NTNU Trondheim, Institutt for fysikk

Examination for FY3452 Gravitation and Cosmology
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Allowed tools: –

Note:
You can obtain 75 points answering all questions correctly. Marks are based on a maximum of 67 points, so 8 are bonus points.

1. Generalities & understanding.

Give concise answers.
a.) Name four experimental tests favouring General Relativity as a correct description of gravity. (2 pts)
b.) Explain why the Christoffel symbols are not tensors and what is their meaning instead. (2 pts)
c.) What are Riemannian normal coordinates? Does the scalar curvature R vanish for such coordinates? (2 pts)
d.) How is the Lagrangian for a free, massless particle defined? (2 pts)
e.) Consider a non-static, isotropic spacetime with metric

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2)$$

describing, e.g., a pulsating or collapsing star. Does such a star emit gravitational waves? (2 pts)
f.) By which property is the event horizon of a black hole defined? (2 pts)
g.) Is a spacetime with $R_{\alpha\beta} = 0$ and $\Lambda = 0$ flat? (2 pts)

a.) Perihelion precession, light deflection, Shapiro time delay, gravitational redshift, cosmological redshift, energy loss due to GW emission in binaries (Hulse-Taylor pulsar), detection of GWs with $c = 1$.

b.) In $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\sigma\mu} V^\sigma$, the LHS is a tensor, but $\partial_\mu V^\nu$ is not. This implies that $\Gamma^\nu{}_{\sigma\mu}$ is also not a tensor; more specifically, it has to transform in such a way that the non-tensorial term of $\partial_\mu V^\nu$ is compensated.
The Christoffel symbols specify how the base vectors change, $\partial_\rho \boldsymbol{e}^\mu = -\Gamma^\mu{}_{\rho\sigma} \boldsymbol{e}^\sigma$.
Alternative: They are a specific case of an affine connection which is torsion-free and metric compatible.

c.) In a (pseudo-) Riemaninan manifold without torsion, we can introduce at any given point P a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu}$, and $\Gamma^\nu{}_{\sigma\mu} = 0$, i.e. only second- (tidal effects) and higher-order derivatives of the metric are non-zero. This corresponds to the idea of the equivalence principle that locally the effects of gravity can be eleminated choosing a freely falling inertial

system.
Since we cannot set the second derivatives of $g_{\mu\nu}$ to zero, the scalar curvature in P is in a general (i.e. curved) spacetime non-zero.

d.) For a massless particle, $\mathrm{d}\tau = 0$, and we have to use another quantity to parameterise their geodesics. Any parameter which ensures that $\mathrm{d}x^\mu(\lambda)/\mathrm{d}\lambda = \text{const.}$ is suited (set of "affine parameters" $\lambda \to a\lambda + b$). Thus the Lagrangian can be chosen as

$$L = g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}.$$

e.) The metric is spherically symmetric and thus the quadrupole moment vanishes. Thus the star does not emit GWs.

f.) The event horizon of a BH is a three-dimensional hypersurface formed by light-rays and thus a null surface. The condition for a null surface defined by the constraint $f(x^\mu) = 0$ is $0 = g^{\mu\nu}(\partial_\mu f)(\partial_\nu f)$.

g.) Since the Riemann tensor has 20 independent components, of which 10 are associated to the Ricci tensor and fixed by the (local) matter distribution, 10 more components are left arbitrary in a Ricci-flat spacetime. They correspond to gravitational waves and/or "non-local" gravitational effects. Thus in general a Ricci-flat spacetime is not flat.

2. Fluid dynamics.
a.) Show that dust, i.e. the fluid elements of a pressureless fluid, moves along geodesics. (4 pts)
b.) Consider an ideal (or perfect) fluid. Assume that in the rest-frame of each fluid element, the fluid has the energy density ρ and the isotropic pressure tensor $P_{ij} = P\delta_{ij}$. Derive from this the stress tensor $T^{\alpha\beta}$ in an arbitrary frame. (2 pts)
c.) Use local energy-momentum conservation to obtain the equation of motion of an ideal fluid in the form (4 pts)

$$u^\alpha A + \frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} B + (g^{\alpha\beta} - u^\alpha u^\beta) C_\beta = 0.$$

d.) Show that this equation corresponds to two independent equations, one expressing local energy conservation and one local momentum conservation. [You don't have to show the meaning (i.e. local energy and momentum conservation) of the two equations.] (4 pts)

a.) We first show that flow mass flow $j^\beta = \rho u^\beta$ is conserved: Use the product rule,

$$0 = \nabla_\alpha T^{\alpha\beta} = \nabla_\alpha(\rho u^\alpha) u^\beta + \rho u^\alpha \nabla_\alpha u^\beta.$$

Then eleminate u^β in the first term by contracting with u_β and using $u_\beta u^\beta = 1$,

$$0 = \nabla_\alpha(\rho u^\alpha) + \rho u^\alpha u_\beta \nabla_\alpha u^\beta.$$

Finally note that $\nabla_\alpha(u_\beta u^\beta) = 0 = 2u_\beta \nabla_\alpha u^\beta$. Thus the second term is zero, and the conservation of $j^\beta = \rho u^\beta$ follows.
Now we use $\nabla_\beta(\rho u^\beta) = 0$ in the first line,

$$0 = \nabla_\alpha T^{\alpha\beta} = \rho u^\alpha \nabla_\alpha u^\beta,$$

and remember that $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$ is the tangent vector at the trajectory $x^\alpha(\tau)$. Thus

$$u^\alpha \nabla_\alpha u^\beta = \frac{\mathcal{D}u}{\mathrm{d}\tau} = 0$$

which is the condition for a geodesics.

b.) We set first $P = 0$. Then $\rho = T_{00}$ and $u^\alpha = (1, \mathbf{0})$ in a local inertial rest frame. This becomes a valid tensor equation setting $T^{\alpha\beta} = \rho u^\alpha u^\beta$.
Alternatively, use the tensor method: Express $T^{\alpha\beta}$ as a linear combination of all relevant tensors, which are in our case the four-velocity u^α plus the invariant tensors of Minkowski space, i.e. the metric tensor and the Levi-Civita symbol. Additionally, we impose the constraint that $T^{\alpha\beta}$ is symmetric, leading to

$$T^{\alpha\beta} = A\rho u^\alpha u^\beta + B\rho\eta^{\alpha\beta}\,. \tag{1}$$

In the rest-frame, $u^\alpha = (1, \mathbf{0})$, the condition $T^{00} = \rho$ leads $A - B = 1$, while $T^{11} = 0$ implies $B = 0$. Thus the stress tensor of dust is

$$T^{\alpha\beta} = \rho u^\alpha u^\beta\,. \tag{2}$$

Next we include the effect of pressure. We know that the pressure tensor coincides with the σ_{ij} part of the stress tensor. Moreover, for a perfect fluid in its rest-frame, the pressure is isotropic $P_{ij} = P\delta_{ij}$. This corresponds to $P_{ij} = -P\eta_{ij}$ and adds $-P$ to T^{00}. Compensating for this gives

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta - P\eta^{\alpha\beta}\,. \tag{3}$$

c.) Local energy-momentum conservation means $\nabla_\beta T^{\alpha\beta} = 0$. Applied to an ideal fluid, this leads to

$$u^\alpha \left[\frac{\mathrm{d}\rho}{\mathrm{d}\tau} + (\rho + P)\nabla_\beta u^\beta\right] + (\rho + P)\frac{\mathcal{D}u^\alpha}{\mathrm{d}\tau} - (g^{\alpha\beta} - u^\alpha u^\beta)\nabla_\beta P = 0. \tag{4}$$

Next we recall from b.) that u^α and the acceleration $\mathcal{D}u^\alpha/\mathrm{d}\tau$ are perpendicular. Thus we have only to show that $\mathcal{P}^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ projects the pressure gradient $\nabla_\beta P$ on the subspace orthogonal to the fluid velocity u^α, as it should be for a pure force:
First, we verify that the operator $\mathcal{P}_\alpha^{\ \beta} = \delta_\alpha^\beta - n_\alpha n^\beta$ with $\boldsymbol{n}$ an unit vector satisfies $\mathcal{P}^2 = \mathcal{P}$,

$$\mathcal{P}_\alpha^{\ \beta}\mathcal{P}_\beta^{\ \gamma} = (\delta_\alpha^\beta - n_\alpha n^\beta)(\delta_\beta^\gamma - n_\beta n^\gamma) = \delta_\alpha^\gamma - n_\alpha n^\gamma = \mathcal{P}_\alpha^{\ \gamma}. \tag{5}$$

Morover, it is $n^\alpha \mathcal{P}_\alpha^{\ \beta} n^\beta = 0$ for all unit vectors $\boldsymbol{n}$; Thus $\mathcal{P}$ projects indeed any vector on the subspace orthogonal to $\boldsymbol{n}$.
Thus we obtain two independent equations, one multiplying with u^α for the component parallel to u^α,

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} + (\rho + P)\nabla_\alpha u^\alpha = 0 \tag{6}$$

and one multiplying with $\mathcal{P}_\beta^{\ \gamma}$ for the components perpendicular to u^α,

$$(\rho + P)\frac{\mathcal{D}u^\alpha}{\mathrm{d}\tau} = -(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta P. \tag{7}$$

3. Kerr black hole (BH).

In Boyer–Lindquist coordinates, the metric of a Kerr black hole is given by

$$\mathrm{d}s^2 = \left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{4Mar\sin^2\vartheta}{\rho^2}\mathrm{d}\phi\mathrm{d}t - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\vartheta^2$$
$$- \left(r^2 + a^2 + \frac{2Mra^2\sin^2\vartheta}{\rho^2}\right)\sin^2\vartheta\mathrm{d}\phi^2,$$

with the abbreviations

$$a = L/M\,, \qquad \rho^2 = r^2 + a^2\cos^2\vartheta, \qquad \Delta = r^2 - 2Mr + a^2.$$

The only non-zero quadratic curvature invariant of this metric is

$$R^{\mu\nu\lambda\kappa}R_{\mu\nu\lambda\kappa} = \frac{48M^2(r^2 - a^2\cos^2\vartheta)[(r^2 + a^2\cos^2\vartheta)^2 - 16r^2a^2\cos^2\vartheta]}{(r^2 + a^2\cos^2\vartheta)^6}$$

a.) What are the physical, what the coordinate singularities of this metric? (2 pts)
b.) Find the position of the stationary limit surface. (3 pts)
c.) Consider now the limit ot a Schwarzschild black hole. Two particles fall radially from infinity towards a point mass M, one starting with energy per unit mass $e = 1$, the other one with $e = 2$. How big is the ratio of their velocities measured by a stationary observer at $r = 6M$? (7 pts)

a.) The metric is singular for $\rho = 0$ and $\Delta = 0$. In the first case, $R^{\mu\nu\lambda\kappa}R_{\mu\nu\lambda\kappa} \propto \rho^{-6}$ diverges and thus the ring $r = 0$ and $\vartheta = \pi/2$ is a physical singularity. In the second case, the curvature invariants are finite and thus $\Delta = 0$ is a coordinate singularity.

b.) A stationary limit surface is defined by $g_{tt} = 0$, since the normalisation condition $\boldsymbol{u}\cdot\boldsymbol{u} = 1$ is inconsistent with $u^\alpha = (u^t, 0, 0, 0)$ and $g_{tt} \leq 0$. Solving

$$g_{tt} = 1 - \frac{2Mr}{\rho^2} = 0, \tag{8}$$

we find two solutions,

$$r_{1/2} = M \pm \sqrt{M^2 - a^2\cos^2\vartheta}. \tag{9}$$

where the outer one corresponds to the position of the stationary limit surface.

c.) An observer with $\boldsymbol{u}_{\text{obs}}$ measure as energy E and velocity v

$$E = \boldsymbol{p}\cdot\boldsymbol{u}_{\text{obs}} = \frac{m}{\sqrt{1 - v^2}}$$

for a particle with four-momentum p^μ and mass m. If the observer is stationary, $u^r_{\rm obs} = u^\vartheta_{\rm obs} = u^\phi_{\rm obs} = 0$, the normalisation condition $\boldsymbol{u}_{\rm obs} \cdot \boldsymbol{u}_{\rm obs} = 1$ gives

$$u^t_{\rm obs} = \left(1 - \frac{2M}{r}\right)^{-1/2}.$$

Thus

$$E = m\boldsymbol{u} \cdot \boldsymbol{u}_{\rm obs} = mg_{\alpha\beta}u^\alpha u^\beta_{\rm obs} = m\left(1 - \frac{2M}{r}\right)^{1/2} u^t = \frac{m}{\sqrt{1-v^2}}.$$

Now we replace u^t by the conserved energy,

$$e = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2M}{r}\right)u^t,$$

to obtain

$$v(e) = \frac{1}{e}\left(e^2 - 1 + \frac{2M}{r}\right)^{1/2}.$$

The ratio of the velocities at $r = 6M$ follows as

$$\frac{v(2)}{v(1)} = \frac{1}{2}\left(\frac{4 - 1 + 1/3}{1 - 1 + 1/3}\right)^{1/2} = \frac{\sqrt{10}}{2}.$$

4. Gravitational waves (GW).

a.) Consider a GW in the harmonic gauge, $\partial_\alpha \bar{h}^{\alpha\beta} = 0$. How many independent elements has its polarisation tensor $\varepsilon^{\alpha\beta}$? (2 pts)

b.) Write down the form of $\bar{h}_{\alpha\beta}(t, \boldsymbol{x})$ for a GW travelling in the positive y direction for the possible physical linear polarisation states. (4 pts)

c.) Estimate the amplitude of a gravitational wave produced by a neutron star-neutron star merger at redshift $z = 0.1$ by dimensional analysis. (6 pts)

a.) The polarisation tensor is symmetric, $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$, and has thus in general 10 independent components. The harmonic gauge condition imposes 4 constraints, leading to 6 independent components.

b.) In order to obtain the physical polarisation states, we have to impose the TT gauge. Then the polarisation tensor is traceless and transverse. The two linear polarisation states of a GW along the y direction are obtained by setting either h_{11} or h_{13} to zero,

$$\bar{h}_{\alpha\beta}(t, \boldsymbol{x}) = h_0^+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cos[\omega(t - y) + \varphi^{(+)}] + h_0^\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cos[\omega(t - y) + \varphi^\times],$$

where h_0^a are the amplitudes and φ^a the phases of the two waves with polarisation a.

c.) The amplitude has to satisfy $h \sim 1/d$ and $h \sim 1/a^n$, where a is the separation; the leading term has hopefully $n = 1$. The scale has to be set by R_s. Thus by dimensional reasons, the amplitude can be approximated by

$$h \simeq \frac{R_s^2}{da}.$$

Let us assume that a BH is formed in the merger. The signal is maximal at coalesence, $a \simeq R_S$, or $h \simeq R_s/d$. Finally, we need to estimate the distance using Hubble's law $z \simeq dH_0/c$ or

$$d \simeq \frac{zc}{H_0} \simeq 0.1\frac{3 \times 10^{19}\text{cm/s}}{7 \times 10^6\text{cm/s}}\text{Mpc} \simeq 430\text{Mpc} \simeq 10^{27}\text{cm}$$

Thus

$$h \simeq \frac{R_s}{d} \simeq \frac{3 \times 10^5\text{cm}}{10^{27}\text{cm}} \simeq 3 \times 10^{-22}.$$

5. FLRW cosmologies.

Consider a FLWR spacetime with metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\sin^2\vartheta\mathrm{d}\phi^2 + \mathrm{d}\vartheta^2)\right]$$

and Ricci tensor.

$$R_{\mu\nu} = \mathrm{diag}\{3\ddot{R}/R, -\frac{C}{1-kr^2}, -Cr^2\sin^2\vartheta, -Cr^2\}$$

with $C = (\ddot{R}R + 3\dot{R}^2 + 2k)$.

a.) Calculate the scalar curvature R of this spacetime. (4 pts)

b.) Explain why R is non-zero even in the "flat" $k = 0$ case. (2 pts)

c.) Consider now the "flat" $k = 0$ case of this metric. Name the Killing vectors and derive the "redshift formula" for a photon. (4 pts)

d.) Consider now the *spatial* part of the FLRW metric for general k. The Ricci tensor satisfies $R_{ab} = 2Kg_{ab}$. What is the property such a spacetime has to satisfy? Write down the Riemann tensor R_{abcd} for the spatial part of the FLRW metric. (2 pts)

a.) We use that for a diagonal metric $g^{\mu\nu} = \mathrm{diag}\{1/g_{\mu\mu}\}$, i.e,

$$g^{\mu\nu} = \mathrm{diag}\{1, -(1-kr^2)/R^2, -1/(Rr\sin\vartheta)^2, -1/(Rr)^2\}$$

Thus

$$R = g^{\mu\nu}R_{\mu\nu} = 3\ddot{R}/R + 3C/R^2 = \frac{6(\ddot{R}R + \dot{R}^2 + k)}{R^2}$$

b.) The spacetime is due to the $R(t)$ term still curved; only 3-space becomes flat for $k = 0$.

c.) The 3-spaces at t = const. and $k = 0$ are $\mathbb{R}^3$: Thus there 3 Killing vectors connected to spatial translations, $\boldsymbol{\xi}_i = \partial_i$, and 3 Killing vectors connect to rotations, $\boldsymbol{J}_i = \varepsilon_{ijk} x_j \partial_k$. [The three Killing vectors leading to cms movement with constant velocity can be omitted.] With $\boldsymbol{p}$ as four-momentum of the photon, it follows $\boldsymbol{\xi}_i \cdot \boldsymbol{p} =$ const. Consider e.g. the x component,

$$\boldsymbol{\xi}_x \cdot \boldsymbol{p} = g_{\mu\nu} \xi_x^\mu p^\mu = -R^2 p^x = \text{const.}$$

and thus $p^x \propto 1/R^2$. Next we use that $\boldsymbol{p}$ is a null vector,

$$(p^t) - a^2 (p^x)^2 = 0$$

or $p^t \propto a p^x \propto 1/R$. An observer at rest, $\boldsymbol{u} = (1, 0, 0, 0)$ measures thus as frequency

$$\omega = \boldsymbol{p} \cdot \boldsymbol{u} = p^t \propto 1/R$$

d.) Such spaces are maximally symmetric, i.e. they have constant scalar curvature. Then $R_{ab} = 2K g_{ab}$ and $R_{abcd} = K(g_{ac} g_{bd} - g_{ad} g_{bc})$.

6. Field theory.

Consider a free, complex scalar field ϕ in Minkowski space.

$$\mathscr{L}_0 = A \partial_\mu \phi^\dagger \partial^\mu \phi + B \phi^\dagger \phi.$$

a.) Explain your choice for A and B. (3 pts)
b.) Find the Noether current of this Lagrangian (4 pts)
c.) Find the interaction terms between the complex field ϕ and the photons A^μ, promoting the global symmetry of $\mathscr{L}_0$ into a local gauge symmetry. (4 pts)

a.) The relativistic eenergy-momentum relation requires $-Am^2 = B$. The requirement that energy is bounded from below requires $A > 0$ and $B \leq 0$. Canonically normalised *real* fields $\phi_{1/2}$ have $A = 1/2$. Combining them into $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$ gives $A = 1$ for the complex field ϕ.

b.) From the formulas given, we have for the Noether current j^μ,

$$0 = \delta \mathscr{L} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a - K^\mu \right) = \partial_\mu j^\mu \,.$$

We can work either with complex fields and U(1) phase transformations

$$\phi(x) \to \phi(x) \mathrm{e}^{\mathrm{i}\alpha} \quad , \quad \phi^\dagger(x) \to \phi^\dagger(x) \mathrm{e}^{-\mathrm{i}\alpha}$$

or real fields (via $\phi = (\phi + \mathrm{i}\phi_2)/\sqrt{2}$) and invariance under rotations SO(2). With $\delta\phi = \mathrm{i}\alpha\phi$, $\delta\phi^\dagger = -\mathrm{i}\alpha\phi^\dagger$, the conserved current is

$$J^\mu = \mathrm{i} \left[\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger) \phi \right]$$

We perform the replacement $\partial_\mu D_\mu = \partial_\mu + \mathrm{i}qA_\mu$ in the kinetic term of the Lagrangian,

$$\mathscr{L} = (D_\mu\phi)^\dagger D^\mu\phi.$$

Expanding gives

$$\mathscr{L} = \partial_\mu\phi^\dagger\partial^\mu\phi \underbrace{-\mathrm{i}qA_\mu\phi^\dagger\partial^\mu\phi + \mathrm{i}qA^\mu(\partial_\mu\phi^\dagger)\phi + q^2A_\mu A^\mu\phi^\dagger\phi}_{\mathscr{L}_I}$$

Some formula:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\sigma\mu}V^\sigma$$

$$\frac{D}{\mathrm{d}\sigma}T^{\mu\dots}_{\nu\dots} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\nabla_\alpha T^{\mu\dots}_{\nu\dots} = 0$$

$$\ddot{x}^c + \Gamma^c{}_{ab}\dot{x}^a\dot{x}^b = 0$$

$$R^\mu{}_{\nu\lambda\kappa} = \partial_\lambda\Gamma^\mu{}_{\nu\kappa} - \partial_\kappa\Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\rho\lambda}\Gamma^\rho{}_{\nu\kappa} - \Gamma^\mu{}_{\rho\kappa}\Gamma^\rho{}_{\nu\lambda},$$

$$0 = \delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a - K^\mu\right).$$

$$T_{\mu\nu} = 2\frac{\partial\mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L}$$

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu$$

$$\text{null surface: } 0 = g^{\mu\nu}(\partial_\mu f)(\partial_\nu f)$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\frac{e^2-1}{2} = \frac{\dot{r}^2}{2} + V_{\text{eff}}$$

$$H^2 = \frac{8\pi}{3}G\rho - \frac{k}{R^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P)$$

$$v \simeq H_0 d$$

$$E(z) = (1+z)E_0$$

$$H_0 \simeq 70\text{km/s/Mpc}$$

$$1\text{Mpc} \simeq 3.1 \times 10^{24}\text{cm}$$

$$R_S \simeq 3\text{km}\frac{M}{M_\odot}$$