# NTNU Trondheim, Institutt for fysikk 

## Examination for FY3451 Astrophysics II

Contact: M. Linares, tel. 95754811
Allowed tools: officially approved pocket calculator

## 1. Multiple choice (answer in Inspera).

A stellar cluster looses continuously the fastest stars with velocities $v>v_{\text {esc }}$. As result, the cluster becomes
a.) hotter
b.) cooler
c.) smaller
d.) larger

## 2. Knowledge and basic understanding.

Answer concise, keywords are enough; for the compactness problem, maximal 5 phrases.

- Name the main source(s) of (internal) pressure in a
a.) main-sequence star,
b.) white dwarf,
c.) neutron star.
a.) Ideal gas pressure of nuclei (i.e. essentially hydrogen and helium) and free electrons contributes roughly equally, while radiation pressure is subdominant.
b.) Degeneracy pressure of electrons.
c.) Degeneracy pressure of neutrons.
- Name the main source of energy generation in a
a.) proto-star,
b.) main-sequence star,
c.) red giant,
d.) active galactic nucleus.
a.) Release of gravitational potential energy during contraction.
b.) Conversion of mass into energy via hydrogen burning, $4 p \rightarrow{ }^{4} \mathrm{He}+2 e^{+}+2 \nu_{e}$.
c.) Conversion of mass into energy, in addition burning of heavier elements ( $\mathrm{He}, \mathrm{CNO}, \ldots$ ).
d.) Release of gravitational potential energy during the accretion of matter.
- Name two observations suggesting the existence of dark matter.

Rotation curves of galaxies, applying virial theorem to galaxy clusters, growth of large-scale structure, aniostropies of the cosmic microwave backgroumd.

- Explain the "compactness problem" of Gamma-Ray Bursts and how it is resolved. pts)

Time variability constrains the source size. Combined with the observed (assumed to be isotropic) luminosity, one can derive the density of photons, and thus the optical depth of multi- MeV photons for pair production, $\tau \sim 10^{13} \gg 1$. The "compactness problem" problem is how the observed photons escaped from such a dense source.
Solution: The threshold energy for $\gamma+\gamma_{\mathrm{BB}} \rightarrow e^{+}+e^{-}$increases as the angle between the threemomentum of 2 photons decreases. Thus a source with a large Gamma factor, which is beamed with $\vartheta \sim 1 / \Gamma$, solves the compactness problem for large enough $\Gamma$.

## 3. The Sun.

a.) Derive a lower limit on the central pressure of the Sun.
b.) Estimate the central temperature of the Sun using the virial theorem.
a.) For an estimate of the central pressure $P_{c}=P(0)$ of a star in hydrostatic equilibrium, we integrate (??) and obtain with $P(R) \approx 0$,

$$
\begin{equation*}
P_{c}=-\int_{0}^{R} \frac{\mathrm{~d} P}{\mathrm{~d} r} \mathrm{~d} r=G \int_{0}^{M} \mathrm{~d} M \frac{M}{4 \pi r^{4}}, \tag{1}
\end{equation*}
$$

where we used the continuity equation to substitute $\mathrm{d} r=\mathrm{d} M /\left(4 \pi r^{2} \rho\right)$ by $\mathrm{d} M$. If we replace furthermore $r$ by the stellar radius $R \geq r$, we obtain a lower limit for the central pressure,

$$
\begin{equation*}
P_{c}=G \int_{0}^{M} \mathrm{~d} M \frac{M}{4 \pi r^{4}}>G \int_{0}^{M} \mathrm{~d} M \frac{M}{4 \pi R^{4}}=\frac{G M^{2}}{8 \pi R^{4}} \tag{2}
\end{equation*}
$$

For the Sun, it follows

$$
\begin{equation*}
P_{c}>\frac{G M^{2}}{8 \pi R^{4}}=4 \times 10^{8} \operatorname{bar}\left(\frac{M}{M_{\odot}}\right)^{2}\left(\frac{R_{\odot}}{R}\right)^{4} . \tag{3}
\end{equation*}
$$

b.) The Sun consists mainly of (ionized) hydrogen. Thus we can estimate the average gravitational potential energy of a single particle inside the Sun as

$$
\left\langle E_{\text {grav }}\right\rangle \sim-\frac{G M_{\odot} m_{p}}{R_{\odot}} \approx-3.2 \mathrm{keV} / c^{2}
$$

For a thermal velocity distribution of a Maxwell-Boltzmann gas we obtain

$$
\left\langle E_{\text {kin }}\right\rangle=\frac{3}{2} k T=-\frac{1}{2}\left\langle E_{\text {grav }}\right\rangle \approx 1.6 \mathrm{keV} / c^{2}
$$

Hence our estimate for the central temperature of the Sun is $T_{\mathrm{c}} \approx 1.1 \mathrm{keV} / \mathrm{c}^{2} \approx 1.2 \times 10^{7} \mathrm{~K}$ (compared to $T_{\mathrm{c}} \sim 1.3 \mathrm{keV} / c^{2}$ in the so-called Solar Standard Model).

## 4. Peak of a SN light-curve.

Light can escape efficiently from the hot ejecta produced in a supernova (SN) only for
times larger than the diffusion time $\tau_{\text {diff }}$.
a.) Estimate the peak time as $t_{\text {peak }}=\tau_{\text {diff }}$ and find the corresponding shock radius using $M=1.4 M_{\odot}$ for the ejected mass, $v_{\text {sh }}=7000 \mathrm{~km} / \mathrm{s}$ for the shock velocity, and $\kappa \simeq 0.1 \mathrm{~cm}^{2} / \mathrm{g}$ for the opacity.
b.) Estimate the drop in the total energy contained in radiation, assuming that the radiation expands adiabatically from the initial radius $r=R_{\mathrm{WD}} \simeq 7000 \mathrm{~km}$ at the time of explosion to $t_{\text {peak }}$.
a.) In the random-walk picture, the number $N$ of steps of size $l_{\text {int }}$ needed to reach the distance $R_{\text {sh }}$ equals $N=R_{\mathrm{sh}}^{2} / l_{\mathrm{int}}^{2}$. Thus

$$
t=\tau_{\mathrm{diff}}=\frac{N l_{\mathrm{int}}}{c}=\frac{R_{\mathrm{sh}}^{2}}{c l_{\mathrm{int}}}=\frac{R_{\mathrm{sh}}^{2} \kappa \rho}{c} .
$$

Next we use for the density the average $\rho=3 M /\left(4 \pi R_{\mathrm{sh}}^{3}\right)$, and that the shock expands initially freely, $R_{\mathrm{sh}}=v_{\mathrm{sh}} t$. Combining everything, it follows

$$
t=\frac{3 \kappa M}{4 \pi v_{\mathrm{sh}} t c}
$$

or

$$
t=\left(\frac{3 \kappa M}{4 \pi v_{\mathrm{sh}} c}\right)^{1 / 2} \simeq 1.8 \times 10^{6} \mathrm{~s} \simeq 20 \text { days }
$$

The shock radius at peak luminosity is $R_{\mathrm{sh}}=v_{\mathrm{sh}} t \simeq 1.2 \times 10^{15} \mathrm{~cm}$.
b.) The energy $E$ stored in radiation is $E=u V=a T^{4} V$. If the radiation contained in the volume $V=4 \pi R^{3} / 3$ expands adiabatically, $\mathrm{d} U=-P \mathrm{~d} V$, it follows

$$
\begin{equation*}
R^{3} \mathrm{~d} u=-3(1+w) u R^{2} \mathrm{~d} R, \tag{4}
\end{equation*}
$$

where we used also $P=w u$ with $w=\left[a T^{4} / 3\right] /\left[a T^{4}\right]=1 / 3$ for radiation. Separating variables and integrating, it follows $u \propto R^{-3(1+w)}=R^{-4}$. Thus the energy in radiation scales as

$$
\begin{equation*}
E=u V \propto 1 / R \tag{5}
\end{equation*}
$$

With $R_{\text {sh }} \simeq 10^{6} R_{\text {WD }}$, we see that the energy initially transferred to radiation is reduced by a factor $10^{-6}$.

## 5. Accretion disks, Eddington accretion and growth of BHs.

a.) Derive the following approximation for the temperature profile $T(r)$ of an accretion disk,

$$
\begin{equation*}
T(r)=\left(\frac{G M \dot{M}}{8 \pi \sigma r^{3}}\right)^{1 / 4} \tag{6pts}
\end{equation*}
$$

b.) Find the corresponding luminosity due to the mass infall $\dot{M}$.
c.) Find the typical time-scale for the growth of BHs accreting at the Eddington rate, if the fraction $\xi$ of the rest mass is converted into radiation.
a.) Consider a mass element $\mathrm{d} M$ falling from a Keplerian orbit at $r+\mathrm{d} r$ to $r$ due to viscous interactions. Half of the gain in potential energy heats up the environment,

$$
\begin{equation*}
\mathrm{d} E_{\text {heat }}=\frac{1}{2}\left(\frac{G M \mathrm{~d} M}{r}-\frac{G M \mathrm{~d} M}{r+\mathrm{d} r}\right) . \tag{6}
\end{equation*}
$$

Assuming that the accretion is stationary, the produced heat equals the emitted radiation. Thus the luminosity of the accretion disk emitted by the shell at radius $r$ is given by

$$
\begin{equation*}
\mathrm{d} L=\frac{\mathrm{d} E_{\text {heat }}}{\mathrm{d} t}=2(2 \pi r) \mathrm{d} r \sigma T^{4}=\frac{1}{2} \frac{G M \dot{M}}{r}[1-(1-\mathrm{d} r / r+\ldots)]=\frac{1}{2} \frac{G M \dot{M} \mathrm{~d} r}{r^{2}}, \tag{7}
\end{equation*}
$$

where we used the Stefan-Boltzmann law and the factor two accounts for the "top" and "bottom" side of the disk. The temperature profile follows as

$$
\begin{equation*}
T(r)=\left(\frac{G M \dot{M}}{8 \pi \sigma r^{3}}\right)^{1 / 4} \tag{8}
\end{equation*}
$$

b.) The total luminosity can be obtained integrating Eq. (??) from the inner to the outer edge of the disc,

$$
\begin{equation*}
L_{\mathrm{tot}}=\frac{1}{2} G M \dot{M} \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \frac{\mathrm{~d} r}{r^{2}}=\frac{1}{2} G M \dot{M}\left(\frac{1}{r_{\mathrm{in}}}-\frac{1}{r_{\mathrm{out}}}\right) \approx \frac{1}{2} \frac{G M \dot{M}}{r_{\mathrm{in}}} \quad \text { for } \quad r_{\mathrm{in}} \ll r_{\mathrm{out}} . \tag{9}
\end{equation*}
$$

c.) The accretion rate is limited by radiation pressure,

$$
L=\xi \dot{M} c^{2}<L_{\mathrm{Edd}}=\frac{4 \pi c G M}{\kappa} .
$$

Setting $L=L_{\text {Edd }}$, it is $\dot{M} / M=$ const and thus the BH mass grows exponentially, $M=M_{0} \exp (\tau)$ with time-scale

$$
\tau=\frac{M}{\dot{M}}=\frac{\xi c \kappa}{4 \pi G} \simeq 4.5 \times 10^{7} \mathrm{yr} \frac{\xi}{0.1} \frac{L_{\mathrm{Edd}}}{L}
$$

## 6. Synchrotron radiation.

Electrons accelerated in a supernova remnant have an energy spectrum $\mathrm{d} N / \mathrm{d} E \propto E^{-\alpha}$. What is the slope of the resulting synchrotron intensity $I_{\nu}$, how does $I_{\nu}$ scale with the magnetic field strength $B$ ? You can assume that a relativistic electron with energy $E$ emits photons with $\omega_{s}=\omega_{\text {cr }}=3 \gamma^{2} \omega_{0} / 2$ and $\omega_{0}=e B / m c$.

The emitted intensity is proportional to

$$
I_{\nu} \mathrm{d} \nu \propto P_{\text {syn }} \frac{\mathrm{d} N}{\mathrm{~d} E}
$$

For ultrarelativistic electrons, we can use $P_{\text {syn }} \propto E^{2} B^{2}$, and thus

$$
I_{\nu} \mathrm{d} \nu \propto E^{2-\alpha} B^{2} \mathrm{~d} E .
$$

With $\gamma=E / m$, we obtain a relation between $E$ and $\omega=2 \pi \nu$, or differentiated

$$
\mathrm{d} \omega_{s} \propto 2 E \mathrm{~d} E B
$$

Inserting everything gives

$$
I_{\nu} \mathrm{d} \nu \propto E^{2-\alpha} B^{2} \frac{\mathrm{~d} \nu}{E B}=E^{1-\alpha} B \mathrm{~d} \nu=\nu^{\frac{1-\alpha}{2}} B^{\frac{1+\alpha}{2}}
$$

## Good luck!

## 1 Some formulas

$$
\begin{gather*}
\frac{\mathrm{d} M(r)}{\mathrm{d} r}=4 \pi r^{2} \rho(r),  \tag{10a}\\
\frac{\mathrm{d} T}{\mathrm{~d} r}=-\frac{3 L(r)\langle\kappa\rangle \rho}{16 \pi r^{2} a c T^{3}}, \quad \frac{\mathrm{~d} P}{\mathrm{~d} r}=-\frac{G M(r) \rho(r)}{r^{2}},  \tag{10b}\\
P=a T^{4} / 3, \quad P=n k T \quad P=K \rho^{\gamma}  \tag{11}\\
L_{\mathrm{Edd}}=\frac{4 \pi c G M}{\kappa}, \quad \kappa_{\mathrm{es}}=0.2(1+X) \mathrm{cm}^{2} / \mathrm{g}  \tag{12}\\
\mathcal{F}=\sigma T^{4}, \quad u=a r^{2} \varepsilon \rho  \tag{13}\\
P_{\text {syn }}=\frac{\mathrm{d} E}{\mathrm{~d} t}=\frac{2}{3} \alpha m^{2}\left(\frac{p_{\perp}}{m} \frac{e B}{m^{2}}\right)^{2} \tag{14}
\end{gather*}
$$

## 2 Physical constants and measurements

Gravitational constant
Planck's constant
velocity of light
Boltzmann constant
electron mass
proton mass
Fine-structure constant
Fermi's constant
Stefan-Boltzmann constant
Radiation constant
Rydberg constant
Thomson cross-section
$G=6.674 \times 10^{-11} \mathrm{~m}^{3} \mathrm{~kg}^{-1} \mathrm{~s}^{-2}=6.674 \times 10^{-8} \mathrm{~cm}^{3} \mathrm{~g}^{-1} \mathrm{~s}^{-2}$
$\hbar=h /(2 \pi)=1.055 \times 10^{-27} \mathrm{erg} \mathrm{s}$
$c=2.998 \times 10^{8} \mathrm{~m} / \mathrm{s}=2.998 \times 10^{10} \mathrm{~cm} / \mathrm{s}$
$k=1.381 \times 10^{-23} \mathrm{~J} / \mathrm{K}=1.38 \times 10^{-16} \mathrm{erg} / \mathrm{K}$
$m_{e}=9.109 \times 10^{-28} \mathrm{~g}=0.5110 \mathrm{MeV} / c^{2}$
$m_{p}=1.673 \times 10^{-24} \mathrm{~g}=938.3 \mathrm{MeV} / c^{2}$
$\alpha=e^{2} /(\hbar c) \approx 1 / 137.0$
$G_{F} /(\hbar c)^{3}=1.166 \times 10^{-5} \mathrm{GeV}^{-2}$
$\sigma=\left(2 \pi^{5} k^{4}\right) /\left(15 c^{2} h^{3}\right) \approx 5.670 \times 10^{-5} \mathrm{erg} \mathrm{s}^{-1} \mathrm{~cm}^{-2} \mathrm{~K}^{-4}$
$a=4 \sigma / c \approx 7.566 \times 10^{-15} \mathrm{erg} \mathrm{cm}^{-3} \mathrm{~K}^{-4}$
$R_{\infty}=1.10 \times 10^{5} \mathrm{~cm}^{-1}$
$\sigma_{T}=8 \pi \alpha_{\mathrm{em}}^{2} /\left(3 m_{e}^{2}\right)=6.652 \times 10^{-25} \mathrm{~cm}^{2}$

## 3 Astronomical constants and measurements

Astronomical Unit
Parsec
Tropical year
Solar radius
Solar mass
Solar luminosity
Solar apparent visual magnitude
Earth equatorial radius
Earth mass
Age of the universe
present Hubble parameter
present CMB temperature
present baryon density
dark matter abundance

$$
\begin{aligned}
& \mathrm{AU}=1.496 \times 10^{13} \mathrm{~cm} \\
& \mathrm{pc}=3.086 \times 10^{18} \mathrm{~cm}=3.261 \mathrm{ly} \\
& \mathrm{yr}=31556925.2 \mathrm{~s} \approx \pi \times 10^{7} \mathrm{~s} \\
& R_{\odot}=6.960 \times 10^{10} \mathrm{~cm} \\
& M_{\odot}=1.998 \times 10^{33} \mathrm{~g} \\
& L_{\odot}=3.84 \times 10^{33} \mathrm{erg} / \mathrm{s} \\
& m=-26.76 \\
& R_{\oplus}=6.378 \times 10^{8} \mathrm{~cm} \\
& M_{\oplus}=5.972 \times 10^{27} \mathrm{~g} \\
& t_{0}=(13.7 \pm 0.2) \mathrm{Gyr} \\
& H_{0}=73 \mathrm{~km} /(\mathrm{s} \mathrm{Mpc})=100 \mathrm{hkm} /(\mathrm{s} \mathrm{Mpc}) \\
& T=2.725 \mathrm{~K} \\
& n_{\mathrm{b}}=(2.5 \pm 0.1) \times 10^{-7} \mathrm{~cm}^{3} \\
& \Omega_{b}=\rho_{b} / \rho_{\mathrm{cr}}=0.0223 / h^{2} \approx 0.0425 \\
& \Omega_{\mathrm{DM}}=\Omega_{m}-\Omega_{b}=0.105 / \mathrm{h}^{2} \approx 0.20
\end{aligned}
$$

## 4 Other useful quantities

| cross section | $1 \mathrm{mbarn}=10^{-27} \mathrm{~cm}^{2}$ |
| :--- | :--- |
| flux conversion | $L=3.02 \times 10^{28} \mathrm{~W} \times 10^{-0.4 M}$ |
| energy conversion | $\mathrm{erg}=624 \mathrm{GeV}$ |

