NTNU Trondheim, Institutt for fysikk

Examination for FY3452 Gravitation and Cosmology
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Allowed tools: –

Note:
You can obtain 75 points answering all questions correctly. Marks are based on a maximum of 67 points, so 8 of them are bonus points.

1. Gravitational waves (GW).
a.) Write down the polarisation tensor $\varepsilon_{\mu\nu}$ for a GW $h_{\mu\nu} \propto \varepsilon_{\mu\nu} \mathrm{e}^{-\mathrm{i}kx}$ in the TT gauge for $D = 5$ spacetime dimensions, where the wave is propagating in the x^1 direction. How many polarisation states has the GW? (4 pts)
b.) In general, a GW in TT gauge can be obtained by applying an appropriate operator to a wave in a more general gauge. Show that the operator

$$P_k^{\ i} P_l^{\ j} - \frac{1}{2} P_{kl} P^{ij}$$

constructed out of $P_i^{\ j} = \delta_i^j - n_i n^j$ has the desired properties. (8 pts)
c.) Explain why gravitational waves do not exist in $D \leq 3$ spacetime dimensions. (2 pts)
d.) Estimate the amplitude of a gravitational wave produced by a black hole-black hole merger in our Galaxy by dimensional analysis; (you can assume as distance $d = 10\,\mathrm{kpc}$ and $M = 10 M_\odot$ as mass for the black holes.) (5 pts)

a.) The polarisation tensor is always symmetric, $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$. In the TT (=transverse-traceless) gauge, the tensor is transverse, $\varepsilon_{0\nu} = \varepsilon_{1\nu} = 0$, and traceless, $\varepsilon_{22} + \varepsilon_{33} + \varepsilon_{44} = 0$. It has therefore only five independent components,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{24} \\ 0 & 0 & \varepsilon_{23} & \varepsilon_{33} & \varepsilon_{34} \\ 0 & 0 & \varepsilon_{24} & \varepsilon_{34} & -\varepsilon_{22} - \varepsilon_{33} \end{pmatrix}.$$

b.) First, we show that $P_i^{\ j} = \delta_i^j - n_i n^j$ projects on the two-dimensional subspace orthogonal to the unit vector $\boldsymbol{n}$ and satisfies $P^2 = P$,

$$P_i^{\ j} P_j^{\ k} = (\delta_i^j - n_i n^j)(\delta_j^k - n_j n^k) = \delta_i^k - n_i n^k = P_i^{\ k}. \quad (1)$$

Morover, it is $n^i P_i^{\ j} v_j = 0$ for all vectors $\boldsymbol{v}$; Thus P projects indeed any vector on the subspace orthogonal to $\boldsymbol{n}$. Since a tensor is a multi-linear map, we have to apply a projection operator on each of the two indices of the polarisation tensor,

$$\varepsilon_{kl}^{\rm T} = P_k^{\ i} P_l^{\ j} \varepsilon_{ij}. \tag{2}$$

The tensor $\varepsilon_{kl}^{\rm T}$ is transverse, $n^k \varepsilon_{kl}^{\rm T} = n^l \varepsilon_{kl}^{\rm T} = 0$, but in general not traceless

$$\varepsilon_k^{{\rm T}\,k} = P_k^{\ i} P^{kj} \varepsilon_{ij} = P_l^{\ i} \varepsilon_{il}. \tag{3}$$

Subtracting the trace, we obtain the transverse, traceless part of ε,

$$\varepsilon_{kl}^{\rm TT} = \left(P_k^{\ i} P_l^{\ j} - \frac{1}{2} P_{kl} P^{ij} \right) \varepsilon_{ij}. \tag{4}$$

c.) In $D = 3$, we have using the transverse condition,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{yy} \end{pmatrix}.$$

Imposing the traceless condition implies $\varepsilon_{yy} = 0$ and the polarisation tensor is identically zero. Thus no propagating gravitational waves exist in $D < 4$.
Alternatively, you can remember that the Riemann tensor vanishes in empty space for $D < 4$.

d.) The amplitude has to satisfy $h \sim 1/d$ and $h \sim 1/a^n$, where a is the separation; the leading term has hopefully $n = 1$. ($n = 1$ is suggested also by the virial theorem, $1/a \propto v^2$.) The scale has to be set by R_s. Thus by dimensional reasons, the amplitude can be approximated by

$$h \simeq \frac{R_s^2}{da}.$$

The signal is maximal at coalesence, $a \simeq R_S$, or

$$h \simeq \frac{R_s}{d} \simeq \frac{3 \times 10^6 \text{cm}}{3 \times 10^{22} \text{cm}} \simeq 10^{-16}.$$

2. Einstein-de Sitter universe as symmetric space.

Maximally symmetric spactimes are spacetimes with constant curvature, satisfying

$$R_{\mu\nu\lambda\kappa} = K(g_{\mu\lambda} g_{\nu\kappa} - g_{\mu\kappa} g_{\nu\lambda})$$

with $K = \text{const}$. Note that we allow in this exercise for an arbitrary spacetime dimension D.
a.) Find the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R. (6 pts)

b.) Show that a maximally symmetric spacetime satisfies the vacuum Einstein equation with a cosmological constant Λ. (This was the first cosmological model, suggested by de Sitter and Einstein.) Derive the connection between Λ, K and D. (6 pts)

a.) Contracting R_{abcd} with g^{ac}, we obtain with $\delta^\mu_\mu = D$ in D dimensions for the Ricci tensor

$$R_{bd} = g^{ac} R_{abcd} = K g^{ac}(g_{ac}g_{bd} - g_{ad}g_{bc}) = K(Dg_{bd} - g_{bd}) = (D-1)Kg_{bd}\,. \tag{5}$$

A final contraction gives as curvature R of a D-dimensional maximally symmetric space

$$R = g^{ab}R_{ab} = K(D-1)\delta^a_a = D(D-1)K\,. \tag{6}$$

b.) Inserting the results into the vacuum Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 0$$

gives

$$(D-1)Kg_{\mu\nu} - \frac{1}{2}D(D-1)Kg_{\mu\nu} - \Lambda g_{\mu\nu} = 0$$

or

$$-\frac{K}{2}[D^2 - 3D + 2 + 2\Lambda/K]g_{\mu\nu} = 0.$$

The bracket has to be zero,

$$0 = D^2 - 3D + 2 + 2\Lambda/K = (D-1)(D-2) + 2\Lambda/K$$

or

$$\Lambda = -\frac{1}{2}(D-1)(D-2)K.$$

(For $D = 4$, it follows $\Lambda/3 = -K$. A comparison with the Friedmann eqaution shows then that $H^2 = 0$ implies $K = -k/R^2$.)

3. Schwarschild metric.

The metric outside a spherically symmetric mass distribution with mass M is given in Schwarzschild coordinates by

$$ds^2 = dt^2\left(1 - \frac{2M}{r}\right) - \frac{dr^2}{1 - \frac{2M}{r}} - r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2)$$

a.) Specifiy the Killing vector fields admitted by this metric. [No calculation needed.] (4 pts)

b.) What is the meaning of the surface $r = 2M$ and $r = 0$? [No calculation needed.] (4 pts)

c.) Two particles fall radially from infinity towards the point mass M. One starts with $e = 1$, the other with $e = 2$, where e is the energy per unit mass. A stationary observer at

$r = 6M$ measures their speed when they pass by. How much faster is the second particle? (10 pts)

a.) The metric is isotropic, i.e. invariant under rotations. Thus the three vector field $K_i = \varepsilon_{ijk} x_j \partial_k$ are Killing vector fields.
The metric is static, i.e. invariant under time translations. Thus the vector field $K_0 = \partial_t$ is the fourth Killing vector field.

b.) The surface $r = 2M$ is an infinite redshift surface and an event horizon. The singularity in the Schwarschild metric is just a coordinate singularity. In contrast, $r = 0$ is a physical singularity: the curvature and thus tidal forces become infinite for $r \to 0$.

c.) An observer with $\boldsymbol{u}_{\rm obs}$ measure as energy E and velocity v

$$E = \boldsymbol{p} \cdot \boldsymbol{u}_{\rm obs} = \frac{m}{\sqrt{1-v^2}}$$

for a particle with four-momentum p^μ and mass m.
If the observer is stationary, $u^r_{\rm obs} = u^\vartheta_{\rm obs} = u^\phi_{\rm obs} = 0$, the normalisation condition $\boldsymbol{u}_{\rm obs} \cdot \boldsymbol{u}_{\rm obs} = 1$ gives

$$u^t_{\rm obs} = \left(1 - \frac{2M}{r}\right)^{-1/2}.$$

Thus

$$E = m\boldsymbol{u} \cdot \boldsymbol{u}_{\rm obs} = m g_{\alpha\beta} u^\alpha u^\beta_{\rm obs} = m\left(1 - \frac{2M}{r}\right)^{1/2} u^t = \frac{m}{\sqrt{1-v^2}}.$$

Now we replace u^t by the conserved energy,

$$e = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2M}{r}\right) u^t,$$

to obtain

$$v(e) = \frac{1}{e}\left(e^2 - 1 + \frac{2M}{r}\right)^{1/2}.$$

The ratio of the velocities at $r = 6M$ follows as

$$\frac{v(2)}{v(1)} = \frac{1}{2}\left(\frac{4 - 1 + 1/3}{1 - 1 + 1/3}\right)^{1/2} = \frac{\sqrt{10}}{2}.$$

4. 2d-Cosmology.
Consider a universe in $D = 2$ dimensions with metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}x^2$$

and filled with a perfect fluid.
a.) Calculate the Christoffel symbols for this metric. (4 pts)

b.) Show that the metric is conformally flat. Consider an observer with a finite life-time. Draw a possible world-line for this observer; indicate the part of the spacetime visible to the observer. (4 pts)

c.) The stress tensor of an ideal fluid is

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - P g_{\mu\nu}$$

with $\rho = \rho(x,t)$ and $P = P(x,t)$. Use local energy-momentum conservation to show that in the fluid rest-frame

$$\dot{\rho} = \frac{\dot{a}}{a}(\rho + P) \quad \text{and} \quad P' = 0$$

holds, where $\dot{f} = df/dt$ and $f' = df/dx$. (6 pts)

a.) We use either the definition or the Euler-Lagrange equation for $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ to determine the Christoffel symbols for this metric.

$$\ddot{t} + a\dot{a}\dot{x}^2 = 0 \Rightarrow \Gamma^0{}_{xx} = a\dot{a}$$

and

$$\ddot{x} + 2\frac{\dot{a}}{a}\dot{t}\dot{x} = 0 \Rightarrow \Gamma^x{}_{tx} = \frac{\dot{a}}{a}$$

b.) Introducing conformal time, $\mathrm{d}\eta = \mathrm{d}t/a$, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}x^2 = a^2(\eta)[\mathrm{d}\eta^2 - \mathrm{d}x^2],$$

i.e. is conformally equivalent to $\mathbb{R}(1,1)$. Thus the light-cone structure is the same as in Minkowski space. Drawing an arbitrary time-like geodesics of finite length as representation of our observer, the area enclosed by the past-light cones starting from birth and death is visible.

c.) We find first the non-vanishing components of $T^{\mu\nu}$,

$$T^{00} = \rho + P - P = \rho \quad \text{and} \quad T^{11} = Pg^{11} = P/a^2$$

Next we evaluate the two equations contained in $\nabla_\mu T^{\mu\nu} = 0$ using the Christoffel symbols,

$$\nabla_\mu T^{\mu 0} = \partial_0 T^{00} + \Gamma^1{}_{10}T^{00} + \Gamma^0{}_{11}T^{11} = \dot{\rho} + H\rho + HP = \dot{\rho} + H(\rho + P) = 0$$

$$\nabla_\mu T^{\mu 1} = \partial_1 T^{11} + 0 + 0 = \frac{1}{a^2}\partial_x P = 0.$$

5. Symmetries.

Consider in Minkowski space a complex scalar field ϕ with Lagrange density

$$\mathscr{L}_1 = \partial_a \phi^\dagger \partial^a \phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2$$

and the photon field with

$$\mathscr{L}_2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

a.) Name the symmetries of the Langrangians. [No calculation needed.] (4 pts)
b.) Derive one conserved current of your choice of the system (4 pts)
c.) Specify the (locally gauge invariant) interaction term $\mathscr{L}_{\rm int}$ between the complex scalar and the photon field. (4 pts)

a.) $\mathscr{L}_1$: space-time symmetries: Translation, Lorentz, (scale invariance). internal: global SO(2) / U(1) invariance. $\mathscr{L}_2$: space-time symmetries: Translation, Lorentz, (scale invariance). internal: local U(1) invariance.

b.) i) Translations: From $\phi_a(x) \to \phi_a(x-\varepsilon) \approx \phi_a(x) - \varepsilon^\mu\partial_\mu\phi(x)$ we find the change $\delta\phi_a(x) = -\varepsilon^\mu\partial_\mu\phi(x)$. The Lagrange density changes similiarly, $\mathscr{L}(x) \to \mathscr{L}(x-\varepsilon)$ or $\delta\mathscr{L}(x) = -\varepsilon^\mu\partial_\mu\mathscr{L}(x) = -\partial_\mu(\varepsilon^\mu\mathscr{L}(x))$. Thus $K^\mu = -\varepsilon^\mu\mathscr{L}(x)$ and inserting both in the Noether current gives

$$J^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\left[-\varepsilon^\nu\partial_\nu\phi(x)\right] + \varepsilon^\mu\mathscr{L}(x) = \varepsilon_\nu T^{\mu\nu}$$

with $T^{\mu\nu}$ as (canonical) energy-momentum stress tensor and four-momentum as Noether charge.
or
ii) Charge conservation: We can work either with complex fields and U(1) phase transformations

$$\phi(x) \to \phi(x)\mathrm{e}^{\mathrm{i}\alpha} \quad , \quad \phi^\dagger(x) \to \phi^\dagger(x)\mathrm{e}^{-\mathrm{i}\alpha}$$

or real fields (via $\phi = (\phi + \mathrm{i}\phi_2)/\sqrt{2}$) and invariance under rotations SO(2). With $\delta\phi = \mathrm{i}\alpha\phi$, $\delta\phi^\dagger = -\mathrm{i}\alpha\phi^\dagger$, the conserved current is

$$J^\mu = \mathrm{i}\left[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right]$$

c.) Replacing the normal with gauge-invariant derivatives in $\mathscr{L}_1$, the Lagrangian is

$$\mathscr{L} = (D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F^2\,.$$

or expanded with $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$,

$$\mathscr{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2 \underbrace{-\mathrm{i}qA_\mu\phi^\dagger\partial^\mu\phi + \mathrm{i}qA^\mu(\partial_\mu\phi^\dagger)\phi + q^2A_\mu A^\mu\phi^\dagger\phi}_{\mathscr{L}_I} - \frac{1}{4}F^2\,.$$

Some formula:

$$\ddot{x}^c + \Gamma^c{}_{ab}\dot{x}^a\dot{x}^b = 0$$

$$R^\mu{}_{\nu\lambda\kappa} = \partial_\lambda \Gamma^\mu{}_{\nu\kappa} - \partial_\kappa \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\rho\lambda}\Gamma^\rho{}_{\nu\kappa} - \Gamma^\mu{}_{\rho\kappa}\Gamma^\rho{}_{\nu\lambda},$$

$$R_{\alpha\beta} = R^\rho{}_{\alpha\rho\beta}$$

$$0 = \delta\mathscr{L} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)}\, \delta\phi_a - K^\mu \right) .$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \,.$$

$$D_\mu = \partial_\mu + \mathrm{i} q A_\mu$$

$$\frac{e^2 - 1}{2} = \frac{\dot{r}^2}{2} + V_{\text{eff}}$$

$$H^2 = \frac{8\pi}{3} G\rho - \frac{k}{R^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P)$$

$$E(z) = (1+z)E_0$$

$$1\text{Mpc} \simeq 3.1 \times 10^{24}\text{cm}$$

$$R_S \simeq 3\text{km}\frac{M}{M_\odot}$$