

with $T^{\mu\nu}$ as (canonical) energy-momentum stress tensor and four-momentum as Noether charge.
or
ii) Charge conservation: We can work either with complex fields and U(1) phase transformations

$$\phi(x) \to \phi(x)\mathrm{e}^{\mathrm{i}\alpha} \quad , \quad \phi^\dagger(x) \to \phi^\dagger(x)\mathrm{e}^{-\mathrm{i}\alpha}$$

or real fields (via $\phi = (\phi + \mathrm{i}\phi_2)/\sqrt{2}$) and invariance under rotations SO(2). With $\delta\phi = \mathrm{i}\alpha\phi$, $\delta\phi^\dagger = -\mathrm{i}\alpha\phi^\dagger$, the conserved current is

$$J^\mu = \mathrm{i}\left[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right]$$

Some formula:

$$\ddot{x}^c + \Gamma^c{}_{ab}\dot{x}^a\dot{x}^b = 0$$

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}\,,$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \, .$$

$$\frac{e^2 - 1}{2} = \frac{\dot{r}^2}{2} + V_{\text{eff}}$$

$$H^2 = \frac{8\pi}{3} G\rho - \frac{k}{R^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3} (\rho + 3P)$$

$$E(z) = (1+z) E_0$$

$$1\text{Mpc} = 3.1 \times 10^{24} \text{cm}$$