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Examination for FY3451 Astrophysics II
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Allowed tools: officially approved pocket calculator

1. Multiple choice.

Assume a star suffers energy losses additional to those of neutrinos, $\varepsilon_{\text{loss}} = (1+\delta)\varepsilon_\nu$. As a result, the star becomes relative to star with $\delta = 0$ (2 pts)

a.) hotter ☒
b.) cooler ☐
c.) smaller ☒
d.) larger ☐

2. Knowledge and basic understanding.

Answer concise, keywords are enough.

• Name the main source of energy generation in/for a (4 pts)

a.) main-sequence star,
b.) type II supernova
c.) light-curve of a supernova
d.) active galactic nucleus.

a.) Conversion of mass into energy via hydrogen burning, $4p \to {}^4\text{He} + 2e^+ + 2\nu_e$.
b.) Release of gravitational potential energy during the collapse.
c.) Radioactive decay of ^{56}Ni and ^{56}Co.
d.) Release of gravitational potential energy during the accretion of matter.

• Name the three main sources of energy transport in stars. Give for two an example of a star for which this transport mechanism is important. (5 pts)

i) radiative energy transfer, i.e. energy transport by photons, ii) conduction, i.e. the transport of heat by the Brownian motion of electrons or atoms, and iii) convection, i.e. macroscopic matter flows.

Conduction plays a prominent role as energy transport only for dense systems, and is therefore only relevant in the dense, final stages of stellar evolution as, e.g., white dwarfs. Convection dominates in (the outer part of) low-mass stars and the center of massive stars. Otherwise radiative energy transfer dominates.

• A photon in a dense medium has the mean-free path $l_0 = 10\,\text{cm}$ for an (isotropic) scattering process. How long does it take to travel the distance $r = R_\odot$? (4 pts)

The distance traveled in a random walk is $\langle x^2 \rangle^{1/2} \simeq \sqrt{N} l_0$. Thus the time is $\tau = N l_0/c = R_\odot^2/(l_0 c) \simeq 500\,\mathrm{yr}$.

• Name the two mechanisms which are currently believed to explain Gamma-Ray Bursts. (2 pts)

Short GRBs: coalescence of a binary system, most likely neutron stars. Long GRBs: supernovae of very massive stars

3. Virial theorem.
The virial theorem connects the internal (or kinetic) and the potential energy of a gravitationally bound system,

$$-3\int_0^M \mathrm{d}m\, \frac{P}{\rho} = U_{\mathrm{pot}}.$$

Consider a star where the pressure is the sum of the pressure of a classical gas and radiation.
a.) Defining β as the fraction the gas contributes to the total pressure, $P_{\mathrm{gas}} = \beta P$, and assuming β uniform in the star, show that its total energy is given by (8 pts)

$$E_{\mathrm{tot}} = \frac{1}{2}\beta U_{\mathrm{pot}} = -\frac{\beta}{2-\beta} U_{\mathrm{int}}.$$

b.) Comment briefly on the limits $\beta \to 1$ and $\beta \to 0$. (4 pts)
c.) If the star contracts maintaining the same uniform β, which fraction of the released gravitational potential energy is radiated away, which fraction is turned into heat? (4 pts)

a.) The internal energy density (per mass) of gas and radiation are given by

$$u_{\mathrm{gas}} = \frac{3}{2}\frac{P_{\mathrm{gas}}}{\rho} = \frac{3}{2}\beta\frac{P}{\rho}$$

and

$$u_{\mathrm{rad}} = 3\frac{P_{\mathrm{rad}}}{\rho} = 3(1-\beta)\frac{P}{\rho}.$$

Adding the two terms up gives

$$u_{\mathrm{rad}} + u_{\mathrm{gas}} = \frac{3}{2}(2-\beta)\frac{P}{\rho}.$$

Inserting

$$\frac{P}{\rho} = \frac{2}{3(2-\beta)}(u_{\mathrm{rad}} + u_{\mathrm{gas}}).$$

into the virial theorem results in

$$U_{\mathrm{pot}} = -3\int_0^M \mathrm{d}m\, \frac{P}{\rho} = -\frac{2}{(2-\beta)}\int_0^M \mathrm{d}m\,(u_{\mathrm{rad}} + u_{\mathrm{gas}}) = -\frac{2}{(2-\beta)} U_{\mathrm{int}}.$$

Thus the total energy is

$$E_{\rm tot} = U_{\rm pot} + U_{\rm int} = \frac{1}{2}\beta U_{\rm pot} = -\frac{\beta}{2-\beta}U_{\rm int},$$

as claimed.

b.) In the limit $\beta \to 1$, we obtain the virial theorem in its standard form, $E_{\rm tot} = U_{\rm pot}/2 = -U_{\rm int}$, valid for a classical gas. In the limit for pure radiation pressure, $\beta \to 0$, $E_{\rm tot} \to 0$, corresponding to an unbound system as expected for relativistic particles.

c.) For a change $\Delta U_{\rm pot}$ in the potential energy, the energy radiated away corresponds to the change in total energy, $\Delta E_{\rm tot} = \beta/2\Delta U_{\rm pot}$. The energy used to heat up the star corresponds to the change in internal energy, $\Delta U_{\rm int} = -[(2-\beta)/2]\Delta U_{\rm pot}$.

4. Pressure in a stellar atmosphere.

a.) Show that the hydrostatic equilibrium equation can be written as

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{g}{\kappa}$$

where τ is the optical depth, g the gravitational acceleration and κ the opacity. (6 pts)
b.) Assuming a constant κ and a negligible thickness of the atmosphere, derive an expression for the pressure in the atmosphere. (6 pts)

a.) Divide

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$$

by $\kappa\rho$,

$$\frac{\mathrm{d}P}{\kappa\rho\mathrm{d}r} = -\frac{g}{\kappa}$$

and recall the definition of the optical depth, $\mathrm{d}\tau = \kappa\rho\mathrm{d}r$.

b.) Using the negligible thickness of the atmosphere, we can set $g = GM/R^2$. Setting also $\kappa_R = \kappa(R)$, we separate variables and integrate,

$$-\kappa_R\int_R^\infty \mathrm{d}P = \frac{GM}{R^2}\int_R^\infty \mathrm{d}\tau.$$

At the position of the atmosphere, $\tau = 2/3$, by definition. The pressure at infinity is zero, and so

$$\kappa_R P_R = \frac{2GM}{3R^2}$$

or

$$P_R = \frac{2GM}{3\kappa_R R^2}.$$

5. High-energy photons.
a.) An experiment measures the differential intensity $I(E)$ of photons emitted by particles accelerated in a supernova remnant between the energy range 10 MeV and 100 GeV. Sketch two differences which in principle allow one to distinguish the two cases that i) photons are emitted mainly by electrons and that ii) photons are mainly produced in pion decays? (4 pts)
b.) The figure below sketches the energy spectrum of photons from a hypothetical active galactic nucleus. Explain briefly the main production mechanism of photons in the different energy regions marked in the figure. (4 pts)

a.) In case i), photons are produced most likely via synchrotron emission of electrons (plus bremsstrahlung at low energies); in case ii) by hadronic interaction of protons on gas (plus bremsstrahlung at low energies). This leads to the following differences:

- the slope of the photon spectrum is $(1-\alpha)/2$ for synchrotron emission, while it is α for hadronic interactions, if α is the slope of the primary spectrum.
- the photon spectrum should be symmetric around $m_\pi/2$ in case of hadronic interactions, n break at $m_\pi/2$ is expected in the case of synchrotron emission.
- in the case of synchrotron emission, the photon luminosity is related to the magnetic field strength, in the case of hadronic interactions to the density of targets (here: gas density).

1. synchrotron emission; optically thick (self-absorption)
2. synchrotron emission; optically thin
3. UV-light from accretion disk, plus reprocessed photons (IR from dust, ...)
4. inverse Compton emission, on CMB, synchroton or external photons

6. Indirect dark matter searches.
One possibility to probe dark matter (DM) is to search for photons produced by DM annihilations in small satellite galaxies of the Milky Way. These dwarf galaxies are strongly dominated by DM; a prime example is the Draco dwarf galaxy with a mass $M = 10^8 M_\odot$ at the distance $d = 76\,\mathrm{kpc}$.
a.) How is the differential photon flux $F(E_\gamma) = \mathrm{d}N_\gamma/(\mathrm{d}E\mathrm{d}A\mathrm{d}t)$ from Draco measured on Earth connected with the thermally averaged annihilation cross section $\langle\sigma_{\mathrm{ann}}v\rangle$, the number density $n_X(r)$ (assuming a spherically symmetric DM distribution in Draco) of DM, and the energy spectrum dN_γ/dE_γ of photons produced per annihilation? (8 pts)
b.) Estimate the number of photons detected per time in the Fermi satellite, using as area $A_{\mathrm{det}} = 1\,\mathrm{m}^2$ for the detector, a DM particle with mass $m_X = 100\,\mathrm{GeV}/c^2$ and a thermally averaged annihilation cross section $\langle\sigma_{\mathrm{ann}}v\rangle = 10^{-26}\mathrm{cm}^3/\mathrm{s}$. Use as total number of photons produced per annihilation $N_\gamma = 100$. To avoid the integral, assume $n_X(r) = \text{const.}$ with $R = 1\,\mathrm{pc}$. (6 pts)

a.) The rate of annihilations per volume and time is given by $\langle\sigma_{\mathrm{ann}}v\rangle n_X^2$. Assuming a spherically symmetric DM distribution in Draco, the rate of annihilations per time is then

$$\langle\sigma_{\mathrm{ann}}v\rangle 4\pi \int_0^R \mathrm{d}r r^2 n_X^2(r).$$

The corresponding rate of photons produced per time and energy is

$$\frac{\mathrm{d}N}{\mathrm{d}E}\langle\sigma_{\mathrm{ann}}v\rangle 4\pi \int_0^R \mathrm{d}r r^2 n_X^2(r).$$

Finally, we obtain the flux by dividing with $4\pi d^2$ as

$$F(E_\gamma) = \frac{\mathrm{d}N_\gamma}{\mathrm{d}E\mathrm{d}A\mathrm{d}t} = \frac{1}{4\pi d^2}\frac{\mathrm{d}N}{\mathrm{d}E}\langle\sigma_{\mathrm{ann}}v\rangle \times 4\pi \int_0^R \mathrm{d}r r^2 n_X^2(r).$$

b.) For constant density, the last factor becomes simply

$$\frac{4\pi R^3}{3} n_x^2 = N_X n_X = \frac{M}{m_X} n_X$$

with $N_X = M/m_X = 1.12 \times 10^{63}/100 \simeq 10^{61}$ and $n_X = N_x/(4\pi R^3/3) \simeq 9 \times 10^4/\mathrm{cm}^3$. The number of photons detected per time is

$$\frac{\mathrm{d}N_\gamma}{\mathrm{d}t} = A_{\mathrm{det}} \frac{\mathrm{d}N_\gamma}{\mathrm{d}A\mathrm{d}t} = \frac{A_{\mathrm{det}}}{4\pi d^2}\, N_\gamma\, \langle \sigma_{\mathrm{ann}} v \rangle n_X\, \frac{M}{m_X}$$

$$\frac{\mathrm{d}N_\gamma}{\mathrm{d}t} = (1.4 \times 10^{-48}) \times\ 100\ \times (9 \times 10^{-22}/\mathrm{s})\ \times (1.1 \times 10^{61}) \simeq 1 \times 10^{-6} \frac{1}{\mathrm{s}} \simeq 40/\mathrm{yr}.$$

[Remark: The chosen value of R is much smaller than the extension of Draco to compensate for the higher density for $r \to 0$.]

Good luck!

1 Some formulas

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r), \qquad \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho(r)}{r^2}, \tag{1a}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3L(r)\langle\kappa\rangle\rho}{16\pi r^2\, acT^3}, \qquad \frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \varepsilon\rho\,. \tag{1b}$$

$$P = aT^4/3, \qquad P = nkT, \qquad U = \frac{3}{2}kT, \qquad P = K\rho^\gamma. \tag{2}$$

$$L_{\mathrm{Edd}} = \frac{4\pi cGM}{\kappa}, \qquad \kappa_{\mathrm{es}} = 0.2(1+X)\mathrm{cm}^2/\mathrm{g} \tag{3}$$

$$\mathcal{F} = \sigma T^4, \qquad u = aT^4 \tag{4}$$

2 Physical constants and measurements

Gravitational constant	$G = 6.674 \times 10^{-11}\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2} = 6.674 \times 10^{-8}\text{cm}^3\,\text{g}^{-1}\,\text{s}^{-2}$
Planck's constant	$\hbar = h/(2\pi) = 1.055 \times 10^{-27}\text{erg}\,\text{s}$
velocity of light	$c = 2.998 \times 10^8$ m/s$= 2.998 \times 10^{10}$ cm/s
Boltzmann constant	$k = 1.381 \times 10^{-23}$ J/K $= 1.38 \times 10^{-16}$ erg/K
electron mass	$m_e = 9.109 \times 10^{-28}\,\text{g} = 0.5110\,\text{MeV}/c^2$
proton mass	$m_p = 1.673 \times 10^{-24}\,\text{g} = 938.3\,\text{MeV}/c^2$
Fine-structure constant	$\alpha = e^2/(\hbar c) \approx 1/137.0$
Fermi's constant	$G_F/(\hbar c)^3 = 1.166 \times 10^{-5}$ GeV^{-2}
Stefan-Boltzmann constant	$\sigma = (2\pi^5 k^4)/(15c^2h^3) \approx 5.670 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4}
Radiation constant	$a = 4\sigma/c \approx 7.566 \times 10^{-15}$ erg cm^{-3} K^{-4}
Rydberg constant	$R_\infty = 1.10 \times 10^5$ cm^{-1}
Thomson cross-section	$\sigma_T = 8\pi\alpha_{\text{em}}^2/(3m_e^2) = 6.652 \times 10^{-25}$ cm^2

3 Astronomical constants and measurements

Astronomical Unit	$\text{AU} = 1.496 \times 10^{13}\,\text{cm}$
Parsec	$\text{pc} = 3.086 \times 10^{18}\,\text{cm} = 3.261\,\text{ly}$
Tropical year	$\text{yr} = 31\,556\,925.2\,\text{s} \approx \pi \times 10^7\,\text{s}$
Solar radius	$R_\odot = 6.960 \times 10^{10}$ cm
Solar mass	$M_\odot = 1.998 \times 10^{33}$ g $= 1.12 \times 10^{57}\,\text{GeV}/c^2$
Solar luminosity	$L_\odot = 3.84 \times 10^{33}$ erg/s
Solar apparent visual magnitude	$m = -26.76$
Earth equatorial radius	$R_\oplus = 6.378 \times 10^8$ cm
Earth mass	$M_\oplus = 5.972 \times 10^{27}$ g
Age of the universe	$t_0 = (13.7 \pm 0.2)$ Gyr
present Hubble parameter	$H_0 = 73\,\text{km}/(\text{s}\,\text{Mpc}) = 100\,h\,\text{km}/(\text{s}\,\text{Mpc})$
present CMB temperature	$T = 2.725$ K
present baryon density	$n_\text{b} = (2.5 \pm 0.1) \times 10^{-7}$ cm^3
	$\Omega_b = \rho_b/\rho_\text{cr} = 0.0223/h^2 \approx 0.0425$
dark matter abundance	$\Omega_\text{DM} = \Omega_m - \Omega_b = 0.105/h^2 \approx 0.20$

4 Other useful quantities

cross section	$1\,\text{mbarn} = 10^{-27}$ cm^2
flux conversion	$L = 3.02 \times 10^{28}\text{W} \times 10^{-0.4M}$
energy conversion	$\text{erg} = 624\,\text{GeV}$