NTNU Trondheim, Institutt for fysikk

Examination for FY3452 Gravitation and Cosmology

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Possible languages for your answers: Bȯkmal, Castellano, English, Nynorsk.
Allowed tools: Pocket calculator, mathematical tables
Some formulas can be found at the end of p.2.

1. Hyperbolic plane H^2.

The line-element of the Hyperbolic plane H^2 is given by

$$ds^2 = y^{-2}(dx^2 + dy^2) \qquad \text{and} \quad y \geq 0\,.$$

a. Write out the geodesic equations and deduce the Christoffel symbols $\Gamma^a{}_{bc}$. (4 pts)
b. Calculate the Riemann (or curvature) tensor $R^a{}_{bcd}$ and the scalar curvature R. (4 pts)

2. Kerr metric.

The metric outside a spherically symmetric mass distribution with mass M and angular momentum J is given by

$$ds^2 = -\left[1 - \frac{2Mr}{\rho^2}\right]dt^2 - \frac{4Mar\sin^2\vartheta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta^2 + \left[r^2 + a^2 + \frac{2Mra^2\sin^2\vartheta}{\rho^2}\right]\sin^2\vartheta d\phi^2\,,$$

with

$$a = \frac{J}{M}\,, \qquad \rho^2 = r^2 + a^2\cos^2\vartheta\,, \qquad \Delta = r^2 - 2Mr + a^2\,.$$

a. Find the outer boundary of the ergosphere, i.e. the surface enclosing the region where no stationary observers are possible in the Kerr metric. (3 pts)
b. Find the two horizons of the Kerr metric. (1.5 pt)
d. Determine the smallest possible unstable circular orbit of a massive particle for $J = 0$. (Hint: Consider the effective potential V_{eff}.) (6 pt)

3. Scalar fields in FLRW metric.

Consider a scalar field ϕ with potential V

$$\mathcal{L} = \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi + V(\phi)$$

in a flat FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(\sin^2\vartheta d\phi^2 + d\vartheta^2)\right]\,.$$

a. Derive the equation of motions for ϕ. (4 pts)
b. Derive the energy-momentum tensor for ϕ. (3 pts)

c. Derive the equation of state $w = P/\rho$ for ϕ assuming that the field ϕ is uniform in space, $\phi(t, \vec{x}) = \phi(t)$. (2 pts)
d. Scalar fields are often used as models for inflation. Give *one* necessary condition that ϕ can drive inflation. (1 pt)

4. Killing vectors.
Consider Minkowski space of special relativity,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,.$$

a. Find all ten Killing vectors and name the conserved symmetries and conserved quantities. (4.5 pt)

5. Radiation from a particle in a gravitational field.
An electron is released at the position $r \gg 2M$ in the gravitational field of a point mass M and moves thereafter on a geodesics. Give either a *short, simple* argument why
a. the electron does not emit radiation. (1 pt)
and
b. the electron does emit radiation. (1 pt)
or
c. decide which one of the alternatives is correct and explain why. (2 pt)

Some formula: Signature of the metric $(-,+,+,+)$.

$$T^{ab} = (\rho + P)u^a u^b + P g^{ab}$$

$$\nabla_i \xi_j + \nabla_j \xi_i = 0$$

$$\ddot{x}^c + \Gamma^c{}_{ab}\dot{x}^a\dot{x}^b = 0$$

$$\frac{2}{\sqrt{|g|}}\frac{\delta S_{\mathrm{m}}}{\delta g^{ab}} = T_{ab}$$

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}\,,$$

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\, g^{ab}\delta g_{ab}$$