NTNU Trondheim, Institutt for fysikk

Examination for FY3452 Gravitation and Cosmology
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Allowed tools: –

Note:
You can obtain 75 points answering all questions correctly. Marks are based on a maximum of 67 points, so 8 of the 75 are bonus points.

1. Generalities & understanding.
Give concise answers.
a.) Name four experimental tests favouring General Relativity as a correct description of gravity. (2 pts)
b.) Explain why the Christoffel symbols are not tensors and what is their meaning instead. (2 pts)
c.) What are Riemannian normal coordinates? Does the scalar curvature R vanish in such a system? (2 pts)
d.) By which property is a stationary limit surface defined? What is the special property (implied by the name) of the ergoregion of a Kerr black hole? (2 pts)
e.) Do geodesics with $\mathrm{d}\phi = \mathrm{d}\vartheta = 0$ in the Kerr metric exist? If yes, where? If no, why not? (2 pts)
f.) Which form of energy/curvature is in a spacetime with $R_{\alpha\beta} = 0$ and $\Lambda = 0$ allowed? (2 pts)

a.) Perihelion precession, light deflection, Shapiro time delay, gravitational redshift, energy loss due GW emission in binaries (Hulse-Taylor pulsar), detection of GWs with $c = 1$.

b.) In $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\sigma\mu} V^\sigma$, the LHS is a tensor, but $\partial_\mu V^\nu$ is not. This implies that $\Gamma^\nu{}_{\sigma\mu}$ is also not a tensor; more specificly, it has to transform such to compensate the non-tensorial term of $\partial_\mu V^\nu$.
The Christoffel symbols specify how the base vectors change, $\partial_\rho \boldsymbol{e}^\mu = -\Gamma^\mu{}_{\rho\sigma} \boldsymbol{e}^\sigma$.

c.) In a (pseudo-) Riemaninan manifold without torsion, we can introduce at any given point P a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu}$, and $\Gamma^\nu{}_{\sigma\mu} = 0$, i.e. only second- (tidal effects) and higher-order derivatives of the metric are non-zero. This corresponds to the idea of the equivalence principle that locally the effects of gravity can be eleminated choosing a freely falling inertial system.
Since we cannot set the second derivatives of $g_{\mu\nu}$ to zero, the scalar curvature in P is in a general (i.e. curved) spacetime non-zero.

d.) A stationary limit surface is defined by $g_{tt} = 0$, since the normalisation condition $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ is inconsistent with $u^\alpha = (1, 0, 0, 0)$ and $g_{tt} \leq 0$.

The ergosphere is the region between the stationary limit surface and the outer horizon of a Kerr black hole: It is the region where one can extracts the rotational energy of the Kerr black hole by an appropriate splitting process $1 \to 2 + 3$.

e.) The $g_{t\phi}$ term in the Kerr metric leads to "frame-dragging", and thus in general the answer is no. An exception is the radial infall from the south or north pole: there, the "frame-dragging" leads to a spin of the particle/gyroscope. (Both possibilities give full points.)

f.) All forms of matter except gravity are contained in $T_{\mu\nu}$ and thus absent. What is left is gravity: propagating modes (i.e. gravitational waves) and static curvature (i.e. the curvature outside a star).
More formally: The Riemann tensor has 20 independent components, 10 can be associated to the Ricci tensor and are fixed by the (local) matter distribution. The other 10 correspond to gravitational waves and/or "non-local" gravitational effects.

2. Test particle and geodesics.
Consider $S = \alpha \int \mathrm{d}\tau$ as action for a free, massive relativistic particle.
a.) Determine the constant α requiring the correct non-relativistic limit. (3 pts)
b.) Show that the Lagrangian $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ with $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ leads to the same geodesics. (3 pts)
c.) How is the Lagrangian for a massless particle defined? (2 pts)

a.) We ask that the action has the correct non-relativistic limit. Then

$$S_0 = \alpha \int_a^b \mathrm{d}\tau = \alpha \int_a^b \mathrm{d}t\, \sqrt{1 - v^2} = \int_a^b \mathrm{d}t\, \left(-m + \frac{1}{2} m v^2 + \mathcal{O}(v^4) \right) , \tag{1}$$

if we set $\alpha = -m$. The mass m corresponds to a potential energy in the non-relativistic limit and has therefore a negative sign in the Lagrangian. Moreover, a constant drops out of the equations of motion, and thus the term $-m$ can be omitted in the non-relativistic limit.

b.) The action S_0 is invariant under reparametrisations

$$S_0 = -m \int_a^b \mathrm{d}\tau = -m \int_a^b \mathrm{d}\tau \left(g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \right)^{1/2} = -m \int_{\sigma(a)}^{\sigma(b)} \mathrm{d}\sigma \left(g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \right)^{1/2} \tag{2}$$

and thus we can absorb the numerical value of $-m$ by an appropriate change $\sigma \to \sigma'$.
Next we show that $L = L_0^2$ gives the same geodescics as L_0. Using the Lagrange equations, it is

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}\tau} \left[2L_0 \frac{\partial L_0}{\partial \dot{x}^\mu} \right] - 2L_0 \frac{\partial L_0}{\partial x^\mu}. \tag{3}$$

Since $L_0 = 1$ does not depend on τ, we can divide by $2L_0$. Thus the two Lagrangians give the same equations of motion for a free relativistic particle.

c.) For a massless particle, $\mathrm{d}\tau = 0$, and we have to use another quantity to parameterise their geodesics. Any parameter which ensures that $\mathrm{d}x^\mu(\lambda)/\mathrm{d}\lambda = \text{const.}$ is suited (set of "affine parameters" $\lambda \to a\lambda + b$). Thus the Lagrangian is

$$L = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}.$$

3. Kerr black hole (BH).

In Boyer–Lindquist coordinates, the metric of a Kerr black hole is given by

$$\begin{aligned} \mathrm{d}s^2 = {} & \left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{4Mar\sin^2\vartheta}{\rho^2}\mathrm{d}\phi\mathrm{d}t - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\vartheta^2 \\ & - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\vartheta}{\rho^2}\right)\sin^2\vartheta\mathrm{d}\phi^2, \end{aligned} \tag{4}$$

with the abbreviations

$$a = L/M\,, \qquad \rho^2 = r^2 + a^2\cos^2\vartheta, \qquad \Delta = r^2 - 2Mr + a^2. \tag{5}$$

The only non-zero quadratic curvature invariant of this metric is

$$R^{\mu\nu\lambda\kappa}R_{\mu\nu\lambda\kappa} = \frac{48M^2(r^2 - a^2\cos^2\vartheta)[(r^2 + a^2\cos^2\vartheta)^2 - 16r^2a^2\cos^2\vartheta]}{(r^2 + a^2\cos^2\vartheta)^6}$$

a.) Name the Killing vectors and the corresponding conserved quantities of this spacetime. (2 pts)
b.) What are the physical, what the coordinate singularities of this metric? (5 pts)
c.) Find the position of the two event horizons. (5 pts)
d.) Calculate the area of the outer event horizon. (6 pts)

a.) The metric $g_{\mu\nu}$ does not depend on t and ϕ, i.e. it is time-independent and axially symmetric. Hence the two Killing vectors are $\xi^\mu = (1,0,0,0)$ and $\eta^\mu = (0,0,0,1)$, ordering coordinates as $\{t, r, \vartheta, \phi\}$. The conserved quantities are the energy and the ϕ component of the angular momentum.

b.) The metric is singular for $\rho = 0$ and $\Delta = 0$. In the first case, $\mathrm{R}^{\mu\nu\lambda\kappa}R_{\mu\nu\lambda\kappa} \propto \rho^{-6}$ diverges and thus $r = 0$ and $\vartheta = \pi/2$ is a physical singularity. In the second case, the curvature invariants are finite and thus $\Delta = 0$ is a coordinate singularity.

c.) The two surfaces defined by the coordinate singularity, $\Delta = r^2 - 2Mr + a^2 = 0$ are candidates for horizons. These are null surface satisfying

$$0 = g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) = g^{rr}(\partial_r f)^2 + g^{\vartheta\vartheta}(\partial_\vartheta f)^2 \tag{6}$$

for an axially symmetric metric. The quadratic equation $\Delta = 0$ has for $a < M$ two solutions,

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{7}$$

which depend only on the coordinate r. The condition defining a horizons becomes simply $g^{rr} = 0$ or $g_{rr} = 1/g^{rr} \to \infty$. Hence, r_- and r_+ define an inner and outer horizon around a Kerr black hole.

d.) The line-element of the outer horizon follows from inserting r_+ together with $\mathrm{d}r = \mathrm{d}t = 0$ into the metric,

$$\mathrm{d}s^2 = \rho_+^2 \mathrm{d}\vartheta^2 + \left(r_+^2 + a^2 + \frac{2Mr_+ a^2 \sin^2\vartheta}{\rho_+^2} \right) \sin^2\vartheta \mathrm{d}\phi^2, \tag{8}$$

Using $r_\pm^2 + a^2 = 2Mr_\pm$, we obtain

$$\mathrm{d}s^2 = \rho_+^2 \mathrm{d}\vartheta^2 + \left(\frac{2Mr_+}{\rho_+} \right)^2 \sin^2\vartheta \mathrm{d}\phi^2. \tag{9}$$

Hence the metric determinant g_2 restricted to the angular variables is given by $\sqrt{g_2} = \sqrt{g_{\vartheta\vartheta} g_{\phi\phi}} = 2Mr_+ \sin\vartheta$. Integrating gives the area A of the horizon as

$$A = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\vartheta \sqrt{g_2} = 8\pi M r_+ = 8\pi M(M + \sqrt{M^2 - a^2}). \tag{10}$$

4. Gravitational waves (GW).

a.) Consider a GW in the harmonic gauge, $\partial_\alpha \bar{h}^{\alpha\beta} = 0$. How many independent elements has its polarisation tensor $\varepsilon^{\alpha\beta}$? (2 pts)

b.) Decide for each of the three following expressions for a GW, if they are valid in the TT gauge; if yes, state the type of polarisation (linear/circular/...) and the direction of wave propagation. . (6 pts)

$$\bar{h}_{\alpha\beta}(t, \boldsymbol{x}) = h_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\omega t & \sin\omega t & 0 \\ 0 & \sin\omega t & -\cos\omega t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

$$\bar{h}_{\alpha\beta}(t, \boldsymbol{x}) = h_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\omega t & \sin\omega t & 0 \\ 0 & \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

$$\bar{h}_{\alpha\beta}(t, \boldsymbol{x}) = h_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos\omega t & 0 \\ 0 & 0 & 0 & \cos\omega t \end{pmatrix} .$$

c.) Derive the smallest spacetimes dimension D where GW can exist. . (4 pts)
d.) Estimate the amplitude of a gravitational wave produced by a black hole-black hole merger in our Galaxy by dimensional analysis; (you can assume as distance $d = 10\,\mathrm{kpc}$ and $M = 30M_\odot$ as mass for the black holes.) (6 pts)

a.) The polarisation tensor is symmetric, $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$, and has thus in general 10 independent components. The harmonic gauge condition imposes 4 constraints, leading to 6 independent components.

b.) In the TT gauge, the polarisation tensor is traceless and transverse. In case i), the polarisation tensor is traceless and transverse, it is a right-handed circularly polarized wave propagating in the z direction. Case ii) is not traceless and thus not a valid expression in the TT gauge. In case iii), the polarisation tensor is traceless and transverse, it is a linear polarized wave propagating in the x direction.

b.) In $D = 3$, we have using the transverse condition,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{yy} \end{pmatrix}.$$

Imposing the traceless condition implies $\varepsilon_{yy} = 0$ and the polarisation tensor is identically zero. Thus no propagating gravitational waves exist in $D < 4$.
Alternatively, you can compare the number n of independent elements of the Ricci tensor (or field equations) to the number m of independent elements of the Riemann tensor. Only for $D \geq 4$, it is $m > n$ and thus in empty space, $R_{\mu\nu} = 0$, the Riemann tensor is not completely fixed and can allow the presence of GWs.

c.) The amplitude has to satisfy $h \sim 1/d$ and $h \sim 1/a^n$, where a is the separation; the leading term has hopefully $n = 1$. The scale has to be set by R_s. Thus by dimensional reasons, the amplitude can be approximated by

$$h \simeq \frac{R_s^2}{da}.$$

The signal is maximal at coalesence, $a \simeq R_S$, or

$$h \simeq \frac{R_s}{d} \simeq \frac{3 \times 10^6\mathrm{cm}}{3 \times 10^{22}\mathrm{cm}} \simeq 10^{-16}.$$

5. Inflation.
Consider in a flat FLWR metric a real scalar field ϕ which has in Minkowski space the Lagrange density

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - V(\phi).$$

a.) Write down the action S and derive the field equation of $\phi(t,\boldsymbol{x})$ in a flat FLWR metric. Interprete the terms of the field equation. (8 pts)
b.) Find the stress tensor $T_{\mu\nu}$ and the equation-of-state $w = P/\rho$ neglecting spatial gradients of ϕ. (8 pts)
c.) Explain briefly (without derivation) under which conditions this field can drive inflation, i.e. a period with accelerated expansion in the early universe. (3 pts)

a.) For a flat FRW metric, it is $g_{\mu\nu} = \mathrm{diag}(1,-a^2,-a^2,-a^2)$ and thus $g^{\mu\nu} = \mathrm{diag}(1,-a^{-2},-a^{-2},-a^{-2})$, and $\sqrt{|g|} = a^3$. Thus the action becomes

$$S = \int_\Omega \mathrm{d}^4x \sqrt{|g|} \left\{ \frac{1}{2} g^{\mu\nu} \nabla_\mu\phi \nabla_\nu\phi - V(\phi) \right\} = \int_\Omega \mathrm{d}^4x \, a^3 \left\{ \frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi) \right\}. \tag{11}$$

Varying the action, exchanging derivatives and variation, and using Gauss' theorem for a partial integration gives

$$\begin{aligned} \delta S_{\mathrm{KG}} &= \int_\Omega \mathrm{d}^4x \, a^3 \left\{ \dot{\phi}\delta\dot{\phi} - \frac{1}{a^2}(\nabla\phi)\cdot\delta(\nabla\phi) - V'\delta\phi \right\} \\ &= \int_\Omega \mathrm{d}^4x \left\{ -\frac{\mathrm{d}}{\mathrm{d}t}(a^3\dot{\phi}) + a\nabla^2\phi - a^3 V' \right\} \delta\phi \\ &= \int_\Omega \mathrm{d}^4x \, a^3 \left\{ -\ddot{\phi} - 3H\dot{\phi} + \frac{1}{a^2}\nabla^2\phi - V' \right\} \delta\phi \overset{!}{=} 0 \, . \end{aligned} \tag{12}$$

Thus the field equation for a Klein-Gordon field in a FRW background is

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V' = 0 \, . \tag{13}$$

The term $3H\dot{\phi}$ acts in an expanding universe as a friction term for the oscillating ϕ field. Moreover, the gradient of ϕ is also suppressed for increasing a; this term can be therefore often neglected in an expanding universe.

b.) Using the relation from the formula section, it is

$$T_{\mu\nu} = 2\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left[\frac{1}{2} g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - V(\phi)\right]. \tag{14}$$

We can describe the scalar field also as an ideal fluid. Equating the two expressions for the stress tensor gives

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathscr{L} \overset{!}{=} (\rho + P)u_\mu u_\nu - P g_{\mu\nu}. \tag{15}$$

Comparing the two independent tensor structures we can identify $P = \mathscr{L}$ and

$$\nabla_\mu\phi\nabla_\nu\phi = (\rho + P)u_\mu u_\nu. \tag{16}$$

Contracting the indices with $g^{\mu\nu}$, remembering $u_\mu u^\mu = 1$ and using $\nabla_\mu\phi\nabla^\mu\phi = 2\mathscr{L} + 2V$ results in

$$\rho = P + 2V. \tag{17}$$

Now we have to calculate only the energy density $\rho = T_{00}$ in order to determine the (isotropic) pressure P and the equation of state $w = P/\rho$. In an FLRW background, the energy density of the field ϕ is given by

$$\rho = T_{00} = \dot{\phi}^2 - \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\boldsymbol{\nabla}\phi)^2 - V(\phi)\right] = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\boldsymbol{\nabla}\phi)^2 + V(\phi). \tag{18}$$

Thus the pressure follows as

$$P = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\boldsymbol{\nabla}\phi)^2 - V(\phi). \tag{19}$$

Neglecting the $(\boldsymbol{\nabla}\phi)^2$ term, the equation of state simplifies to

$$w = \frac{P}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \in [-1 : 1]. \tag{20}$$

c.) To have long enough inflation, $\dot{\phi}^2 \ll 2V(\phi)$, we need large $V(\phi)$, small slope V'/V and curvature V''/V, where the relevant scale is set by the Planck mass. Thus

$$\frac{V'}{(8\pi G)^{1/2}V} \ll 1 \quad \text{and} \quad \frac{V''}{8\pi G V} \ll 1. \tag{21}$$

Some formula:

$$\ddot{x}^c + \Gamma^c{}_{ab}\dot{x}^a\dot{x}^b = 0$$

$$R^\mu{}_{\nu\lambda\kappa} = \partial_\lambda\Gamma^\mu{}_{\nu\kappa} - \partial_\kappa\Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\rho\lambda}\Gamma^\rho{}_{\nu\kappa} - \Gamma^\mu{}_{\rho\kappa}\Gamma^\rho{}_{\nu\lambda},$$

$$0 = \delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a - K^\mu\right).$$

$$T_{\mu\nu} = 2\frac{\partial\mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L}$$

$$\text{null surface: } 0 = g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) \tag{22}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\frac{e^2-1}{2} = \frac{\dot{r}^2}{2} + V_{\rm eff}$$

$$H^2 = \frac{8\pi}{3}G\rho - \frac{k}{R^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P)$$

$$E(z) = (1+z)E_0$$

$$1\text{Mpc} \simeq 3.1 \times 10^{24}\text{cm}$$

$$R_S \simeq 3\text{km}\frac{M}{M_\odot}$$