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STUD. TECHN. TORMOD GJESTLAND

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English title: *"Gauge theories at finite temperature"*

This work has been carried out at NTNU, under the supervision of Jens O. Andersen

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Jens O. Andersen

Associate Professor/Professor at Department of Physics

Abstract

This thesis is written with the main purpose of deriving the pressure in pure $SU(3)$ gauge theory to two-loop, before fitting a running coupling to lattice data in the deconfined phase, computed by Boyd et al.

The pressure and the fitted coupling were extrapolated to $50T_c = 7500\text{MeV}$. At high temperatures the pressure slowly converged towards the ideal-gas pressure, just as expected. The coupling was found to be small enough for the theory to be treated perturbatively at the end of this temperature range. Here, it was approximately $\frac{g^2}{4\pi} \simeq 3 \cdot \alpha_e = \frac{3}{137}$ at $T = 7500\text{MeV}$.

The problem was solved by first deriving the functional representation of the partition function, and showing that the corrections could be found in terms of Feynman diagrams, at weak coupling. Moreover, the finite-temperature Feynman rules were shown to be obtainable by modifying the ones representing the S-matrix elements. This was first used to derive the pressure in QED to two-loop to make some aquaintance with the theory, before finding the pressure in a hot gluon plasma to two-loop. Both these derivations are found to order g^3 in the coupling by including dressed propagators.

Finally, MatLab was used to fit the pressure to lattice data.

Preface

This thesis is written with the main purpose of deriving the pressure in pure-gluon plasma to a two-loop approximation. It represents 20 weeks of work at the Norwegian University of Science and Technology, corresponding to 30 study points. Before this semester I had only taken one introductory course in quantum field theory. This was soon found to be insufficient for starting directly on the QCD calculations. I tried to read several books on finite-temperature field theory to become acquainted with the imaginary-time formalism. Each time I failed to fully understand the concepts, due to the overwhelming notational simplifications the authors had introduced, and lack of explicit calculations. All the examples found were on scalar field theory, and not at all easy to transfer to gauge theories. The books were simply not written for undergraduates.

As most physicists, I always aspire to understand the underlying concepts of the theories I work with. To be able to understand the concepts on which the imaginary-time formalism was built, I simply had to go back to scratch and re derive the functional representation of the partition function in chapter four and five. Furthermore, all the Feynman rules emerging from the weak-coupling expansion are also found in these chapters. Chapter four and five are the foundation on which this thesis is built, and represent my own little guide to thermal field theory. After deriving these chapters, thermal field theory finally made sense. It has during the rest of this semester, become my one and only favourite theory!

All my previous work on quantum field theory had been done at zero temperature. Moreover, the only Feynman diagrams i had calculated were scattering amplitudes corresponding to S-matrix elements. Due to this, I somehow thought that the calculations were easier to perform in the real-time formalism using Minkowski vectors, before converting the expressions back to Euclidean space. Unfortunately I seemed to clinch on to what was already familiar to me. However, at the end of the semester, I started using the modified Feynman rules for the imaginary-time diagrams directly, and found that they made all the calculations considerably simpler and easier to interpret. This is why the last section of chapter three, four, five and six are made using the imaginary-time Feynman rules. I wish I had five more months to proceed with further work on this theory.

Tormod Gjestland
17.06.2007
Trondheim

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My work on this thesis could not be accomplished without my supervisor Jens O. Andersen. I would like to thank him for inspiration, help and support throughout the entire semester. It has been great!

I also thank Michael Strickland for providing the lattice data from Boyd et al. Several people has contributed during these past five years, to my knowledge on quantum theory. I would like to thank the most important ones, next to my supervisor, namely Kåre Olaussen and Ingjald Øverbø.

Introduction

The main purpose of this thesis is to derive the pressure in a pure-glue plasma in a two-loop approximation, and fit this pressure to lattice data, using a temperature-dependent coupling. To do so, the modern approach to the theory of statistical physics at finite temperature will be used, namely the functional representation of the partition function.

In the first chapter, the concepts of quasiparticles and quasiparticle models are introduced. The first section is written with the purpose of introducing the idea behind the quasiparticles, discussing their origin and field of use. This is done to become more acquainted with the idea behind the models which are introduced later in the chapter. The rest of chapter one is a summary of the article found in the reference [1]. Some quasiparticle models are here introduced, to make the reader more comfortable with the concepts of quasiparticle descriptions of pure gauge theories.

The second chapter is a calculation of frequently appearing sums, when working with finite temperature field theory. They appear in all the Feynman diagrams emerging from the imaginary-time formalism, and are in this chapter found explicitly by the method of contour integration in the complex plane.

Chapter three consists of a one-loop calculation of the photon self-energy tensor, both at finite temperature representing corrections to the imaginary-time photon propagator, and zero temperature representing both corrections to the real-time photon propagator and the vacuum part of the imaginary-time propagator[1]. The finite temperature result is used explicitly in both the chapters four and five. It is also indirectly used in chapter six. The concept of dimensional regularization is also introduced in chapter three.

Chapter four and five are dedicated to the concepts and derivations of the imaginary-time formalism. The functional representation of the partition function will be found. It will also be shown that it can be expanded in terms of Feynman diagrams (vacuum diagrams). These two chapters represent a foundation for the rest of the thesis, and contain all the rules used when calculating finite temperature quantities. In chapter five, the functional representation of the partition function will be used to derive a two-loop approximation to the pressure in a gas of interacting photons and fermions. This is done to become more acquainted with the calculations concerning gauge theories. At the end of chapter five, a quasiparticle description of the pressure in such a gas is also found to order e^3 in the coupling by introducing effective propagators.

[1]Imaginary-time diagrams represent corrections to the partition function, and real-time diagrams represent transition amplitudes at $T = 0$. Why they are given these names will be made clear in chapter four.

Finally, in chapter six, the two-loop approximation to the partition function in pure-glue will be found. This will be done by using all the rules derived for gauge fields and finite-temperature sums in chapter two through five. Moreover, dressed propagators are at the end of this chapter introduced to find the complete two-loop approximation to the pressure in a gas of pure-glue to order g^3 in the coupling. This result is in chapter seven, used to fit a running coupling constant to lattice data, by using simple programming in MatLab. The running coupling will again be used to extrapolate the relative pressure to high temperatures. Note, all the calculations in this thesis are made in natural units, i.e $\hbar, c, \varepsilon_0, \mu_0, k_b = 1$.

A summary and a conclusion will be given in chapter eight.

Contents

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Chapter 1

Quasiparticle models

1.1 Quasiparticles

When bare particles propagate through a medium, they acquire different properties than they would have had as freely propagating particles in vacuum, at zero temperature. What is meant by a bare particle, is a particle in vacuum when all interactions are switched off. When particles at finite temperature propagate in a medium consisting of interacting particles, several interaction processes take place. This makes the properties of the system differ from that of an ideal gas, where all particles are looked upon as though they were completely independent of eachother, thus freely propagating bare particles. These interactions are called *medium effects* [7], since they are due to the medium in which the particles propagate.

A way to incorporate these interaction-related properties into the collective description of a given system, is to modify the dispersion relation of the bare particles by adding different quantities. The dispersion relation gives the possible excitation of each particle within the system. Since the modified dispersion relations differ from the original ones, the new "particles", which possible excitations are described by the modified dispersion relations are called quasiparticles. In this way, possible excitations of the total system can be obtained from the one-quasiparticle excitations.

These modifications, no matter how simple they are, can help describing rather complex quantum systems that seems close to unsolveable when looking at for instance all the Feynman diagrams representing the interaction contributions to the free energy.

Later in this thesis, it is shown that the massless gauge boson in $U(1)$ gauge theory gains an effective mass due to medium effects, when propagating through a cloud of electrons and positrons at finite temperature. The interactions taking place when the photons propagate through a system in presence of fermions, give rise to different terms in the perturbation series describing the system. Many of these terms can be reproduced by modifying the photon propagator, and from the poles of the propagator one can obtain the modified dispersion relation [4].

Such observations lead to a quasiparticle description of many-body systems, where collective excitations of systems consisting of bare particles which interact, are described by excitations of the corresponding freely propagating quasiparticles. In this way, both the energy emerging from the inertia of the particles themselves, and the medium effects, is implemented into the dispersion relation. This can be illustrated mathematically by expressing the Hamiltonian of a system of gauge bosons in natural units as follows

$$H = g\sum_i |k|_i n_i + H_{int} \rightarrow g\sum_i \sqrt{k_i^2 + m^2} \cdot n_i, \tag{1.1}$$

where g is the degeneracy factor, n_i and k_i is the number of particles and the energy of the one-particle state i. The right-hand side is given in terms of the modified dispersion relation of the quasiparticles. The interaction term on the left-hand side is parametrized by the effective mass on the right-hand side.

From the above, one can see that the quasiparticles are not *fundamental* particles as for example the electrons and positrons, but rather a mathematical tool to help describing the behaviour of relatively complex many-body quantum systems in an elegant way.

In QCD, non-perturbative or *residual* interaction processes take place due to confinement below the critical temperature $T_c \simeq 150MeV$. To describe these efffects, a thermal mass is not sufficient to reproduce the Monte Carlo lattice simulations, which are up until this date, the most successful tools when describing such systems below or close to T_c, due to confinement. These strong interactions can be taken care of by introducing a *bag pressure*, $-B(T)$, which concept is based on the MIT bag model. It is looked upon as the external pressure that keep the quarks and gluons confined to a finite region of space (in small bags), representing the particles of the standard model. This term is inserted into the Hamiltonian as a temperature dependent vacuum-energy density.

1.2 Pure-glue plasma

This section is a recapitulation on some of the already existing quasiparticle descriptions of pure glue in the deconfined phase. It is a summary to get aquainted with some of the different models, their advantages and disadvantages. It is based on the articles found in the references [1, 7].

Pure-glue plasma with a temperature dependent degeneracy factor

The gluons are as one knows, massless vector bosons. Such bosons have two helicity states. However, in QCD the gauge bosons interact with eachother. This means that one does not need the presence of fermions to obtain interactions within a system consisting of gluons. From the introduction to this chapter one can then imagine that the gluons will aquire thermal masses at finite temperature due to medium effects. Moreover, massive vector bosons have three helicity states, $S = 0, \pm 1$. One can then try to see if the thermal mass makes the gluons act as a massive vector boson.

A way to make an effective gluon description out of this idea, is to implement these temperature-dependent variations of the degeneracy factor G, by assuming that it is a function of T. $G \rightarrow G(T)$. Consequently, the effective mass is also temperature dependent.

Given the above, the dispersion relation for a gluon propagating through a medium in thermal equilibrium at temperature T becomes [1]:

$$\omega(k,T) = \sqrt{k^2 + m^2(T)}. \tag{1.2}$$

The grand-canonical partition function, Z, for a system of gluons omitting zero net chemical potential[1], given the dispersion relation above is [5]

$$Z = \mathrm{Tr}\Big\{e^{-\beta \hat{H}}\Big\}. \tag{1.3}$$

The pressure in such a gas is according to [5],

$$P(T) = \frac{T}{V}\ln Z. \tag{1.4}$$

From this, one obtains [1, 5]

$$P(T) = \frac{G(T)}{(2\pi)^3}\int_0^\infty d^3k \frac{k^2}{3\omega(k,T)} n(k,T), \tag{1.5}$$

where $G(T)$ is the generalized temperature-dependent degeneracy factor and $n(k,T)$ is the boson distribution function,

$$n(k,T) = \frac{1}{e^{\beta\omega} - 1}. \tag{1.6}$$

When the pressure and temperature are given, the net energy density $\varepsilon(T)$ can be found from the thermodynamic relations,

$$U(S,V) = TS - PV$$

$$F(T,V) = U - TS = -PV. \tag{1.7}$$

where U is the internal energy, S is the entropy and T, V is the temperature and volume of the system, respectively. It is assumed that the net chemical potential, $\mu = 0$. By differentiating the two expressions in Eq. (1.7), one can identify the relations

$$dU = TdS - PdV$$

$$dF = dU - TdS - SdT = -SdT - PdV. \tag{1.8}$$

From the last differential, one can find that the entropy is given by

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \left(\frac{dP}{dT}\right)_V V. \tag{1.9}$$

Now, by inserting this into the internal energy and dividing by the constant volume on each side, one obtains the energy density,

$$\frac{U}{V} = \varepsilon(T) = T\frac{dP}{dT} - P(T). \tag{1.10}$$

[1] This implies that there are as many particles as there are anti-particles within the system. Since the gluons are represented by real scalar fields, they are their own anti-particles. This means that the chemical potential is always zero in pure gauge theories.

However, there is another expression for the energy density via the grand canonical partition function [2, 5], namely

$$\varepsilon(T) = -\frac{1}{V}\frac{\partial}{\partial\beta}\ln Z = \frac{1}{VZ}Tr\{He^{-\beta H}\}. \tag{1.11}$$

For the Eqs. (1.10) and (1.11) to be thermodynamically consistent, meaning that they are equal to eachother, a constraint arises as will be shown later on [1]. From Eq. (1.5), Eq. (1.10) takes the form

$$\varepsilon(T) = \frac{G(T)}{(2\pi)^3}\int d^3k\left[\omega(k,T)n(k,T) - TM(T)\frac{dM}{dT}\frac{1}{\omega(k,T)}n(k,T)\right] + T\frac{P(T)}{G(T)}\frac{dG}{dT}. \tag{1.12}$$

However, from Eq. (1.11), the internal energy density is defined to be

$$\varepsilon(T) = \frac{G(T)}{(2\pi)^3}\int d^3k\omega(k,T)n(k,T). \tag{1.13}$$

This is where the previously mentioned constraint appears. A relation between the temperature dependent mass and the degeneracy factor must be established to make Eq. (1.10) consistent with Eq. (1.11). It follows that

$$\begin{aligned} -\int d^3kTM(T)\frac{dM}{dT}\frac{1}{\omega(k,T)}n(k,T) + T\frac{P(T)}{G(T)}\frac{dG}{dT} &= 0 \\ \Rightarrow \frac{dG}{dT}\int d^3k\frac{k^2}{\omega}n(k,T) = M(T)\frac{dM}{dT}3G(T)\int d^3k\frac{1}{\omega}n(k,T). \end{aligned} \tag{1.14}$$

Here, the last line can be integrated with respect to T on both sides, and solved for $G(T)$. If $G(T)$ and $M(T)$ satisfy this constraint, also referred to as the self-consistency condition, the quasiparticle picture is said to be *thermodynamically consistent*. This must of course be fulfilled, or the theory will break down. It would not make sence if one could obtain two different answers by using two different approaches to find the energy density. Eq. (1.14), can then be used to check whether the functions extracted from lattice data are thermodynamically consistent. From Eqs. (1.5) and (1.13), given that Eq. (1.14) is fulfilled, the functions $M(T)$ and $G(T)$ can be extracted given the energy density $\varepsilon(T)$ and the pressure $P(T)$ from lattice data for a pure $SU(3)$ gauge theory.

As seen from Fig. 1.1, the degeneracy factor $G(T)$ does not converge towards $\frac{3}{2}\cdot 16 = 24$ in either the high- or low-temperature limit. This means that an originally massless vector boson does not behave like a massive one even if it gains a thermal mass due to medium effects. In fact, both the mass and the degeneracy factor have a rather unphysical behaviour close to the phase transition $T = T_c$. However, in the high temperature limit, its deviation from $G = 16$, is rather small. One would expect the degeneracy factor to converge towards 16 for high temperatures, since this is the degeneracy factor of a free massless gluon (there are eight different kinds of gluons, with two polarizations), and the mass seems to drop to zero on the temperature scale, i.e $\frac{M}{T} \to 0$.

Because of the high degeneracy factor expected from this previous model, there is a reason to believe that there are some non-perturbative effects, affecting the system in the region close to T_c. In fact, one of the main reasons for the degeneracy factor to obtain such a

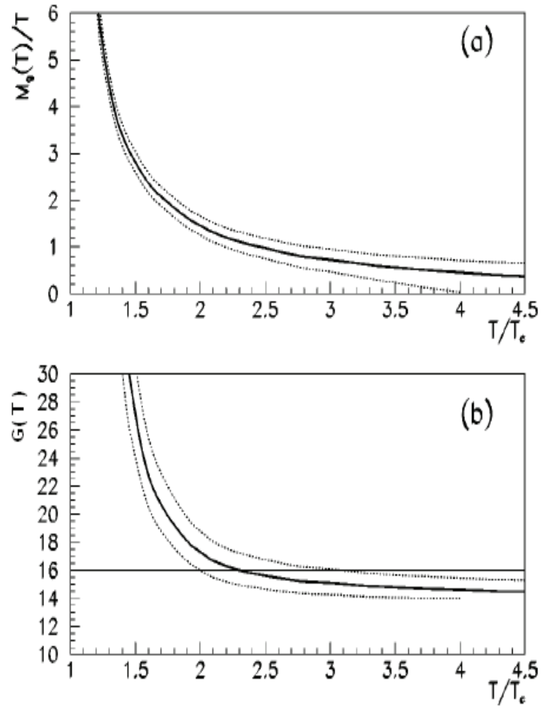

Figure 1.1: $M(T)$ and $G(T)$ extracted from lattice data via Eqs. (1.11) (1.5) and (1.10). This figure is taken from [1].

strange behaviour in the low temperature limit, is because the gluons in this region generate large thermal masses compared to the temperature, as can also be seen from Fig. (1.1). This leads to Boltzmann suppression. Since the energy density is given from lattice data, Eq. (1.11) explains one of the reasons why the degeneracy factor increases rapidly when the ratio $\frac{M(T)}{T}$ increases.

1.3 Pure-glue plasma with a temperature dependent mass and bag constant

To get rid of the unphysical behaviour of the degeneracy factor $G(T)$ for low temperatures, one can recall the introduction to this chapter. It was stated that non-perturbative effects can be taken care of by introducing a bag pressure $-B(T)$. Now, another model can be obtained. In this model, the degeneracy factor will be kept constant at $G = 16$ due to the fact that there are eight gluons with two helicity states. The reason for why it is assumed to be constant is because of its almost constant level at temperatures $T > 2.3T_c$, obtained from Fig. 1.1. Furthermore, one can assume that the increase in degeneracy in the previous model was due to non-perturbative corrections only.

Now, the bag constant gives a pressure contribution, $P_{bag} = -B(T)$. The Hamiltonian describing the system is

$$H = G\sum_i \omega(k_i, T)n_i + V \cdot B(T), \tag{1.15}$$

where the sum goes over all single-particle states i, occupied by n_i particles with energy ω_i. G is a temperature independent degeneracy factor. This results in an energy density from Eq. (1.11)

$$\varepsilon(T) = \frac{U}{V} = \frac{G}{(2\pi)^3}\int d^3k\omega(k,T)n(k,T) + B(T) \tag{1.16}$$

The pressure comes out to be

$$P(T) = -\frac{T}{V}\ln Z = \frac{G}{(2\pi)^3}\int_0^\infty d^3k\frac{k^2}{3\omega(k,T)}n(k,T) - B(T) \tag{1.17}$$

The thermodynamic identity takes care of the self-consistency condition to make the system thermodynamically consistent. Via, Eq. (1.7), the constraint appearing is:

$$\frac{dB}{dT} = \frac{G}{(2\pi)^3}M(T)\frac{dM}{dT}\int d^3k\frac{n(k,T)}{\omega(k,T)}. \tag{1.18}$$

Now, Eq. (1.16) and Eq. (1.17) can be used to extract the functions $M(T)$ and $B(T)$, given the self-consistency contraint in Eq. (1.18). In Fig. 1.2, one can see that the data seem

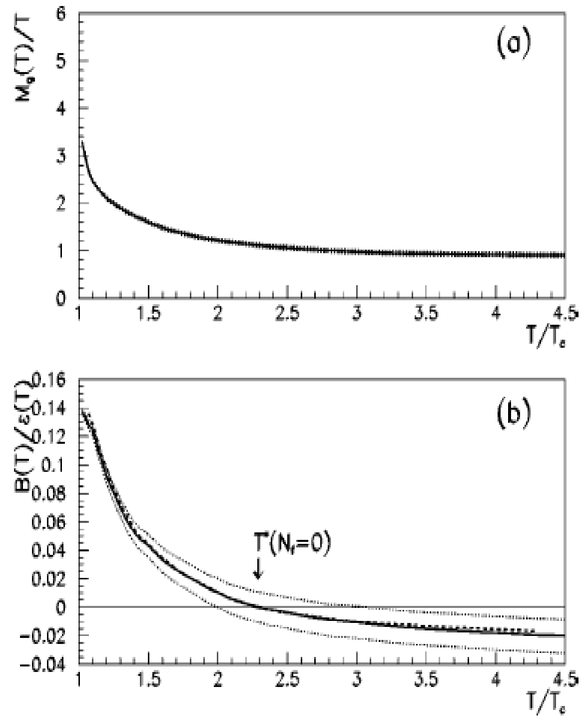

Figure 1.2: The functions $\frac{B(T)}{\varepsilon(T)}$ and $M(T)$ extracted from lattice data via Eq. (1.16) and Eq. (1.17). Figure taken from [1].

to fit better than in Fig. 1.1. In the low temperature limit, the mass does not increase as rapidly as it did in Fig. 1.1. At T_c, the thermal mass reaches a value of approximately

$3.15T_c$, while it was greater than $6T_c$ in the case where the bag constant was omitted. One can also remark that the masses in the high-temperature limit is slightly larger in Fig. 1.2 than in Fig. 1.1. This is partly because the degeneracy factor was said to be 16 and not $\simeq 14.5$ as the high temperature limit of Fig. 1.1 implied [1]. $\frac{B(T)}{\varepsilon(T)}$ is also a well behaved continous fuction with finite values for all temperatures within the region.

This implies that there are effects taking place near T_c that are non-perturbative and therefore cannot be described by the thermal masses of the gluons, but should rather be parametrized via an extra term $B(T)$.

Chapter 2

Matsubara sums

The calculations appearing in finite-temperature field theory are very similar to the calculations of S-matrix elements in perturbative quantum field theory at $T = 0$. As one knows, the S-matrix elements can be represented by Feynman diagrams, so can the elements of the partition function. The close relation between these two theories will be made clearer later in this thesis. One of the main differences between the two formalisms, is the substitution it in the functional representation of the transition amplitude goes to the real quantity τ in the partition function. This is why the two are referred to as the real-time and imaginary-time representations, respectively. Another difference is that the fields which the partition function is represented by, have to be (anti-) periodic[1] in τ with a periode $\beta = \frac{1}{T}$. This leads to a modification of the zeroth component of the 4-momentum in the real-time formalism (S-matrix calculations), p_0. To get from the real-time diagrams to the ones representing the imaginary-time (Partition function) calculations, the substitution $p_0 \rightarrow i\omega_n$ must be made, where $\omega_n = 2\pi nT$ for bosons, $\omega_n = 2\pi(n + \frac{1}{2})T$ for fermions and $n = 0 \pm 1 \pm 2 ... \pm \infty$. ω_n is called the Matsubara frequency.

When calculating Feynman diagrams in the real-time formalism, integrals on the form

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2}, \tag{2.1}$$

are constantly appearing. In the imaginary-time formalism, these integrals, given the substitution $p_0 \rightarrow i\omega_n$, become

$$-i \int \frac{d^3p d\omega_n}{(2\pi)^4} \frac{1}{\omega_n^2 + \underbrace{\vec{p}^2 + m^2}_{\omega^2}}. \tag{2.2}$$

This however, is only valid in the limit $T \rightarrow 0$, since

$$2\pi nT = \omega_n,$$
$$\frac{\Delta\omega_n}{2\pi} = \Delta nT \rightarrow 0. \tag{2.3}$$

[1]Periodic for bosons and anti-periodic for fermions

However, at finite T, the integral over ω_n in Eq. (2.2) becomes discretizised as ΔnT becomes finite. The integral in Eq. (2.2) then becomes

$$-iT\int\frac{d^3p}{(2\pi)^3}\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+\omega^2}. \tag{2.4}$$

This means that all integrals on the form

$$\int\frac{d^Dp}{(2\pi)^D}f(p_0,\vec{p}) \tag{2.5}$$

will be substituted with

$$i\int\frac{d^{D-1}p}{(2\pi)^{D-1}}T\sum_{n=-\infty}^{\infty}f(p_0=i\omega_n,\vec{p}), \tag{2.6}$$

to obtain imaginary-time expressions from the real-time Feynman diagrams.

An elegant way to calculate a sum on the form

$$T\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+\omega^2}=T\sum_{n=-\infty}^{\infty}R(\omega_n), \tag{2.7}$$

is to convert the sum into a countour integral in the complex plane by using the residu theorem. The residue theorem states [14],

$$\oint_C f(z)dz=2\pi i\sum_i \mathrm{Res}\{(f(z),z_i)\}, \tag{2.8}$$

where C is the path of the contour, and $\{z_i\}$ are the residues enclosed by the contour as shown in Fig. 2.1.

2.1 Bosons

By looking at the bosonic Matsubara frequencies $\omega_n=2\pi nT$, as the four'th component of a Minkowski four-vector, $\omega_n\to i\cdot z$ the argument of the sum in Eq. (2.7) becomes

$$R(iz)=-\frac{1}{z^2-\omega^2}. \tag{2.9}$$

Now, to rewrite the sum in Eq. (2.7) in terms of a contour integral one should find a proper function, $f(z)$, to insert into Eq. (2.8) and in this way forces Eq. (2.7) to become the right-hand side of Eq. (2.8). Furthermore, for a simple pole at z_0, which means that $f(z)\propto\frac{1}{z-z_0}$ for z close to z_0, the residue is defined to be

$$\mathrm{Res}\{f(z),z_0\}=\lim_{z\to z_0}(z-z_0)f(z). \tag{2.10}$$

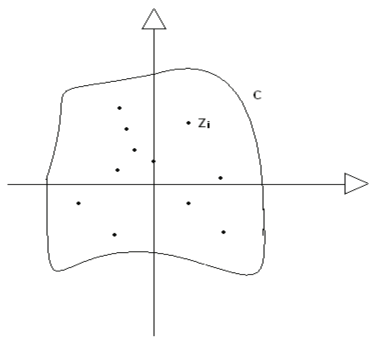

Figure 2.1: A figure of the contour C. The enclosed residues are drawn as dots in the region enclosed by C.

Now, if one chooses $f(z) = R(iz) \cdot G(z)$, $G(z)$ has to be a periodic function with simple poles at the points $z_n = 2\pi niT$ on the imaginary axis, proportional to $\frac{1}{z-z_n}$ around z_n. Given the properties above, the function $G(z)$ can either be $G(z) = \frac{1}{\sinh\frac{\beta}{2}z}$ or $G(z) = \coth\frac{\beta}{2}z$ [3]. The last function will be chosen due to the fact that it is bounded for large real values of z, i.e $\lim_{z\to\infty} G(z) = 1$ which will be an important property in further calculations.

Returning to Eq. (2.8), the left-hand side will be some constant Ω, times the contour integral over $R(iz) \times G(z)$, i.e

$$\Omega \cdot \oint_C \frac{\coth\frac{\beta}{2}z}{z^2 - \omega^2} dz, \tag{2.11}$$

given the proper path of integration C. To check whether it gives the correct values for Eq. (2.7) to become the right hand side of the residue theorem, a contour $C = C_n$ can be placed around the n'th singularity along the imaginary axis $z = 2\pi inT$ as shown in Fig. 2.3, and integrated in counter clockwise direction. If the integrand is correct, this should give the n'th term of the sum in Eq. (2.7).

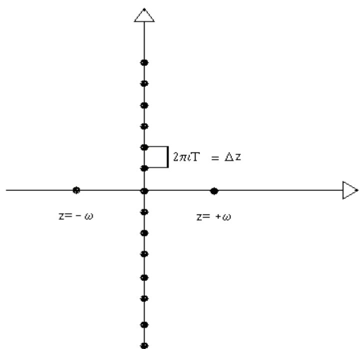

Figure 2.2: A figure illustrating the positions of the residues of the function $f(z)$.

$$\Omega \cdot \oint_{C_n} \frac{\coth\frac{\beta}{2}z}{z^2 - \omega^2} dz = 2\pi i\Omega \lim_{z\to 2\pi inT} \frac{(z - 2\pi inT)}{z^2 - \omega^2} \frac{\cosh\frac{\beta}{2}z}{\sinh\frac{\beta}{2}z}. \tag{2.12}$$

To evaluate $\coth\frac{\beta}{2}z$ for $z \to 2\pi i n T$, its periodic properties will be exploited.

$$\lim_{z\to 0}\coth(\frac{\beta}{2}z) \simeq \lim_{z\to 0}\left[\frac{2}{\beta z}\right]; \quad \coth\frac{\beta}{2}(z+2\pi i n T) = \coth\frac{\beta}{2}z$$
$$\Rightarrow$$
$$\lim_{z\to 2\pi i n T}\coth\frac{\beta}{2}z \simeq \lim_{z\to 2\pi i n T}\left[\frac{1}{\frac{\beta}{2}(z-2\pi i n T)}\right]. \tag{2.13}$$

This turns Eq. (2.12) into

$$\Omega\cdot\oint_{C_n}\frac{\coth\frac{\beta}{2}z}{z^2-\omega^2}dz = 2\pi i\Omega\lim_{z\to 2\pi i n T}\frac{(z-2\pi i n T)}{z^2-\omega^2}\frac{1}{\frac{\beta}{2}(z-2\pi i n T)} = -4\pi i\Omega\left[\frac{1}{(2\pi n T)^2+\omega^2}\right]\cdot T. \tag{2.14}$$

Comparing this to Eq. (2.7), one finds that $\Omega = -\frac{1}{2}\cdot\frac{1}{2\pi i}$.

This means that the matsubara sum can be converted into a contour integral via the residu theorem given the proper contour, L,

$$-\frac{1}{2\pi i}\oint_L\frac{1}{z^2-\omega^2}\frac{1}{2}\coth\frac{\beta}{2}z dz = T\sum_n\frac{1}{\omega_n^2+\omega^2}. \tag{2.15}$$

where the sum now runs over all n corresponding to singularities enclosed by L. The integral along L is made in the counter clockwise direction. From Fig. 2.2, it is easy to see that if the contour encloses all the singularities on the imaginary axis, the sum in Eq. (2.7) and the integral in Eq. (2.15) will be equal.

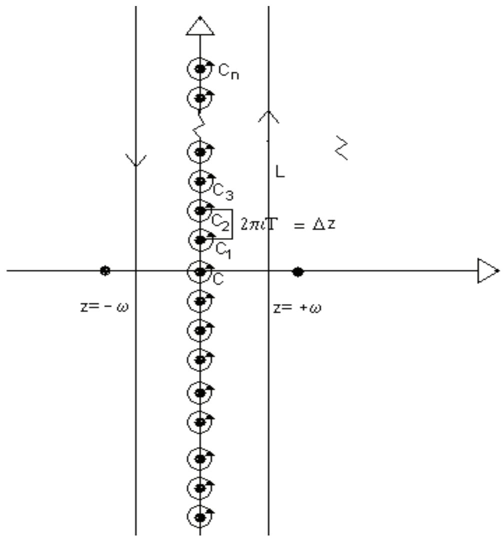

Figure 2.3: A sketch illustrating the contours.

It follows from Fig. 2.3 and Eq. (2.14), that

$$-\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\oint_{C_n}\frac{1}{z^2-\omega^2}\frac{1}{2}\coth\frac{\beta}{2}zdz = -\frac{1}{2\pi i}\oint_L\frac{1}{z^2-\omega^2}\frac{1}{2}\coth\frac{\beta}{2}zdz = T\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+\omega^2}. \tag{2.16}$$

Since

$$\lim_{n\to\infty}\mathrm{Res}\{\frac{1}{z^2-\omega^2}\coth\frac{\beta}{2}z, z_n\} = 0, \tag{2.17}$$

the singularities at infinity hardly contribute. Moreover, one can see that if one chooses to complete the ends of the rectangle given by the closed curve L by two line integrals going from $z = \pm i\infty \pm \varepsilon$ to $z = \pm i\infty \mp \varepsilon$, the denominator of the integrand in the integral on the left hand side of Eq. (2.15), is infinity along these paths, hence the line-integrals give zero contribution. To be more specific, this is true if one chooses to place the path of these ends between two singularities on the imaginary axis. To make sure that there are no corrections to the infinite sum, one can place a closed curve around the next singularity, integrate along this closed path and hence, due to Eq. (2.17) this does not contribute.

From the above, one can draw the conclusion that the integral over the closed curve given by the rectangle L is equal to the integral over the two vertical lines in Fig. 2.3, given the direction along the arrows. This gives,

$$\begin{aligned}
-\frac{1}{2\pi i}\oint_L\underbrace{\frac{1}{z^2-\omega^2}\frac{1}{2}\coth\frac{\beta}{2}z}_{f(z)}dz &= -\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}f(z)dz - \frac{1}{2\pi i}\int_{i\infty-\varepsilon}^{-i\infty-\varepsilon}f(z)dz \\
&= T\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+\omega^2}.
\end{aligned} \tag{2.18}$$

By simplification,

$$\begin{aligned}
& f(z) = -f(-z) \\
\Rightarrow\ & -\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}\left[f(z)-f(-z)\right]dz \\
=\ & -2\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}\left[f(z)\right]dz.
\end{aligned} \tag{2.19}$$

This expression is still not easy to handle, but it can be evaluated by calculating the residues in $z = \pm\omega$. The integrals over each of the two lines defining L can be found by integrating over two semi-circles in the right and left half plane respectively as implied by Fig. 2.4. The integral along the path shown in Fig. 2.4 is given by

$$\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}\left[f(z)\right]dz + \lim_{R\to\infty} i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}Re^{i\theta}d\theta f(\varepsilon+Re^{i\theta}) = 2\pi i\mathrm{Res}\{f(z),\omega\}, \tag{2.20}$$

since $z = \varepsilon + Re^{i\theta}$ along the circular path. Now, the last integral on the left hand side has to be evaluated and is eventually found to be zero by the following argument:

$$\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}Re^{i\theta}d\theta f(\varepsilon+Re^{i\theta})\right| = \left|Ri\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{(Re^{i\theta})^2-\omega^2}\coth\frac{\beta}{2}(\varepsilon+Re^{i\theta})d\theta\right|$$

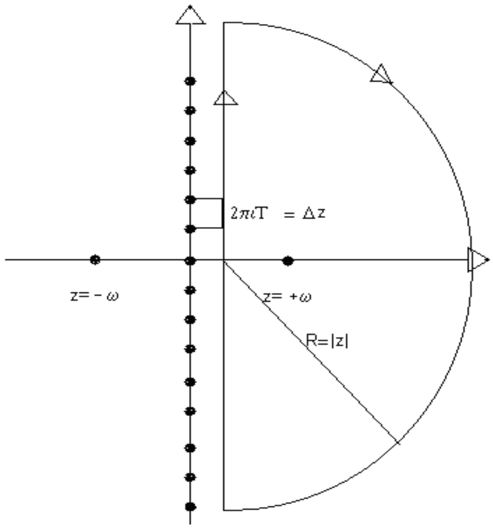

Figure 2.4: Closing the contour in the right half plane.

$$
\begin{aligned}
&\leq \; |Ri| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\frac{1}{(Re^{i\theta})^2 - \omega^2}| \cdot |\coth \frac{\beta}{2}(\varepsilon + Re^{i\theta})| \cdot |d\theta| \\
&\simeq \; \frac{1}{R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\coth \frac{\beta}{2}(\varepsilon + Re^{i\theta})| \cdot |d\theta| \\
&= \; \frac{1}{R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \left[1 + \frac{2}{e^{\beta(\varepsilon + Re^{i\theta})} - 1} \right] \right| d\theta \\
&\leq \; \frac{1}{R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[|1| + \left| \frac{2}{e^{\beta(\varepsilon + Re^{i\theta})} - 1} \right| \right] d\theta \\
&= \; 0 + \frac{1}{R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\sqrt{1 + e^{\beta(\varepsilon + R\cos\theta)}(e^{\beta(\varepsilon + R\cos\theta)} - 2\cos(\beta R \sin\theta))}} d\theta \\
&= \; 0, \qquad (2.21)
\end{aligned}
$$

for all $\varepsilon > 0$ in the limit $R \to \infty$. This means that the part of the integral, L, that lays in the right half plane can be expressed in terms of the residue in $z = \omega$:

$$\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} f(z)dz = -2\pi i \frac{1}{4\omega} \left[1 + \frac{2}{e^{\beta\omega} - 1} \right] \qquad (2.22)$$

From Eq. (2.19), the total integral along the rectangle L, given the above, can be expressed in terms of the residue in the right half plane. One then obtains the final result,

$$T \sum_n \frac{1}{\omega_n^2 + \omega^2} = -\frac{2}{2\pi i} \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} f(z)dz = \frac{1}{2\omega} \left[1 + \frac{2}{e^{\beta\omega} - 1} \right]. \qquad (2.23)$$

The first expression of the right hand side, is the pure vacuum contribution, i.e the zero-temperature part. The second part is

$$\frac{1}{\omega} n_b(\omega, T), \qquad (2.24)$$

where n_b is the boson distribution function.

2.2 Fermions

For fermions, it was stated in the introduction to this chapter, that the fermion fields in the imaginary-time formalism had to be anti-periodic in β. This leads to the possible Matsubara frequencies $\omega_n = 2\pi(n+\frac{1}{2}), n = 0, \pm 1, \pm 2 ... \pm \infty$. This gives another $G(z)$ over which the argument in Eq. (2.9) is integrated. In fact, since the only difference is that the possible frequencies are shifted by

$$[\omega_n]^{fermions} - [\omega_n]^{bosons} = \frac{1}{2} 2\pi T, \tag{2.25}$$

$G(z) = \tanh \frac{\beta}{2} z$. This function is also bounded for large real values of z, and the exact same procedure as for the bosons is followed to find the sum.

The integrand of the contour integral has the singularities shown in Fig. 2.5, and the contour is taken along the same path as for the bosons in Fig. 2.4. It follows from Eqs. (2.22) and (2.23) that

$$T \sum_n \frac{1}{\omega_n^2 + \omega^2} = \frac{-2}{2\pi i} \oint \frac{dz}{z^2 - \omega^2} \frac{1}{2} \tanh \frac{\beta}{2} z = \frac{1}{2\omega} \tanh \frac{\beta}{2} \omega = \frac{1}{2\omega} \left[1 - \frac{2}{e^{\beta\omega} + 1} \right] \tag{2.26}$$

Here, the first part is to be interpreted as the first part in the expression for bosons. The second part is

$$\frac{1}{\omega} n_f(\omega, T), \tag{2.27}$$

where n_f is the fermion distribution function.

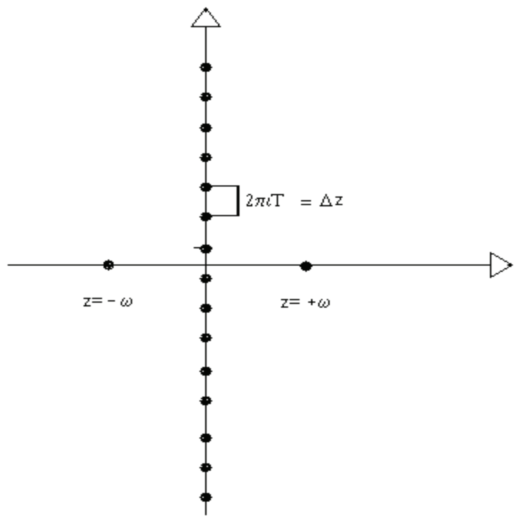

Figure 2.5: A figure illustrating the singularities of the function $\frac{1}{z^2-\omega^2} \tanh \frac{\beta}{2} z$ in the complex plane.

Chapter 3

The photon polarization tensor

In a system of propagating interacting particles at finite temperature, the particles acquire somewhat different properties than they would have had if they were independent and freely propagating in vacuum at $T = 0$. They can be looked upon as a set of freely propagating quasiparticles. These quasiparticles are treated as though they were real particles, but as implied have some interaction related properties incorporated. These properties can for instance be such as charge screening or effective thermal masses.

As seen from the first chapter of this thesis, the quasiparticle models open up for the possibility to describe collective behaviour of many-particle systems in a simple way, quite successfully. These models are the modern description of the quantum many-body problem, and is applicable to systems with a large number of degrees of freedom. Such systems are far too complex to solve using ordinary quantum mechanics. It is in this field of physics, where the number of particles, N, within a given system is large and varying, that the quasiparticle descriptions are the most successful models when describing the collective physical behaviour.

The collective modes of the system can be found directly from the poles of the effective propagators of its constituents, sometimes obtainable from perturbation theory. In gauge theories, there are as many particles as anti-particles. This means that the net chemical potential is zero. Moreover, the masses of the bare gauge bosons are zero. This makes it possible to characterize the collective excitations by two quantities, namely the coupling constant g and the temperature of the system, T [12]. For perturbation theory to be applicable, the coupling must be small, i.e $g \ll 1$. In QED the coupling is weak, but in low temperature QCD perturbation theory breaks down due to the strong coupling. However, at extremely high temperatures the theory can be treated perturbatively due to asymptotic freedom.

To get aquainted with the finite temperature formalism, and to establish a foundation of intuition when working with gauge theories, the better established and less complicated theory, QED will be used.

Firstly, a gas of electrons, positrons and photons will be considered. The photon propagator $iD^{\mu\nu}$ acquires corrections due to interaction processes both at finite and zero temperature. This is where one goes from the description of bare particles, to physically observable particles. The theory might not describe how the particles really look when they are separated from all and everything and all interactions are shut off, but still; this is how the particles behave in an interacting environment, thus how one observes them.

The photon propagator in an interacting environment recieves corrections due to interaction with the fermions. The correction is in the real-time formalism described by the Feynman diagrams algebraically written as

$$\begin{aligned} iD(q)_{\mu\nu} \quad \rightarrow \quad & iD(q)_{\mu\nu} + iD(q)_{\mu\lambda} i\Pi^{\lambda\gamma} iD(q)_{\gamma\nu} + \\ & + iD(q)_{\mu\lambda} i\Pi^{\lambda\gamma} iD(q)_{\gamma\delta} i\Pi^{\delta\rho} iD(q)_{\rho\nu} + \ldots\ldots\ldots + \\ & + iD(q)_{\mu\lambda} i\Pi^{\lambda\gamma} iD(q)_{\gamma\delta} i\Pi^{\delta\rho} iD(q)_{\rho\zeta} \ldots\ldots\ldots i\Pi^{\xi\pi} D_(q)_{\pi\nu}, \end{aligned} \tag{3.1}$$

where $i\Pi^{\lambda\gamma}$ is called the photon polarization tensor [4], or the self-energy tensor. The one-loop diagram approximation to this tensor is simply expressed using the Feynman diagrams in Fig. 3.1. The diagrams imply that a photon recieves corrections due to creation and annihilation processes of virtual fermion-anti fermion pairs. To go from the first diagram

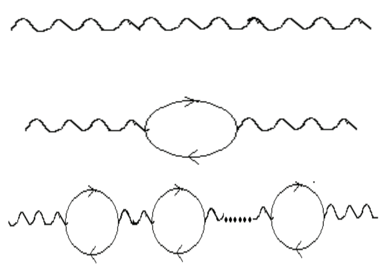

Figure 3.1: The free photon propagator, the second order correction and a correction of arbitrary order.

in Fig. 3.1 to the second one, one includes a factor

$$\begin{aligned} i\Pi^{\mu\nu}(q^2) &= -(-ie)^2 \int d^4p \mathrm{Tr}\Big\{\gamma^\mu \frac{i}{\not p - m} \gamma^\nu \frac{i}{\not p' - m}\Big\} \\ &= -e^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\Big\{\gamma^\mu(\not p + \not q + m)\gamma^\nu(\not p + m)\Big\}}{[(p+q)^2 - m^2 + i\varepsilon]\,[p^2 - m^2 + i\varepsilon]}, \end{aligned} \tag{3.2}$$

as the first correction in the series described in Eq. (3.1). Here, p is the four-momentum running through the lower half of the bubble, $p' = p + q$ is the four momentum running through the upper half and q is the external four momentum. This tensor is everything one needs to find the corrected poles and other properties of the photon propagator, and represents the fermion loops in Fig. 3.1. The interpretation of Eq. (3.1) is illustrated in terms of its Feynman diagram in Fig. 3.1. It is simply the propagator plus the propagator with one loop, two loops, three loops etc. etc., where all the diagrams are expressed using the one-loop diagram in Fig. 3.1. I.e the two-loop diagram in Fig. 3.2 and higher order contributions to the self-energy tensor are neglected.

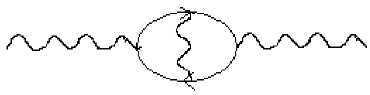

Figure 3.2: Two-loop contribution.

Firstly, to simplify the expression for the polarization tensor, the trace in the numerator of Eq. (3.2) will be found explicitly. This is done by rewriting the traces via the relations [4],

$$\begin{aligned} \mathrm{Tr}\Big\{\gamma^\mu\gamma^\nu\Big\} &= 4\eta^{\mu\nu} \\ \mathrm{Tr}\Big\{\gamma^\mu\gamma^\lambda\gamma^\delta\gamma^\nu\Big\} &= 4(\eta^{\mu\lambda}\eta^{\delta\nu} - \eta^{\mu\delta}\eta^{\lambda\nu} + \eta^{\mu\nu}\eta^{\lambda\delta}). \end{aligned} \tag{3.3}$$

All traces containing an odd number of γ-matrices vanish, since all the matrices γ^μ are antisymmetric. Furthermore, the Feynman slash notation is defined as

$$\not{p} = \gamma^\mu p_\mu, \tag{3.4}$$

where the notation indicates Einstein's sum convention. Now, by using Eq. (3.3), the trace in Eq. (3.2) can be written as

$$\begin{aligned} Tr\{\gamma^\mu(\underbrace{\not{p}+\not{q}}_{\not{r}}+m)\gamma^\nu(\not{p}+m)\} &= \mathrm{Tr}\Big\{\gamma^\mu\not{r}\gamma^\nu\not{p}\Big\} + m^2\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} \\ &= 4(p^\mu r^\nu - \eta^{\mu\nu}(p\cdot r - m^2) + p^\nu r^\mu) \\ &= 4(p^\mu q^\nu + 2p^\nu p^\mu - \eta^{\mu\nu}(p\cdot q - m^2) + p^\nu q^\mu). \end{aligned} \tag{3.5}$$

There are no obvious symmetries manifested by the integrand of Eq. (3.2). This makes the integral very hard to solve directly. Since symmetric integration can simplify the expression considerably, the denominator of Eq. (3.2) is rewritten by the use of the parametrization [4],

$$\frac{1}{AB} = \int_0^1\int_0^1 dxdy\frac{\delta(x+y-1)}{[Ax+By]^2}. \tag{3.6}$$

This "trick" is in many books referred to as Feynman's trick, and is exploited here to find a suitable change of integration variables, which again makes one able to solve the integral in Eq. (3.2) by symmetric integration. By defining

$$\begin{aligned} A &\equiv (p^2 - m^2 + i\varepsilon), \\ B &\equiv ((p+q)^2 - m^2 + i\varepsilon), \end{aligned} \tag{3.7}$$

and inserting this into Eq. (3.6), it follows after a little re-organizing of the terms in the denominator that

$$\frac{1}{AB} = \int_0^1 \frac{dx}{((p+(1-x)q)^2 - a^2)^2}, \tag{3.8}$$

where

$$-a^2 = (1-x)xq^2 - m^2 + i\varepsilon. \tag{3.9}$$

Now, making the substitution

$$K^\alpha = p^\alpha + (1-x)q^\alpha \Rightarrow d^D K = d^D p, \tag{3.10}$$

which is allowed without changing the limits in dimentional regularization (there is no defined cutoff parameter in the system). This substitution makes the denominator of Eq. (3.2) become

$$\int_0^1 \frac{dx}{(K^2-a^2)^2}. \tag{3.11}$$

Furthermore, by inserting the parametrization of the denominator into Eq. (3.2) and recalling the substitution defined by Eq. (3.10), the expression for the polarization tensor takes the form

$$i\Pi^{\mu\nu} = \lim_{D\to 4} De^2 \int \frac{d^D K}{(2\pi)^4} \frac{(2K^\mu K^\nu - 2q^\mu q^\nu x(1-x) - \eta^{\mu\nu}(K^2 - m^2 - x(1-x)q^2))}{(K^2-a^2)^2}, \tag{3.12}$$

which is much more convenient to work with than Eq. (3.2). Hence the notation implies integration over x from zero to one.

3.1 Zero-temperature calculations

At $T=0$, p^0 takes on continuous values $-\infty \leq p^0 \leq \infty$. Eq. (3.12) can therefore easily be integrated by the use of some simple considerations and dimensional regularization. To make the integral in Eq. (3.12) as easy to solve as possible one can introduce the integrals:

$$\begin{aligned}
J^{\mu\nu}_{2,D} &= \int \frac{d^D K}{(2\pi)^D} \frac{K^\mu K^\nu}{(K^2-a^2)^2} \\
&= \frac{\eta^{\mu\nu}}{D} \int \frac{K^2}{(2\pi)^D} \frac{1}{(K^2-a^2)^2} \\
J_{2,D} &= \int \frac{d^D K}{(2\pi)^D} \frac{K^2}{(K^2-a^2)^2} \\
I_{N,D} &= \int \frac{d^D K}{(2\pi)^D} \frac{1}{(K^2-a^2)^N}
\end{aligned} \tag{3.13}$$

Now, Eq. (3.12) can be written in terms of $I_{2,D}$. The exact expression for $I_{N,D}$ is obtained from the arguments:

$$\begin{aligned}
I_{N,D} &= \int_{-\infty}^{\infty} dK^0 \underbrace{\int \frac{d^{D-1} K}{(2\pi)^D} \frac{1}{(K^2-a^2)^N}}_{f(K^0)} \\
&= -\int_{3+3'} f(K^0) dK^0 - \int_{2+2'} f(K^0) dK^0 \\
&= \int_{-(3+3')} f(K^0) dK^0 + \int_{-(2+2')} f(K^0) dK^0.
\end{aligned} \tag{3.14}$$

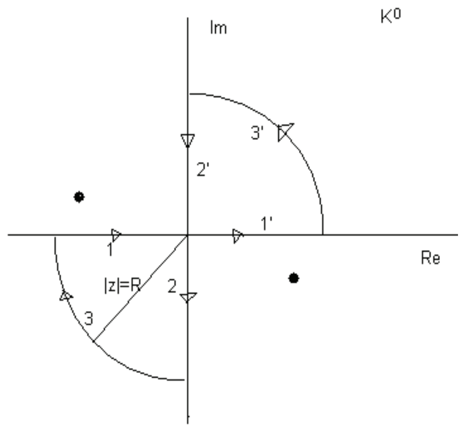

Figure 3.3: A figure illustrating the positions of the residues and contours of the function $f(K^0)$.

Fig. 3.3 is an illustration of how the path of integration over the function $f(K^0)$ can be deformed in the complex plane. The integral over the quarter of a circle in the third quadrant as $R \to \infty$, goes to zero by the argument

$$\begin{aligned}
&= \int_3 dK^0 \int \frac{d^{D-1}K}{(2\pi)^D} \frac{1}{(K^2-a^2)^N}. \\
&= \lim_{R\to\infty} \int \frac{d^{D-1}K}{(2\pi)^D} i \int_{-\frac{\pi}{2}}^{-\pi} Re^{i\theta} d\theta \frac{1}{((R^2 e^{2i\theta}) - (\vec{K}^2 + a^2))^N} \\
&\simeq \lim_{R\to\infty} \int \frac{d^{D-1}K}{(2\pi)^D} i \int_{-\frac{\pi}{2}}^{-\pi} Rd\theta \frac{1}{R^{2N}} \\
&= 0, \quad \text{for} \quad N > \frac{1}{2}. \qquad (3.15)
\end{aligned}$$

This means that for $N > \frac{1}{2}$, a Wick rotation in the complex plane leaves $I_{N,D}$ invariant. The line integral over $K^0 \in [-\infty, \infty]$ goes to $iK^0 \in [-\infty, \infty]$, where $K^0 \in \mathbf{R}$ which means that the path of integration is rotated by the angle $\frac{\pi}{2}$ in the complex plane (Along the negative direction of the line composited by (2+2') in Fig. 3.3). This is done to be able to treat K^2 as the negative length of an ordinary Euclidian vector ($K'^2 = -((K^0)^2 + (K^i)^2)$), instead of a Minkowski vector ($K^2 = (K^0)^2 - (K^i)^2$). One can now convert the integral into a D- dimensional integral in spherical coordinates. The integral is converted according to

$$\begin{aligned}
\int_{-\infty}^{\infty} dK^0 \int \frac{d^{D-1}K}{(2\pi)^D} \frac{1}{(K^2-a^2)^N} &= i \int \frac{d^D K'}{(2\pi)^D} \frac{1}{(-K'^2 - a^2)^N} \\
&= i(-1)^N \int \frac{d^D K'}{(2\pi)^D} \frac{1}{(K'^2 + a^2)^N} \\
&= i(-1)^N \int d\Omega \int \frac{K'^{D-1} dK'}{(2\pi)^D} \frac{1}{(K'^2 + a^2)^N}, \qquad (3.16)
\end{aligned}$$

where the integral over $d\Omega$ is the D-dimensional angular integral.

This integral is found by the use of

$$
\begin{aligned}
\left[\int_{-\infty}^{\infty} dx e^{-x^2}\right]^D &= \int d\Omega \int X^{D-1} dX e^{-X^2} \\
\pi^{\frac{D}{2}} &= \int d\Omega \cdot \frac{1}{2}\Gamma(\frac{D}{2}) \\
\Rightarrow \int d\Omega \cdot &= 2\pi^{\frac{D}{2}} \frac{1}{\Gamma(\frac{D}{2})},
\end{aligned} \tag{3.17}
$$

where D is the number of dimensions. Now, the last integral in Eq. (3.16) can be written as

$$
\begin{aligned}
I_{N,D} &= i(-1)^N \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int \frac{K'^{D-1} dK'}{(2\pi)^D} \frac{1}{(K'^2 + a^2)^N} \\
&= \frac{i(-1)^N}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(N - \frac{D}{2})}{\Gamma(N)} (a^2)^{\frac{D}{2} - N},
\end{aligned} \tag{3.18}
$$

where the last line was obtained by recognizing the Beta function [8]. By rewriting $I_{N,D}$, one obtains

$$
I_{N-1,D} = \frac{N-1}{(\frac{D}{2} - N + 1)} a^2 I_{N,D}. \tag{3.19}
$$

Now, Eq. (3.13) can be expressed in terms of $I_{2,D}$:

$$
\begin{aligned}
J^{\mu\nu}_{2,D} &= \frac{\eta^{\mu\nu}}{D} J_{2,D} \\
J_{2,D} &= \int \frac{d^D K}{(2\pi)^D} \frac{K^2 - a^2 + a^2}{(K^2 - a^2)^2} \\
&= I_{1,D} + a^2 I_{2,D} \\
&= \left[\frac{a^2}{(\frac{D}{2} - 1)} + a^2\right] I_{2,D}
\end{aligned} \tag{3.20}
$$

Furthermore, Eq. (3.12) can be expressed in terms of $I_{2,D}$ via the relations in Eq. (3.20)

$$
\begin{aligned}
i\Pi^{\mu\nu} &= \lim_{D\to 4} De^2 \int dx \Big[2\frac{\eta^{\mu\nu}}{D}\left[\frac{a^2}{(\frac{D}{2} - 1)} + a^2\right] - 2q^\mu q^\nu x(1-x) - \\
&\quad \eta^{\mu\nu}\left[\left[\frac{a^2}{(\frac{D}{2} - 1)} + a^2\right] - m^2 - x(1-x)q^2\right] I_{2,D} \\
&= -De^2 \left[2q^\mu q^\nu - 2\eta^{\mu\nu} q^2\right] \int_0^1 dx x(1-x) I_{2,D}(q,x) \\
&= -De^2 \left[2q^\mu q^\nu - 2\eta^{\mu\nu} q^2\right] \int_0^1 dx x(1-x) \frac{i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(2 - \frac{D}{2})}{\Gamma(2)} (a^2)^{\frac{D}{2} - 2}.
\end{aligned} \tag{3.21}
$$

Our world is assumed to contain roughly four dimensions, so the limit of the expression above has to be taken as D approaches four. I.e:

$$\begin{aligned}\lim_{\varepsilon\to 0} i\left[\Pi^{\mu\nu}(q^2)\right]_{D=4-\varepsilon} &= -4e^2\left[2q^\mu q^\nu - 2\eta^{\mu\nu}q^2\right]\int_0^1 dxx(1-x)\frac{i}{(4\pi)^{\frac{4-\varepsilon}{2}}}\frac{\Gamma(2-\frac{4-\varepsilon}{2})}{\Gamma(2)}(a^2)^{\frac{4-\varepsilon}{2}-2}\\ &= -i\left[q^\mu q^\nu - \eta^{\mu\nu}q^2\right]\underbrace{\frac{8e^2}{(4\pi)^2}\int_0^1 dxx(1-x)\left(\frac{2}{\varepsilon}-\ln a^2+\gamma_e\right)}_{\Pi(q^2)}, \quad (3.22)\end{aligned}$$

where γ_e is the Euler-Mascheroni constant [4], from the Taylor expansion

$$\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma_e + \mathcal{O}(\varepsilon). \quad (3.23)$$

As $\varepsilon \to 0$, this expression is clearly divergent due to the first term, which is proportional to $\frac{1}{\varepsilon}$. To interpret what this divergence means physically, the effective propagator given by Eq. (3.1) must be found. By writing the polarization tensor on the form

$$\Pi^{\mu\nu}(q^2) = -\left[q^\mu q^\nu - \eta^{\mu\nu}q^2\right]\Pi(q^2) \quad (3.24)$$

Now, in Feynman gauge Eq. (3.1) becomes

$$iD_{\mu\nu}(q) = -i\frac{\eta_{\mu\nu}}{q^2} - i\frac{\eta_{\mu\lambda}}{q^2}\underbrace{i\Pi^{\lambda\rho}(-i)\frac{\eta_{\rho\nu}}{q^2}}_{\Gamma^\lambda_\nu} - i\frac{\eta_{\mu\lambda}}{q^2}\Gamma^\lambda_\rho\Gamma^\rho_\nu + ..., \quad (3.25)$$

where

$$\begin{aligned}\Gamma^\rho_\nu &= \left(\eta^\rho_\nu - \frac{q^\rho q_\nu}{q^2}\right)\Pi(q^2),\\ \Gamma^\lambda_\rho\Gamma^\rho_\nu &= \Gamma^\lambda_\nu\Pi(q^2)^2. \quad (3.26)\end{aligned}$$

It follows from the last line that it is easy to rewrite Eq. (3.25) in terms of a geometric series

$$\begin{aligned}iD_{\mu\nu}(q) &= -i\frac{\eta_{\mu\nu}}{q^2} - i\frac{\eta_{\mu\lambda}}{q^2}(\Gamma^\lambda_\nu)(1+\Pi(q^2)+\Pi^2(q^2)+\Pi^3(q^2)+....)\\ &= -i\frac{\eta_{\mu\nu}}{q^2} - i\frac{\eta_{\mu\lambda}}{q^2}(\left(\eta^\rho_\nu - \frac{q^\rho q_\nu}{q^2}\right)(\Pi(q^2)+\Pi^2(q^2)+\Pi^3(q^2)+...)\\ &= -i\frac{1}{q^2(1-\Pi(q^2))}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - i\frac{q_\mu q_\nu}{q^4}. \quad (3.27)\end{aligned}$$

A propagator in general gauge is

$$iD_{\mu\nu} = -i\frac{1}{q^2}(\eta_{\mu\nu} - (1-\alpha)\frac{q_\mu q_\nu}{q^2}). \quad (3.28)$$

All physically observable quantities are gauge invariant. Gauge invariance is one of the fundamental symmetries of the QED Lagrangian. From the generalized propagator, it means that all terms proportional to $(1-\alpha)$ disappear. This means that when one sums over all $q_\mu q_\nu$ when connecting the propagator to two vertices, this term should vanish and hence, the result would be the same if one neglected the term completely. Now, the total propagator in Eq. (3.27), neglecting all terms that does not contribute to observable quantities, can be written as follows

$$iD_{\mu\nu} = \frac{-i}{q^2(1-\Pi(q^2))}\eta_{\mu\nu}. \tag{3.29}$$

Since $\Pi(q^2)$ does not have a singularity at $q^2 = 0$, the propagator's pole is unchanged thus the photon remains massless [4]. Anyway, the divergence in $\Pi(q^2)$ due to the $\frac{1}{\varepsilon}$ term, is not really a physically interesting quantity, because it is not observable. It simply means that the observable charge of the electron is infinitely much larger than its bare charge. However, a quantity that is measureable is the momentum dependency. To find an explicit expression for this, one has to compare it to a reference frame. A suitable frame is the value of Π when the photon is on its mass shell, i.e $q^2 = 0$ [4]. This means that one renormalize the problem, by requiring the condition that the effect should disappear when the photon is on its mass shell. Now, one can define

$$\Pi(q^2) \to F(q^2) = \Pi(q^2) - \Pi(0). \tag{3.30}$$

Given this substitution, Eq. (3.22) becomes

$$\begin{aligned} i\Pi^{\mu\nu} &= -i\left[q^\mu q^\nu - \eta^{\mu\nu}q^2\right]\left[\Pi(q^2) - \Pi(0)\right], \\ &= i\frac{2\alpha}{\pi}\left[q^\mu q^\nu - \eta^{\mu\nu}q^2\right]\int_0^1 dx x(1-x)\left(\ln\frac{a^2}{m^2}\right), \end{aligned} \tag{3.31}$$

where alpha is the fine structure constant [4], and a is defined in Eq. (3.9). Now, if one looks at the scattering process in Fig. 3.4 where two electrons scatter via emission and absorption of a photon, the expression contains a factor e for each vertex, and one photon

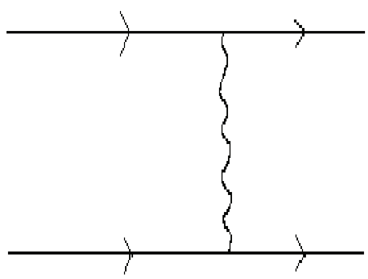

Figure 3.4: One of the Feynman diagrams illustrating electron-electron scattering.

propagating from one vertex to the other. If one now substitute the propagator with the effective propagator one described by Eq. (3.29), one sees that

$$-ie^2\frac{\eta^{\mu\nu}}{q^2} \to \frac{-ie^2}{q^2(1-F(q^2))}\eta_{\mu\nu}. \tag{3.32}$$

From this, it follows that the substitution gives rise to an effective momentum-related charge $e(q^2)$, given by

$$e^2 \to \frac{e^2}{(1-F(q^2))}. \tag{3.33}$$

Eq. (3.33) implies that the effective charge is momentum dependent which is very interesting. In fact this effect has been measured at CERN. The effect is called charge screening and is due to polarization of vacuum. Due to quantum fluctuations, vacuum behaves like a dielectricum. This is because virtual electron-positron pairs pop out of vacuum and align themselves in a way similar to the charges in a dielecticum thus screens the charge under consideration.

From the validity of Wick rotation, one can see from Eq. (2.2) that this is in fact equivalent to the zero-temperature part of the imaginary-time polarization tensor. This means that at $T = 0$, given that the expression for the Feynman diagrams allows Wick rotation, the zero-temperature part of the imaginary-time representation diagrams and the corresponding real-time S-matrix elements are equivalent.

3.2 The zero-momentum polarization tensor at finite temperature

At finite temperature, the phase-space integral is substituted according to Eq. (2.6), remembering that the formalism used in the expression requires Euclidean D-dimensional vectors (imaginary-time formalism). From Eq. (2.6), one can see that the substitution from real-time to imaginary-time Feynman diagrams is really nothing but a Wick rotation and a discretization of p_0. The latter turns Eq. (3.12) into the following expression,

$$i\Pi^{00}(q_0 = 0, \vec{q} \to 0) = \lim_{D\to 4} -De^2 i \int \frac{d^{D-1}K}{(2\pi)^{D-1}} T \sum_n \frac{2\omega_n^2 - (K^2 + m^2)}{(K^2 + m^2)^2}, \tag{3.34}$$

where $K^2 = \omega_n^2 + \vec{K}^2$, and $\omega_n = 2\pi T(n + \frac{1}{2})$ since the fermion fields are anti-periodic in τ with a periode β. $\vec{K}$ is defined to be the spatial-momentum vector. This can be simplified, and one obtains

$$i\Pi^{00}(q_0 = 0, \vec{q} \to 0) = \lim_{D\to 4} De^2 i \int \frac{d^{D-1}K}{(2\pi)^{D-1}} T \sum_n \left[\frac{2\omega^2}{(K^2 + m^2)^2} - \frac{1}{(K^2 + m^2)} \right], \tag{3.35}$$

where $\omega^2 = \vec{K}^2 + m^2$. One can now define the sums,

$$\begin{aligned} T\sum_n \frac{1}{\omega_n^2 + \omega^2} &= \frac{1}{2\omega} \tanh \frac{\beta}{2}\omega = I_1' \\ T\sum_n \frac{1}{(\omega_n^2 + \omega^2)^2} &= \frac{1}{4\omega^3} \left[\tanh \frac{\beta}{2}\omega - \frac{\beta\omega}{2\cosh^2 \frac{\beta}{2}\omega} \right] = I_2' = -\frac{d}{d\omega^2} I_1'. \end{aligned} \tag{3.36}$$

By renaming the spatial momentum $\vec{K} \to K$, since this is the only variable left in the integrand, Eq. (3.35) can be written as

$$\begin{aligned} i\Pi^{00}(q_0 = 0, \vec{q} \to 0) &= \lim_{D\to 4} De^2 i \int \frac{d^{D-1}K}{(2\pi)^{D-1}} \left[I_1'(K) - 2\omega^2 I_2'(K) \right] \\ &= \lim_{D\to 4} \frac{De^2 i}{(2\pi)^{D-1}} \int d\Omega \int dK K^{D-2} \left[\frac{\beta}{4\cosh^2 \frac{\beta}{2}\omega} \right]. \end{aligned} \tag{3.37}$$

This integral is not possible to solve in general, but by assuming zero fermion mass it turns into

$$\begin{aligned} i\Pi^{00} &= \lim_{D\to 4} -\frac{De^2 i}{(2\pi)^{D-1}}\int d\Omega \int dK K^{D-2}\left[\frac{\beta}{4\cosh^2\frac{\beta}{2}K}\right] \\ &= -4\frac{e^2 i}{(2\pi)^3}4\pi\beta\int dK K^2 \frac{1}{e^{\frac{\beta}{2}K}+e^{-\frac{\beta}{2}K}} \\ &= -\frac{i16\pi e^2}{(2\pi^2)^3}\beta\int dK K^2\frac{e^{\beta K}}{(e^{\beta K}+1)^2}. \end{aligned} \tag{3.38}$$

Since the integral is solveable in exactly four dimensions using, $D \to 4$ has been inserted directly before integrating. One integration by parts and the substitution $\beta K \to x$, leads to

$$\begin{aligned} i\Pi^{00}(q_0 = 0, \vec{q} \to 0) &= -\frac{i4e^2}{\pi^2\beta^2}\int dx x\sum_{n=1}^{\infty} e^{-nx}(-1)^{n+1} \\ &= -\frac{i4e^2}{\pi^2\beta^2}\frac{1}{2}\zeta(2) \\ &= -i\frac{1}{3}e^2T^2, \end{aligned} \tag{3.39}$$

where $\zeta(n)$ is Riemann's zeta function. The only arguments of the zeta used in this thesis is for $n = 2$ and 4. These are given by [5]

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90}. \tag{3.40}$$

Furthermore, the transverse modes

$$i\Pi^{ij}(q_0 = 0, \vec{q} \to 0) \propto \delta^{ij}\int_0^\infty d^{D-1}K\left[\frac{2}{D-1}K^2 I_2' - I_1\right], \tag{3.41}$$

can be found assuming zero fermion mass, by using the identity $-I_2' = \frac{d}{dK^2}I_1' \Rightarrow \int 2KI_2' dK = -I_1'$,

$$\begin{aligned} i\Pi^{ij}(q_0 = 0, \vec{q} \to 0) &\propto \delta^{ij}\int_0^\infty dK K^{D-2}\left[\frac{2}{D-1}K^2 I_2' - I_1\right] \\ &\propto \delta^{ij}\left[\int_0^\infty dK\frac{1}{D-1}K^{D-1}2KI_2 - \int_0^\infty K^{D-2}I_1 dK\right] \\ &= 0. \end{aligned} \tag{3.42}$$

The last line followed from the second by integration by parts. Now, the corrected photon propagator $iD_{\mu\nu}$ at finite temperature can be written down. The photon propagator in the real-time formalism is essentially the expectation value of a product of two gauge-field components on the form

$$iD_{\mu\nu} = \langle A_\mu(\vec{p})A_\nu(\vec{p})\rangle = -i\frac{\eta_{\mu\nu}}{p^2}, \tag{3.43}$$

in the Feynman gauge. In the following chapters, it will be made clear that in the imaginary-time formalism the propagator in Feynman gauge is

$$M_{\mu\nu} = \frac{\delta_{\mu\nu}}{P^2}, \tag{3.44}$$

where P is a Euclidean four-component vector and $P_0 = \omega_n$. This means that the dressed imaginary-time propagator in this gauge is

$$\frac{\delta_{\mu\nu}}{P^2} \to \frac{\delta_{\mu\nu}}{P^2} + \frac{\delta_{\mu\lambda}}{P^2}\Pi^{\lambda\rho}\frac{\delta_{\rho\nu}}{P^2} + ... = \frac{\delta_{\mu\nu}}{P^2 + m^2\delta^{\mu 0}\delta^{\nu 0}\delta_{\omega_n 0}}, \tag{3.45}$$

where

$$m^2 = -\Pi^{00}. \tag{3.46}$$

From Eq. (3.43) and (3.45), it follows that the propagator of the gauge field component A^0 recieves a correction in form of a mass insertion. It results in Debye screening [3] of the static electric field, thus it gives rise to a temperature-dependent, screened effective potential between two point charges in a hot medium [3, 12]. Since all A^is are unaffected by the medium effects, the magnetic field

$$\vec{B} = \nabla \times \vec{A}, \tag{3.47}$$

remains unchanged.

The modified propagator opens up for a quasiparticle description of QED at finite temperature where infra red (IR) divergences (divergences given by poles as $\vec{P} \to 0$) related to the zero Matsubara frequency mode are removed for the longitudinal photons, due to the thermal mass generated by medium effects. This will be made clearer in the following chapters. It is important to know that this is not a pure quantum effect as the charge screening in the scattering amplitude at $T = 0$, but rather a statistical effect due to interaction processes between the particles in a gas of gauge bosons and fermions. This means that the interpretation of the fermion bubbles in Fig. (3.1) is at finite temperature real fermion anti-fermion pairs rather than virtual ones. Consequently, if there is more than one flavour of fermions in the gas $\Pi^{00} \to -N_f \frac{1}{3} e^2 T^2$, since one has to sum over all the fermion flavours present. Since all the fermion masses were said to be negligible, this sum only contributes with a factor N_f. The reason for why the fermion masses are negligible, is because they are always associated with the fermion Matsubara frequencies, which for fermions are always nonzero due to the fact that the fields have to be anti-periodic in τ. This means that as long as the fermion mass $m_f \ll T$, it can be neglected.

Chapter 4

The partition function

The partition function can be obtained, using a path integral representation of field theory similar to the S-matrix expansion. By representing it in this way, the corrections follow naturally by expansion in terms of Feynman diagrams. First, the imaginary-time approach will be justified by starting with the transition amplitude defined by the matrix element

$$\langle\phi'|e^{-i\hat{H}\frac{t}{2}}e^{-i\hat{H}\frac{t}{2}}|\phi\rangle = \langle\phi'|\phi\,(t)\rangle, \tag{4.1}$$

where $\hat{H}$ is the Hamiltonian operator, and the vectors $|\phi\rangle$ are eigenvectors of the field operator $\hat{\phi}(\mathbf{x})$ defined below in Eq. (4.4). The right hand side follows from the time-dependent Scho dinger equation. Eq. (4.1) is defined as the amplitude for a system to change from state $|\phi\rangle$ to $|\phi'\rangle$ during a time interval t, without any information about which intermediate states the system has occupied during the time interval itself.

To make an interesting theory out of this amplitude, from a thermodynamic point of view, a set of convenient basis vectors must first be defined for the operators involved.

First the eigenvalue problem

$$\hat{\phi}(\mathbf{x}_i, 0)|\phi(\mathbf{x}_i)\rangle = \phi(\mathbf{x}_i)|\phi(\mathbf{x}_i)\rangle, \tag{4.2}$$

will be considered. Here $\hat{\phi}(\mathbf{x}_i, 0)$ is the time-independent Schrødinger-picture field operator at lattice point $\mathbf{x}_i$ in space. This implies that the vectors $|\phi_i\rangle$ that satisfy Eq. (4.2) and generate the eigenvalues $\phi(\mathbf{x}_i)$ form a complete set of eigenvectors for the field operator at the point $\mathbf{x}_i$. These vectors can only describe the field at point i, and is not really convenient to use for evaluating a system of finite size. However, if one goes to the equal-time commutators

$$\left[\hat{\phi}(\mathbf{x}_1), \hat{\phi}(\mathbf{x}_2)\right] = 0, \tag{4.3}$$

it is possible to define set of vectors $|\phi\rangle$ that satisfies the equation

$$\hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle, \tag{4.4}$$

where

$$|\phi\rangle \equiv |\phi(\mathbf{x}_1)\rangle \cdot |\phi(\mathbf{x}_2)\rangle \cdots |\phi(\mathbf{x}_N)\rangle, \tag{4.5}$$

since the vectors representing the states are independent of eachother and therefore find themselves in different spaces, i.e they represent independent degrees of freedom. The complete set of vectors, satisfying Eq. (4.4) gives the relation of completeness,

$$\prod_i \int d\phi(\mathbf{x}_i)|\phi\rangle\langle\phi| = \mathbf{1}, \tag{4.6}$$

given that the states are properly normalized.

The Euler-Lagrange equations for a classical field system are obtained by finding the extrema of the action,S, when varying the fields and their derivatives independentently [9]. S is defined to be

$$S = \int dt \int d^3x\mathcal{L}, \tag{4.7}$$

where $\mathcal{L}$ is the Lagrangian density. In a classical one-particle system, the extrema of the action is found by varying the spatial coordinates (time dependent), and their time derivatives. This gives a close relation between the spatial coordinates in a discrete system and the fields in quantum field theory (in quantum theory the solutions of the classical Euler-Lagrange equations are quantized by requiring the validity of the canonical commutators).

By looking at the properties of the coordinate operators in the Schrödinger picture (time-independent operators), one can see that they are not parametrized by an internal parameter, i.e x,y and z are totally independent of all other quantities. One has the eigenvalue problem [6]

$$\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle, \tag{4.8}$$

which leads to the relation of completeness

$$\int d^3x|\mathbf{x}\rangle\langle\mathbf{x}| = \mathbf{1}, \tag{4.9}$$

which is a three dimensional integral. A field in the Schödinger picture has the relation of completeness given by Eq. (4.6), which is an infinite dimensional integral. From this, one can picture the analogy that the fields can be looked upon as coordinates in an infinite-dimensional space. It can also be seen directly from Eq. (4.4) that there are an infinite number of degrees of freedom in quantum field theory. This is because there is a complete set of solutions for the eigenvalue problem in Eq. (4.4) at each point in space and as far as one knows, space is continuous.

The same arguments as above are repeated for the eigenstates of the conjugate momentum operator $\hat{\pi}(x)$, $|\pi\rangle$. The conjugate momentum is defined from the Lagrangian density as

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial_0\phi(x)}. \tag{4.10}$$

Only the results will be listen below

$$\int \prod_i \frac{d\pi_i}{2\pi}|\pi\rangle\langle\pi| = \mathbf{1}, \tag{4.11}$$

$$\langle\pi^a|\pi^b\rangle = \prod_x \delta(\pi^a(\mathbf{x}) - \pi^b(\mathbf{x})). \tag{4.12}$$

Now, by dividing the total time interval t into N equal time intervals, i.e $\Delta t = \frac{t}{N}$, the transition amplitude in Eq. (4.1) can be rewritten as follows

$$\langle \phi' | e^{-i\hat{H}\Delta t} e^{-i\hat{H}\Delta t} e^{-i\hat{H}\Delta t} \cdots e^{-i\hat{H}\Delta t} | \phi \rangle. \tag{4.13}$$

Furthermore, for every exponential operator at time t_n insert on the left side, a relation of completeness for the conjugate momentum vectors, and on the right side a relation of completeness for the field vectors. One can look at these operations as if one assumes that the system is in state $|\phi_N\rangle, |\pi_N\rangle$ at time t_n, then take the expectation value of the exponential operator between the conjugate- and the field vector at this time, then project the field vector at this time onto the congugate momentum vector at time t_{n-1}, then do the same procedure at this instantaneous point in time, before integrating over all possible instantenous states at each point in time to find the total overlap.

These projections allows one to evaluate the exponential operators at a single point in time [7]. All such operations leave the system invariant since a relation of completeness only contributes with a factor **1**, per definition. Assuming that $|\phi'\rangle = |\phi\rangle = |\phi_a\rangle$, Eq. (4.1) describes the transition amplitude for a system to return to its initial state after a given time t.

By performing the operations described above, Eq. (4.13) given that $|\phi'\rangle = |\phi\rangle = |\phi_a\rangle$, becomes [7]

$$\begin{aligned} = \quad & \int \langle \phi_a | (\prod_i \frac{d\pi_N^i}{2\pi} d\phi_N^i) |\pi_N\rangle \langle \pi_N | e^{-i\hat{H}\Delta t} | \phi_N \rangle \\ & \times \langle \phi_N | (\prod_i \frac{d\pi_{N-1}^i}{2\pi} d\phi_{N-1}^i) |\pi_{N-1}\rangle \langle \pi_{N-1} | e^{-i\hat{H}\Delta t} | \phi_{N-1} \rangle \langle \phi_{N-1} | \cdots \\ & \cdot\cdot \prod_i \frac{d\pi_k^i}{2\pi} d\phi_k^i |\pi_k\rangle \langle \pi_k | e^{-i\hat{H}\Delta t} | \phi_k \rangle \langle \phi_k | \cdot\cdot \\ & \cdot\cdot \prod_i \frac{d\pi_1^i}{2\pi} d\phi_1^i |\pi_1\rangle \langle \pi_1 | e^{-i\hat{H}\Delta t} | \phi_1 \rangle \langle \phi_1 | \phi_a \rangle. \end{aligned} \tag{4.14}$$

The expression is completely equivalent to Eq. (4.1) given that $|\phi'\rangle = |\phi\rangle$, because the only difference between the two representations is that Eq. (4.14) is Eq. (4.1) multiplied by one.

From quantum mechanics (QM) one has the relation [6]

$$\langle \mathbf{x} | \mathbf{p} \rangle = e^{i\mathbf{p}\mathbf{x}}. \tag{4.15}$$

As previously stated, the Lagrangian formalism implies a close relation between the coordinates in single-particle theory and the fields in field theory. Moreover, in field theory the Lagrangian *density*, $\mathcal{L}$, is varied with respect to the fields and their derivatives to find the extrema of the action, while in discrete (single-particle) systems the Lagrangian itself L is varied with respect to the coordinates and their time-derivatives. The relation between the two is

$$L = \int d^3x \mathcal{L}. \tag{4.16}$$

In a discrete system, the conjugate momentum is defined as

$$p_i = \frac{\partial L}{\partial_o x_i}. \tag{4.17}$$

By looking at the two definitions above and Eq. (4.10), one can draw the conclusion that the field theoretical analogy to the argument of the exponential in Eq. (4.15), is the conjugate momentum at space-time point x times the corresponding coordinate represented by the field at the same point integrated over all space, since the conjugate momentum is represented as a density. One can easily see that it has to represent a density, since it is defined from the Lagrangian density via Eq. (4.10). It is easier to see the relation when writing it as follows

$$\underbrace{\frac{\partial L}{\partial_o x_i(t)} \cdot x_i(t)}_{Mechanics} \rightarrow \underbrace{\int d^3x \frac{\partial \mathcal{L}}{\partial_0 \phi(\mathbf{x}, t)} \phi(\mathbf{x}, t)}_{Field\ theory}, \tag{4.18}$$

where the right hand side is summed over all the fields if there are more than one field present in the theory. From this, and the fact that the states are eigenvectors of the time-independent field and conjugate momentum operators, the projection of a field vector onto a conjungate-momentum field vector becomes [3],

$$\langle \phi_{k+1} | \pi_k \rangle = e^{i \int d^3x \pi_k(\mathbf{x}) \phi_{k+1}(\mathbf{x})}. \tag{4.19}$$

Moreover, as $\Delta t \to 0$, the approximation

$$\langle \pi_i | e^{-i\hat{H}\Delta t} | \phi_i \rangle \simeq \langle \pi_i | 1 - i\hat{H}\Delta t | \phi_i \rangle = e^{-i\Delta t \int d^3x \mathcal{H}_i + \frac{\pi_i(\mathbf{x})\phi_i(\mathbf{x})}{\Delta t}}, \tag{4.20}$$

can be made, where $\mathcal{H}$ is the Hamiltonian density (Note that this is no longer an operator, but the expectation value of an operator). Now, Eq. (4.14) can be rewritten as

$$\langle \phi_a | e^{-i\hat{H}t} | \phi_a \rangle = \int \prod_{i,k} \frac{d\phi_k^i d\pi_k^i}{2\pi} e^{i\Delta t \sum_k \int d^3x \pi_k \frac{(\phi_{k+1} - \phi_k)}{\Delta t} - \mathcal{H}_k} \delta(\phi_a^i - \phi_1^i), \tag{4.21}$$

where the deltafunction gives the constraint $\phi_1 = \phi_a$, and the first projection in Eq. (4.14) (to the left on the right-hand side), gives the constraint $\phi_{N+1} = \phi_a$. Now, one has obtained a smart simplification by inserting the complete sets of states. It simply opened up for the possibility to evaluate the Hamiltonian at a simple point in time, in stead of doing the full evaluation of the transition amplitude with time-dependent fields. Taking the continuum limit of Eq. (4.21) as $\Delta t \to 0$, leads to

$$\begin{aligned} \Gamma &= \langle \phi_a | e^{-iHt} | \phi_a \rangle = \int \prod_{i,k} \frac{d\phi_k^i d\pi_k^i}{2\pi} e^{i \int_0^t dt \int d^3x \pi(\mathbf{x},t) \frac{d\phi(\mathbf{x},t)}{dt} - \mathcal{H}} \delta(\phi_a^i - \phi_1^i) \\ &= \int \prod_{\mathbf{x},t} \frac{d\phi(\mathbf{x},t) d\pi(\mathbf{x},t)}{2\pi} e^{i \int_0^t dt \int d^3x \pi(\mathbf{x},t) \frac{d\phi(\mathbf{x},t)}{dt} - \mathcal{H}} \delta(\phi_a(\mathbf{x}) - \phi(\mathbf{x}, 0)) \end{aligned} \tag{4.22}$$

As stated previously, there is a restriction in Eq. (4.21) and that is $\phi_{N+1} = \phi_1 = \phi_a$. In a continuous theory, this is simply $\phi(\mathbf{x}, 0) = \phi(\mathbf{x}, t) = \phi_a(\mathbf{x})$.

From [3], the partition function for a system with net zero chemical potential can be written as

$$Z = \mathrm{Tr}\left\{ e^{-\beta \hat{H}} \right\} = \int \prod_{\mathbf{x}} d\phi_a(\mathbf{x}) \langle \phi_a | e^{-\beta \hat{H}} | \phi_a \rangle, \tag{4.23}$$

When looking at the first expression in Eq. (4.22), one can see that there is a remarkable resemblance between the transition amplitude and Eq. (4.23). If one let $it \to \tau = \beta$ in the first expression for Γ in Eq. (4.22), and integrate over all possible states $|\phi_a\rangle$ they are in fact equivalent. The functional representation of the transition amplitude must then be equivalent to the partition function if the same substitutions are made. One can just let the integration variable in the exponent $i\int_0^t dt \rightarrow \int_0^\beta d\tau$ and integrate over all possible configurations $|\phi_a\rangle$. This only changes the constraints in the functional representation. From before, they were $\phi(\mathbf{x}) = \phi(\mathbf{x}, 0) = \phi_a(\mathbf{x}, t)$. Now after integrating over all $\phi(\mathbf{x})$, and changing the time variable they become $\phi(\mathbf{x}, 0) = \phi(\mathbf{x}, \beta)$ hence the only constraint left is that the fields must be periodic in β, thus they must not be in a specific state $|\phi_a\rangle$ at the imaginary times $\tau = 0, \tau = \beta$.

From the above, one obtains the general formula

$$\begin{aligned} Z &= \int \prod_{\mathbf{x}} d\phi_a(\mathbf{x})\Gamma \\ &= \int_{periodic} \prod_{x} \frac{d\phi(\mathbf{x},\tau)d\pi(\mathbf{x},\tau)}{2\pi} e^{\int_0^\beta d\tau \int d^3x \pi(\mathbf{x},\tau)\frac{d\phi(\mathbf{x},\tau)}{-id\tau} - \mathcal{H}}, \end{aligned} \tag{4.24}$$

for the partition function, where the delta function in Eq. (4.22) was integrated over to give the periodic constraint. The most important formula for further calculations in this thesis is finally identified, and only some simplifications are required to make it more compact and elegant.

From Eq. (4.14) it can be seen that the functional integration over the πs does not have any constraints, thus they are simply integrated over all space. This observation leads to the conclusion that the π dependencies in Eq. (4.22) can be integrated over and removed from the functional representation of the partition function, i.e

$$\mathcal{H} = \frac{1}{2}(\pi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{4.25}$$

$$Z = \int_{periodic} \prod_{\mathbf{x},t} \frac{d\phi(\mathbf{x},\tau)d\pi(\mathbf{x},\tau)}{2\pi} e^{\int_0^\beta d\tau \int d^3x \pi(\mathbf{x},\tau)\frac{d\phi(\mathbf{x},\tau)}{-id\tau} - \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right)}, \tag{4.26}$$

Returning to the case where the time was discrete, only now it is not the time, but the imaginary-time variable τ which is discretizised, i.e $\phi(\mathbf{x}, \tau_i) \equiv \phi_i$, one obtains

$$Z = \int_{periodic} \prod_{\mathbf{x},i} \frac{d\phi(\mathbf{x})_i d\pi(\mathbf{x})_i}{2\pi} e^{\Delta\tau \sum_i \int d^3x \pi(\mathbf{x})_i \frac{\phi(\mathbf{x})_{i+1} - \phi(\mathbf{x})_i}{-i\Delta\tau} - \frac{1}{2}(\pi(\mathbf{x})_i^2 + (\nabla\phi(\mathbf{x})_i)^2 + m^2\phi(\mathbf{x})_i^2)}. \tag{4.27}$$

Furthermore, let the space under consideration extend a cubic box of total volume L^3. This box is again divided into M^3 infinitesimal boxes of volume a^3 at the lattice points denoted by k. The time (imaginary-time) is as before divided into N intervals. Moreover, the field at lattice point k at "time" i is denoted by ϕ_i^k.

$$\begin{aligned} Z &= \int_{periodic} \left[\prod_{k',i'} d\phi_{i'}^{k'} \frac{d\pi_{i'}^{k'}}{(2\pi)} \right] e^{\Delta\tau \sum_{i,k} a^3 \pi_i^k (-i \frac{\phi_{i+1}^k - \phi_i^k}{\Delta\tau}) - \frac{1}{2}(\pi_i^k)^2) - G(\phi_k^i)}, \\ &= \int_{periodic} \prod_{k',i'} d\phi_{i'}^{k'} e^{-\Delta\tau a^3 \sum_{i,k} G(\phi_k^i)} \times \\ &\times \frac{(d\pi_{i'}^{k'})}{2\pi} e^{-\Delta\tau \sum_{i,k} a^3 \left[\frac{1}{2} \left(\pi_i^{k\ 2} - 2i\pi \frac{1}{\Delta\tau}(\phi_{i+1}^k - \phi_i^k) - \left(\frac{(\phi_{i+1}^k - \phi_i^k)}{\Delta\tau} \right)^2 \right) + \frac{1}{2} \left(\frac{(\phi_{i+1}^k - \phi_i^k)}{\Delta\tau} \right)^2 \right]}, \end{aligned} \tag{4.28}$$

$$Z = \int_{periodic} \prod_{k,i} d\phi_i^k e^{-\Delta\tau a^3 G(\phi_k^i)} \cdot \frac{(d\pi_i^k)}{2\pi} e^{-\Delta\tau a^3 \left[\frac{1}{2}(\pi_i^k - i\frac{1}{\Delta\tau}(\phi_{i+1}^k - \phi_i^k))^2 + \frac{1}{2} \left(\frac{\phi_{i+1}^k - \phi_i^k}{\Delta\tau} \right)^2 \right]}, \tag{4.29}$$

where $G(\phi)$ is defined to be

$$G = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{4.30}$$

Now, by performing the linear shift $\pi_k^{i\prime} \to \Delta\tau a^3 \left(\pi_i^k - \frac{i}{\Delta\tau}(\phi_{i+1}^k - \phi_i^k) \right)$

$$Z = \int_{periodic} \prod_{k,i} d\phi_i^k e^{-\Delta\tau a^3 G(\phi_k^i)} \frac{(d\pi_i'^k)}{2\pi\sqrt{\Delta\tau a^3}} e^{-\frac{1}{2}(\pi_i'^k)^2 - a^3 \Delta\tau \frac{1}{2} (\frac{(\phi_{i+1}^k - \phi_i^k)}{\Delta\tau})^2} \tag{4.31}$$

$$\begin{aligned} &= \left(\frac{1}{\sqrt{2\pi\Delta\tau a^3}}\right)^{NM} \int_{periodic} \prod_{k,i} d\phi_i^k e^{-\Delta\tau a^3 G(\phi_k^i)} e^{-a^3\Delta\tau \frac{1}{2}(\frac{(\phi_{i+1}^k - \phi_i^k)}{\Delta\tau})^2} \\ &= \text{Constant} \cdot \int_{periodic} \prod_{\mathbf{x},\tau} d\phi(\mathbf{x},\tau) e^{-\int_0^\beta d\tau d^3x (\frac{1}{2}(\frac{\partial\phi}{\partial\tau})^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2)} \end{aligned} \tag{4.32}$$

$$= \int_{periodic} \prod_{\mathbf{x},\tau} d\phi(\mathbf{x},\tau) e^{\int_0^\beta d\tau d^3x \mathcal{L}}, \tag{4.33}$$

since

$$\mathcal{L}(it \to \tau) = -\left[\left(\frac{\partial\phi}{\partial\tau} \right)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right]. \tag{4.34}$$

All in all, Eq. (4.33) is the most commonly used form of Eq. (4.23) [3]. Moreover, it was stated that since Eq. (4.23) implies integration over all possible configurations $|\phi_a\rangle$, the constraint does not require the field at $\tau = 0$ and $\tau = \beta$ to take one specific value at the end points. It simply requires the fields to be periodic in τ with a periode β. This means that $\phi(\mathbf{x}, 0) = \phi(\mathbf{x}, \beta)$. From now on, the point $(\mathbf{x}, \tau)$ will from now on be incorrectly referred to as the space-time coordinate x.

The partition function is to be identified as a statistical distribution. All physical quantities defined from it are averaged over the ensemble defined by the Lagrangian density. This is why the normalization constant denoted by *constant* is irrelevant for all the physics obtained by Z and is set to one in the last line of Eq. (4.33).

4.1 The partition function for a pure $U(1)$ gauge theory

For a pure $U(1)$ gauge theory defined by the Lagrangian $\mathcal{L}$ in Eq. (4.42), the partition function is essentially defined by Eq. (4.33). What is meant by essentially, is that there are some restrictions on the fields related to the fact that all A^μ are not independent of eachother.

The observables in electro-magnetic quantum theory is the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. For each set of observables $\mathbf{E}$ and $\mathbf{B}$ there are within a given class of gauges, an infinite number of A_μ which give the exact same observables [11]. This can be seen from Eq. (4.36), since the Lagrangian density and also Maxwells equations [11] are invariant when gauge transforming $A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x)$ [11]. Due to this, one must introduce a deltafunction that single out one of the gauges which correspond to a set $\mathbf{E}$ and $\mathbf{B}$, and simply factor out the gauge volume as an irrelevant constant in front of the partition function.

This is in [4] taken care of by introducing a gauge condition as an integral over a deltafunction. In this thesis, the generalized covariant Lorentz gauge is used througout the following chapters. It states [4]

$$F = \partial_\mu A^\mu - \omega(x) = 0, \tag{4.35}$$

where $\omega(x)$ is some arbitrary scalar function. This is referred to as a gauge fixing function [4]. For the QED Lagrangian to be invariant under local $U(1)$ phase transformations on the Dirac fields, a corresponding transformation must be applied to the gauge fields. The Lagrangian is invariant if the fields transform according to [10]

$$\begin{aligned} \Psi \rightarrow \Psi' &= e^{ie\alpha(x)}\Psi \\ A^\mu \rightarrow A^{\mu\prime} &= A^\mu + \partial^\mu \alpha(x). \end{aligned} \tag{4.36}$$

This means that the gauge condition under such a transformation becomes

$$F' = \partial_\mu A^\mu + \partial_\mu \partial^\mu \alpha(x) - \omega(x). \tag{4.37}$$

Now, one can define the integral

$$\prod_x \int d\alpha(x) \delta(\partial_\mu A^\mu + \partial_\mu \partial^\mu \alpha(x) - \omega(x)). \tag{4.38}$$

Since $\alpha(x)$ is some arbitrary scalar function, it does not have any constraints defining the path of integration. It is simply integrated over all possible values in every point. This means that both its derivative and second derivative can take on any value for all x. This allows for the change of variables

$$\begin{aligned} \alpha(x)' &= \partial_\mu \partial^\mu \alpha(x) \\ d\alpha(x)' &= \det(\partial_\mu \partial^\mu) d\alpha(x) \end{aligned} \tag{4.39}$$

This leads to

$$\prod_x \int d\alpha(x)' \frac{1}{\det(\partial_\mu \partial^\mu)} \delta(\partial_\mu A^\mu + \alpha' - \omega(x)) = \frac{1}{\det(\partial_\mu \partial^\mu)}. \tag{4.40}$$

Now, Eq. (4.33) can be given the gauge constraint by inserting a simple integral which value is unitary.

$$Z = \int_{periodic} \prod_{x\lambda} dA_\lambda(x) d\alpha \delta(\partial_\mu A^\mu + \partial_\mu \partial^\mu \alpha(x) - \omega(x)) \det(\partial_\mu \partial^\mu) e^{\int_0^\beta d\tau \int d^3x \mathcal{L}}. \tag{4.41}$$

$\mathcal{L}$ is defined as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}(\partial_\nu A_\mu - \partial_\mu A_\nu)(\partial^\nu A^\mu - \partial^\mu A^\nu). \tag{4.42}$$

By performing a linear shift on the gauge fields

$$A_\mu \to A_\mu - \partial_\mu \alpha(x), \tag{4.43}$$

all A_μs related to one-another by the linear shift leaving the observables invariant, are substituted with the new field A_μ, over which one integrate. Now, one obtains the partition function

$$Z = \int_{periodic} \prod_{x\lambda} d\alpha(x) dA_\lambda(x) \delta(\partial_\mu A^\mu - \omega(x)) \det(\partial_\mu \partial^\mu) e^{\int_0^\beta d\tau \int d^3x \mathcal{L}}, \tag{4.44}$$

where the integral over all $\alpha(x)$ only contribute with an infinite normalization constant, which is to be interpreted as the gauge volume. If this was not done, too many degrees of freedom would have been counted for the system, since it is the observables that are of interest. Moreover, the deltafunction takes care of the gauge-fixing and makes the system obey the Lorentz gauge condition.

To make this equation consistent with Eq. (4.23), the substitution $A_0 \to iA_0$ must also be made according to [3]. An alternative approach to justify this substitution will be made below. It follows from the argument that the imaginary-time Lagrangian density must be equal to the negative real-time Hamiltonian density ($-\mathcal{H}(t = \tau) = \mathcal{L}(t = -i\tau)$) when making the substitution $it \to \tau$ and expressing the conjugate-momentum fields π_μ in the Hamiltonian density in terms of the fields A_μ. For a scalar field theory, the $it \to \tau$ substitution was the only thing one had to do, but in a $U(1)$ gauge theory, some other peculiar substitutions must be made. By looking at Eq. (4.33), the exponent is

$$\mathcal{L}(t = -i\tau) = -\frac{1}{2}\left[(\partial_\tau \phi(x))^2 + (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2\right]. \tag{4.45}$$

The Hamiltonian describing such a system is

$$\mathcal{H}(t) = \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \tag{4.46}$$

In the canonical formalism,

$$\pi = \frac{\partial \mathcal{L}}{\partial_0 \phi} = \partial_0 \phi. \tag{4.47}$$

The Hamiltonian becomes

$$\mathcal{H}(t=\tau) = \frac{1}{2}\left[(\partial_\tau\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right] = -\mathcal{L}(i\tau). \tag{4.48}$$

To make the QED Lagrangian consistent with this observation, one can start with the Lagrangian density in the real-time formalism

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\left(\partial_\nu A_\mu - \partial_\mu A_\nu\right)\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) = \frac{1}{2}\left[\mathbf{E}^2 - \mathbf{B}^2\right] \\ \mathcal{H} &= \frac{1}{2}\left[\mathbf{E}^2 + \mathbf{B}^2\right],\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}E^i(t) &= -(\partial_0 A^i + \nabla_i A^0) \\ B^i(t) &= \epsilon_{ijk}\partial_j A^k\end{aligned} \tag{4.50}$$

It follows from these relations, that

$$-E^{i\ 2}(t=-i\tau) = (-i\partial_\tau A^i - \nabla_i A^0)(i\partial_\tau A^i + \nabla_i A^0) \underset{A^0\to iA^0}{=} E^{i\ 2}(t=\tau). \tag{4.51}$$

Again, from this it follows that

$$\mathcal{L}(t=-i\tau, A_0 = iA_0) = -\mathcal{H}(t=\tau). \tag{4.52}$$

This is why the substitution $A_0 \to iA_0$ must be made. Now, according to [3, 12], the integral

$$Z = \int_{periodic} \prod_{x\mu} dA^\mu(x)\delta(F) det\left(\frac{\partial F}{\partial\alpha}\right) e^{\int_0^\beta d\tau \int d^3x\mathcal{L}}, \tag{4.53}$$

where F is the gauge fixing function in Eq. (4.37), is the proper expression for the partition function. Moreover, they state that the conversion from the real-time Lagrangian is done by making the substitutions

$$A^0 \to iA^0, \quad k^0 \to ik^0. \tag{4.54}$$

This is exactly the same expression as Eq. (4.44), and the substitutions from the real-time to the imaginary-time Lagrangian are stated for in the Eqs. (4.24) and (4.45) through (4.52). The substitution of the zeroth component of the four momenta is equivalent to requiring periodicity in τ, after the substitution $it \to \tau$ has been made.

Because my previous work in field theory was done in Minkowski space, all the calculations concerning differentials within the Lagrangian densities, and Feynman diagrams will be calculated in the real-time formalism before converting back to the imaginary-time formalism.

Returning to the partition function in Eq. (4.44),

$$Z = \int_{periodic} \prod_x dA^\mu(x)\delta\left(\partial_\mu A^\mu + \omega(x)\right) \det\left(\partial_\mu\partial^\mu\right) e^{\int_0^\beta d\tau \int d^3x\mathcal{L}}. \tag{4.55}$$

These integrals are not at all easily calculable. Problems arise mostly due to the function $\omega(x)$, which is an arbitrary function of the space-time coordinate x. As long as one does not know anything about the form of this function, it is trivial that the expression for the partition function cannot be solved. To get rid of this problem, one can simply integrate over all possible functions $\omega(x)$, assuming a gaussian distribution around $\omega(x) = 0$ for all x [4],

$$\begin{aligned} Z &= \int_{periodic} \prod_{x\mu} dA^\mu(x) d\omega(x) \delta\left(\partial_\mu A^\mu + \omega(x)\right) \det\left(\partial_\mu \partial^\mu\right) e^{\int_0^\beta d\tau \int d^3x \mathcal{L}} e^{-\int_0^\beta d\tau \int d^3x \frac{\omega(x)^2}{2\alpha}}, \\ &= \int_{periodic} \prod_{x} dA^\mu(x) \det\left(\partial_\mu \partial^\mu\right) e^{\int_0^\beta d\tau \int d^3x \mathcal{L} - \frac{(\partial_\mu A^\mu)^2}{2\alpha}}. \end{aligned} \tag{4.56}$$

Now, by defining the action

$$S = \int_0^\beta d\tau \int d^3x \mathcal{L} - \frac{(\partial_\mu A^\mu)^2}{2\alpha}, \tag{4.57}$$

where all fields contained in $\mathcal{L}$ are periodic, and transformed according to the previously mentioned constraints.

For a free $U(1)$ gauge theory in the real-time formalism, the Lagrangian density is defined to be

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(\partial_\nu A_\mu - \partial_\mu A_\nu)(\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &= -\frac{1}{2}((\partial_\mu A_\nu)^2 - \partial_\mu A_\nu \partial^\nu A^\mu). \end{aligned} \tag{4.58}$$

From the construction of Euler-Lagrange equations defining the equations of motion for the free gauge fields, the Hamilton principle assumes that the fields vanish at the boundaries confining the region in which the fields are defined. This means that the fields vanish in the limit $x, y, z = \pm\infty$. One must also remember the required periodicity in β. Taking these statements into consideration, one integration by parts turns the action in Minkowski space into

$$S_{eff} = -\frac{1}{2}\int_0^\beta d\tau \int d^3x A_\lambda(-\partial_\mu \partial^\mu \eta^{\alpha\lambda} + \frac{\alpha - 1}{\alpha}\partial^\alpha \partial^\lambda)A_\alpha. \tag{4.59}$$

For notational convenience, a matrix $\mathbf{M}$ will according to the previous expression be defined as

$$M \equiv \begin{array}{c} \\ A_0 \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} A_0 & A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} -\Box + \frac{\alpha-1}{\alpha}\partial^0\partial^0 & \frac{\alpha-1}{\alpha}\partial^0\partial^1 & \frac{\alpha-1}{\alpha}\partial^0\partial^2 & \frac{\alpha-1}{\alpha}\partial^0\partial^3 \\ \frac{\alpha-1}{\alpha}\partial^0\partial^1 & \Box + \frac{\alpha-1}{\alpha}\partial^1\partial^1 & \frac{\alpha-1}{\alpha}\partial^1\partial^2 & \frac{\alpha-1}{\alpha}\partial^1\partial^3 \\ \frac{\alpha-1}{\alpha}\partial^0\partial^2 & \frac{\alpha-1}{\alpha}\partial^1\partial^2 & \Box + \frac{\alpha-1}{\alpha}\partial^2\partial^2 & \frac{\alpha-1}{\alpha}\partial^2\partial^3 \\ \frac{\alpha-1}{\alpha}\partial^0\partial^3 & \frac{\alpha-1}{\alpha}\partial^1\partial^3 & \frac{\alpha-1}{\alpha}\partial^2\partial^3 & \Box + \frac{\alpha-1}{\alpha}\partial^3\partial^3 \end{pmatrix} \end{array}, \tag{4.60}$$

where $\Box = \partial_\mu \partial^\mu = \partial_t^2 - \sum_i \partial_i^2$. Now, to make the matrix consistent with the imaginary time formalism, all 0'th components of the Minkowskian operators involved are made imaginary.

According to [3], the right signs of the new imaginary variables are: $ix^0 \to x^4 \equiv \tau, A^0 \to iA^0$.

$$M \equiv \begin{array}{cc} & \begin{array}{cccc} A_0 & A_1 & A_2 & A_3 \end{array} \\ \begin{array}{c} A_0 \\ A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} \Box + \frac{\alpha-1}{\alpha}\partial_\tau\partial_\tau & -\frac{\alpha-1}{\alpha}\partial_\tau\partial_1 & -\frac{\alpha-1}{\alpha}\partial_\tau\partial_2 & -\frac{\alpha-1}{\alpha}\partial_\tau\partial_3 \\ -\frac{\alpha-1}{\alpha}\partial_\tau\partial_1 & \Box + \frac{\alpha-1}{\alpha}\partial_1\partial_1 & \frac{\alpha-1}{\alpha}\partial_1\partial_2 & \frac{\alpha-1}{\alpha}\partial_1\partial_3 \\ -\frac{\alpha-1}{\alpha}\partial_\tau\partial_2 & \frac{\alpha-1}{\alpha}\partial_1\partial_2 & \Box + \frac{\alpha-1}{\alpha}\partial_2\partial_2 & \frac{\alpha-1}{\alpha}\partial_2\partial_3 \\ -\frac{\alpha-1}{\alpha}\partial_\tau\partial_3 & \frac{\alpha-1}{\alpha}\partial_1\partial_3 & \frac{\alpha-1}{\alpha}\partial_2\partial_3 & \Box + \frac{\alpha-1}{\alpha}\partial_3\partial_3 \end{pmatrix} \end{array}, \tag{4.61}$$

where $\Box$ now is represented by

$$\Box \equiv -\partial_\tau^2 - \partial_i^2. \tag{4.62}$$

Now, the effective integral can be written on the compact form

$$S_{eff} = -\frac{1}{2}\int_0^\beta \int d^3x A_\lambda M^{\lambda\alpha} A_\alpha. \tag{4.63}$$

And

$$Z \propto \int_{periodic} \prod_{x\mu} dA_\mu(x) \det(\Box) e^{-\frac{1}{2}\int_0^\beta \int d^3x A_\lambda M^{\lambda,\alpha} A_\alpha}. \tag{4.64}$$

Since $\mathbf{M}$ is a symmetric matrix, it can be diagonalized by a simple orthogonal transformation [2]. This means that the basis in which the four vector $\mathbf{A}$ is defined, can be transformed by an orthogonal transformation into a system $\mathbf{A'}$ where $\mathbf{M}$ is diagonal. By rewriting the exponent of the partition function,

$$\mathbf{A}^T\mathbf{M}\mathbf{A} = \mathbf{A}^T\mathbf{O}^T\mathbf{O}\mathbf{M}\mathbf{O}^T\mathbf{O}\mathbf{A}, \tag{4.65}$$

nothing is changed since all orthogonal matrices obey the relation $\mathbf{O}^T = \mathbf{O}^{-1}$. Furthermore, by defining **A'=OA**, this can be rewritten as follows

$$\mathbf{A}'^T\mathbf{O}\mathbf{M}\mathbf{O}^T\mathbf{A}'. \tag{4.66}$$

By doing the proper transformation, the new vector $\mathbf{A}'$ is expressed in terms of the eigenvectors of the matrix M, with the corresponding eigenvalues represented by the diagonal matrix $\mathbf{D}$. This turns the expression above into

$$\mathbf{A}'^T\mathbf{D}\mathbf{A}'. \tag{4.67}$$

Since an orthogonal transformation does not change the measure $dA^\mu(x)$, one can simply change the integration variables directly and hence, $\int dA^\mu(x) = const \int dA^{\mu\prime}$. If the orthogonal matrix is in the subgroup $SO(4)$, the constant is unitary.

Now it follows directly that

$$
\begin{aligned}
Z &\propto \int_{periodic} \prod_{x\mu} dA'_\mu(x) \det(\Box) e^{-\frac{1}{2}\int_0^\beta \int d^3x \mathbf{A^{T\prime} O^T M O A'}} \\
&= \int_{periodic} \prod_{x\mu} dA'_\mu(x) \det(\Box) e^{-\frac{1}{2}\int_0^\beta \int d^3x \mathbf{A^{T\prime} D A'}} \\
&= \det(\Box) \int_{periodic} \prod_{x\lambda} dA'_\lambda(x) e^{-\frac{1}{2}\int_0^\beta \int d^3x A'_\lambda D^{\lambda\lambda} A'_\lambda}, \qquad (4.68)
\end{aligned}
$$

where, D is the diagonal matrix $O^T MO$. Furthermore, the Fourier expansion of the fields can be defined as [3]

$$
\begin{aligned}
\mathbf{A}(x) &= \sqrt{\frac{1}{\beta V}} \sum_{n,\vec{p}} e^{i(\omega_n \tau - \vec{p}\cdot\vec{x})} \mathbf{A_n}(\vec{p}) \\
\mathbf{A}'(x) &= \sqrt{\frac{1}{\beta V}} \sum_{n,\vec{p}} e^{i(\omega_n \tau - \vec{p}\cdot\vec{x})} O^T \mathbf{A_n}(\vec{p}) \\
&= \sqrt{\frac{1}{\beta V}} \sum_{n,\vec{p}} e^{i(\omega_n \tau - \vec{p}\cdot\vec{x})} \mathbf{A'_n}(\vec{p}). \qquad (4.69)
\end{aligned}
$$

To switch from integration over the field variables on the left-hand side of Eq. (4.69) (From now referred to as the space-time representation), to the ones on the right-hand side (From now referred to as the energy-momentum representation), the differentials must be conserved. This change of integration variables leads to the new differentials

$$
\begin{aligned}
|dA^\mu(x)|^2 &= \sum_{n,\vec{p}} \left| \frac{dA^\mu(x)}{dA_n^\mu(\vec{p})} \right|^2 |dA_n^\mu(\vec{p})|^2 \\
&= \frac{1}{\beta V} \sum_{n,\vec{p}} |dA_n^\mu(\vec{p})|^2 \\
dS^2 &= \sum_x |dA^\mu(x)|^2 \\
&= \frac{1}{\beta V} \sum_x \sum_{n,\vec{p}} |dA_n^\mu(\vec{p})| \\
\sum_x |dA^\mu(x)|^2 &= \frac{\beta V}{ba^3 \beta V} \sum_{n,\vec{p}} |dA_n^\mu(\vec{p})|^2 \\
\prod_x dA^\mu(x) &= \prod_{n\vec{p}} \frac{1}{\sqrt{ba^3}} dA_n^\mu(\vec{p}), \qquad (4.70)
\end{aligned}
$$

where the last line followed from a discretization of space-time. Space is divided into $M^3 = \frac{L^3}{a^3}$ cubes of volume a^3 and the imaginary-time into $N = \frac{\beta}{b}$ intervals of length b. Thus taking the continuous limit, the measure is conserved, and the prefactor only contributes with an infinite normalization constant which of reasons that were previously mentioned, does not affect the physics.

Now, it is finally time to find the partition function

$$\begin{aligned}
Z &\propto \int_{periodic} \prod_{x\mu} dA'_\mu(x)\det(\Box)e^{-\frac{1}{2}\int_0^\beta \int d^3x A'_\rho(x) D^{\rho\rho} A'(x)_\rho} \\
&\propto \int_{periodic} \prod_{\lambda n' \vec{k}'} dA'_{\lambda n'}(\vec{k}')\det(\Box)e^{-\frac{1}{2}\sum_{n,\vec{k}} A'_{\lambda -n}(-\vec{k}) D^{\lambda\lambda}(\omega_n,\vec{k}) A'_{\lambda n}(\vec{k})} \\
&\propto \prod_{\lambda,n,\vec{k}} \int_{periodic} dA'_{\lambda n}(\vec{k})\det(\Box)e^{-\frac{1}{2}A'_{\lambda -n}(-\vec{k}) D^{\lambda\lambda}(\omega_n,\vec{k}) A'_{\lambda n}(\vec{k})} \\
&\propto \prod_{\lambda,n,\vec{k}} \frac{1}{\sqrt{D^{\lambda\lambda}(\omega_n,\vec{k})}} \int \prod_x d\overline{C}(x) dC(x) e^{-\int_0^\beta d\tau \int d^3x \overline{C}\Box C} \\
&= N\prod_{n,\vec{k}} \frac{1}{\sqrt{\det(\mathbf{D}(\omega_n,\vec{k}))}} \int_{periodic} \prod_x d\overline{C}(x) dC(x) e^{-\int_0^\beta d\tau \int d^3x \overline{C}(x)\Box C(x)}. \quad (4.71)
\end{aligned}$$

Since the determinant of a given matrix is invariant under orthogonal transformations, this can be expressed as

$$Z = N \cdot \det(\Box) \cdot \prod_{n,\vec{k}} \frac{1}{\sqrt{\det(\mathbf{M}(\omega_n,\vec{k}))}}, \quad (4.72)$$

where the Eqs. (4.61) and (4.69) define the matrix,

$$\mathbf{M}(\vec{k},n) = \begin{pmatrix}
\frac{\omega_n^2}{\alpha} + k_1^2 + k_2^2 + k_3^2 & -\frac{(\alpha-1)}{\alpha}\omega_n k_1 & -\frac{(\alpha-1)}{\alpha}\omega_n k_2 & -\frac{(\alpha-1)}{\alpha}\omega_n k_3 \\
-\frac{(\alpha-1)}{\alpha}\omega_n k_1 & \omega_n^2 + \frac{k_1^2}{\alpha} + k_2^2 + k_3^2 & -\frac{(\alpha-1)}{\alpha} k_1 k_2 & -\frac{(\alpha-1)}{\alpha} k_1 k_3 \\
-\frac{(\alpha-1)}{\alpha}\omega_n k_2 & -\frac{(\alpha-1)}{\alpha} k_1 k_2 & \omega_n^2 + k_1^2 + \frac{k_2^2}{\alpha} + k_3^2 & -\frac{(\alpha-1)}{\alpha} k_2 k_3 \\
-\frac{(\alpha-1)}{\alpha}\omega_n k_3 & -\frac{(\alpha-1)}{\alpha} k_1 k_3 & -\frac{(\alpha-1)}{\alpha} k_2 k_3 & \omega_n^2 + k_1^2 + k_2^2 + \frac{k_3^2}{\alpha}
\end{pmatrix}. \quad (4.73)$$

In Eq. (4.71), N is just some irrelevant normalization contant. Moreover C and $\overline{C}$ are complex normalized Grassmann fields usually referred to as ghost-fields, introduced to define the determinant [4],

$$\det(\Box) = \int d\overline{C} dC e^{-\int_0^\beta d\tau \int d^3x \overline{C}(x)\Box C(x)} = \prod_{n,\vec{k}} (\omega_n^2 + \vec{k}^2). \quad (4.74)$$

The right hand side followed from the fact that the ghosts are defined in the same basis as the gauge fields, to subtract the extra bosonic degrees of freedom. Since the ghosts are bosonic fields with Grassmann commutators, they violate the laws of spin statistics [4]. Eq. (4.74) followed from the rules for Grassmann integration found in the references [4, 7]. Grassmann variables are anti-commuting quantities, i.e

$$\{C_i, C_j\} = \{C_i, C_i\} = \{\overline{C}_i, C_j\} = \{\overline{C}_i, \overline{C}_j\} = \{\overline{C}_i, \overline{C}_i\} = 0. \quad (4.75)$$

Furthermore, if the ghost fields are properly normalized,

$$\int \prod_i dC_i C_i = 1$$
$$\int \prod_i dC_i = 0 \tag{4.76}$$

and from Eq. (4.75) it follows that higher powers of $C_i^n = 0$ for all $n > 1$. Now, the only integral one needs to be able to determine the determinant in the partition function is

$$\det(\mathbf{L}) \equiv \int \prod_i d\overline{C}_i dC_i e^{-\overline{C}_a L^{ab} C_b}. \tag{4.77}$$

This can be solved by considering the simple integral with two independent two-dimensional Grassmann vectors

$$\int d\overline{C}_1 dC_1 d\overline{C}_2 dC_2 e^{-\overline{C}_i C_i} = \int d\overline{C}_1 dC_1 d\overline{C}_2 dC_2 C_2 \overline{C}_2 C_1 \overline{C}_1 = 1. \tag{4.78}$$

By changing variables $C_i = L^{ij} C'_j$, and remembering the anticommuting relations, the integral above becomes

$$\begin{aligned} & Constant \int d\overline{C}_1 dC'_1 d\overline{C}_2 dC'_2 (L^{21} C'_1 + L^{22} C'_2)\overline{C}_2((L^{11} C'_1 + L^{12} C'_2)\overline{C}_1 \\ = \; & Constant \det \mathbf{L} \int d\overline{C}_1 dC'_1 d\overline{C}_2 dC'_2 C'_2 \overline{C}_2 C'_1 \overline{C}_1 = 1, \end{aligned} \tag{4.79}$$

where the determinant was obtained from anticommuting the variables into their proper position. For this equation to be satisfied, i.e for the differential to be conserved, $constant = \frac{1}{\det \mathbf{L}}$. By defining $C'_a = L^{ab} C_b$, Eq. (4.77) can be written as

$$\det(\mathbf{L}) = \det \mathbf{L} \int \prod_i d\overline{C}_i dC'_i e^{-\overline{C}_i C'_i} = \det \mathbf{L}. \tag{4.80}$$

4.2 Final expression for the pure $U(1)$ gauge theory partition function

Finally, the partition function can be obtained. It becomes

$$\begin{aligned} Z &= N \prod_{n,\vec{k}} \frac{(\omega_n^2 + \vec{k}^2)}{\sqrt{((\omega_n^2 + \vec{k}^2))^4}} \\ \ln(Z) &= -\sum_{n,k} \ln(\omega_n^2 + \vec{k}^2) + constants, \end{aligned} \tag{4.81}$$

hence the zeroth order contribution is gauge invariant, despite the gauge variance of the matrix $\mathbf{M}$. This expression will be calculated explicitly in the following chapter. However, in presence of fermions at finite temperature, the photon propagator has a correction which results in an effective squared mass $\Pi^{\mu\nu}$ due to statistical effects. This was shown in the last section of chapter three. The polarization tensor only has one component that is nonzero, namely Π^{00}. This means that the field A^0 is adressed a mass $m = \frac{eT}{\sqrt{3}}$ at finite temperature. In this case, the matrix $\mathbf{M}$ recieves a correction, more prezicely $M^{00} \to M^{00\prime} = M^{00} + m^2\delta_{\omega_n,0}$. This results a different contribution from the "free" photons, namely

$$\ln(Z) = konst - \frac{1}{2}\sum_{n,k}\left[\ln(\omega_n^2 + \vec{k}^2) + \ln(\omega_n^2 + \vec{k}^2 + m^2\delta_{\omega_n,0})\right]. \tag{4.82}$$

For a large volume V, with periodic boundaries this becomes

$$\ln(Z) = -\frac{V}{2}\int\frac{d^3k}{(2\pi)^3}\left[\sum_n[2\ln(\omega_n^2 + \vec{k}^2)] + \underbrace{\ln(\vec{k}^2)}_{P} + \ln(\vec{k}^2 + m^2)).\right] \tag{4.83}$$

The term denoted by P does not contribute, since it only gives an infinitely large constant without T or m dependence and is set to zero througout the entire report. It does nothing more than to shift the vacuum energy by a constant and is not a measurable quantity.

The effect described above, where the longitudinal photons acquire thermal masses, does not exist without the presence of fermions. It is only stated here to show that the photons at finite temperature acquire thermal masses due to interactions with the fermions. This again opens up for a quasiparticle description of the free photon partition function.

Chapter 5

The QED partition function at finite temperature

In QED, the photons are represented by the gauge field from the previous chapter. In this chapter, the theory already worked out for the gauge field will be exploited to make the calculations considerably simpler. The theory adopted is mostly the rules for Grassmann integration in Eqs. (4.74)-(4.80), the transformations in Eqs.(4.65)-(4.70), and the final expression for the free $U(1)$ gauge field partition function in Eqs. (4.81)-(4.83).

When working at weak coupling, one can expand the expression for Z in powers of the coupling constant. It was in the previous chapter shown that the partition function can be obtained by modifying the transition amplitude for a system to return to its initial state after a given time interval t. It is well known that at weak coupling, this transition amplitude can be found by perturbation theory. The perturbation series describing the transition amplitude can again be represented in terms of Feynman diagrams. In this chapter, it will first be shown that the perturbative terms of the partition function also can be represented by a series of Feynman diagrams. The only difference between the two theories are some small modifications of the Feynman rules. Finally, a two-loop approximation of the QED partition function will be found.

For a non interacting system of photons and fermions, the Lagrangian density is

$$\mathcal{L}_0 = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \equiv \overline{\Psi}(i\partial\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{5.1}$$

where as before,

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu. \tag{5.2}$$

The Lagrangian above does not contain any cross terms involving both A's and Ψ's. That is why it is called a free theory. Both the fields can be treated as though they found themselves in two different spaces with no overlap. The Lagrangian is clearly invariant under global $U(1)$ phase transformations on the fermion fields, i.e it has a conserved charge current [2, 4, 10].

However, to make the theory invariant under local phase transformations as described in Eq. (4.36), the covariant derivative must be introduced. The interacting description of this theory is given by

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \equiv \overline{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{5.3}$$

where $D = \partial_\mu - ieA_\mu$ [10]. This Lagrangian only describes a theory consisting of one fermion flavour, but all the interesting calculations in this chapter are made by assuming that the fermion mass is negligible on the temperature scale. In this limit, the final result is obtained from the single-flavour contribution, by multiplying the latter with the extra degrees of freedom one obtains from introducing additional flavours.

The Lagrangian can then be written in terms of a noninteracting part $\mathcal{L}_0$ and an interacting part as follows

$$\mathcal{L} = \mathcal{L}_0 + e\overline{\Psi}\gamma^\mu A_\mu \Psi. \tag{5.4}$$

As is well-known, $e \ll 1$ thus the introduction to this chapter implies that the last term can be treated perturbatively.

5.1 The partition function for gas consisting of free fermions and photons

By returning to the previous chapter, adopting Eq. (4.64) and re-introducing the ghost fields, the non interacting partition function becomes

$$Z_0 = \prod_x \int d\overline{C}(x)dC(x)d\overline{\Psi}(x)d\Psi(x)dA_\mu(x)\times$$
$$e^{\int_0^\beta d\tau \int d^3x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\sigma}(\partial_\mu A_\mu)^2 - \overline{C}(x)\Box C(x) + \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi)}, \tag{5.5}$$

underlying that the notation implies

$$\prod_{\alpha,\mu} \int d\overline{\Psi}_\alpha(x)d\Psi_\alpha(x)dA_\mu(x) \equiv \int d\overline{\Psi}(x)d\Psi(x)dA_\mu(x), \tag{5.6}$$

where α is the spinor index.

The part of this expression containing the gauge fields can be represented by the matrix $\mathbf{M}(\omega_n, \vec{p})$ defined by Eq. (4.73). This representation will be adopted here to make the notation more compact. Moreover, the change of integration variables from the space-time representation of the fields, to the energy-momentum representation will be made. This is consistent with Eq. (4.70).

The partition function can then be written as follows

$$Z_0 = \det(\Box) \prod_{n,\vec{p}} \int dA_{\mu n}(\vec{p}) e^{-\frac{1}{2}A_{\mu -n}(-\vec{p})M^{\mu\nu}A_{\nu n}(\vec{p})} \int d\overline{\Psi}_n(\vec{p})d\Psi_n(\vec{p})e^{\overline{\Psi}_n(\vec{p})(\not{p}-m)\Psi_n(\vec{p})}, \tag{5.7}$$

where the Fourier expansions of the spinor fields used are

$$\Psi(x) = \frac{1}{\sqrt{\beta V}} \sum_{n,\vec{p}} e^{-i(\omega_n\tau - \vec{p}\cdot\vec{x})}\Psi_n(\vec{p})$$

$$\overline{\Psi}(x) = \frac{1}{\sqrt{\beta V}} \sum_{n,\vec{p}} e^{i(\omega_n \tau - \vec{p}\cdot\vec{x})} \overline{\Psi}_n(\vec{p}). \tag{5.8}$$

From these Fourier expansions and Eq. (5.7), one can also see that $\not{p}$ must be redefined by substituting $\gamma^0 \to i\gamma^0$ and $p_0 = \omega_n$ to make the notation consistent with the actual meaning of the expression.

The first integrals over A_μ, C and $\overline{C}$ in Eq.(5.7) were according to Eq. (4.81) found to be

$$Z_g = \prod_{n,\vec{p}} \frac{(\omega_n^2 + \vec{p}^2)}{\sqrt{\det(\mathbf{M}(n,\vec{p}))}}. \tag{5.9}$$

Furthermore, the second term in Eq. (5.7) is evaluated in the following way: Ψ and $\overline{\Psi}$ are two independent Grassmann fields due to their anti-commutation relations. The rules for Grassmann integration are stated in Eq. (4.80). The fermion term in Eq. (5.7) looks quite messy, but one can make a convenient substitution of variables as was previously made to obtain Eq. (4.80). When transforming $\Psi_n(\vec{p})$, the "Grassmann Jacobian" follows from Eq. (4.80)

$$-(\not{p} - m)\Psi_n(\vec{p}) = \eta_n(\vec{p}) \Rightarrow d\Psi \to \det(-(\not{p} - m))d\eta_n(\vec{p}). \tag{5.10}$$

This, and the fact that it follows from integration over normalized Grassmann variables that $\int d\overline{\alpha} d\alpha e^{-\overline{\alpha}\alpha} = 1$, makes the last term in Eq.(5.7) become

$$\prod_{n,\vec{p}} \int d\overline{\eta}_n(\vec{p}) d\eta_n(\vec{p}) e^{-\overline{\eta}_n(\vec{p})\eta_n(\vec{p})} \det(-(\not{p} - m)) = -1^D \det(\not{p} - m) = \det(\not{p} - m). \tag{5.11}$$

Hence, the negative sign in the determinant does not affect the answer, since it can be pulled outside the determinant and multiplied with itself an even number of times, since all γ's are even-even matrices and D represents their dimension.

This means that for a system consisting of freely propagating photons and N_f flavours of fermions, the partition function is given by the expression

$$\ln(Z_0) = \left[\sum_{n,\vec{p}} \ln(\omega_n^2 + \vec{p}^2) - \frac{1}{2} \ln(\det(\mathbf{M}(n,\vec{p}))) \right] + N_f \sum_{\{n\},\vec{p}} \ln(\det(\not{p} - m)), \tag{5.12}$$

where the first two and the last term requires bosonic and fermionic values of $p_0 = \omega_n$, respectively. Fermionic and bosonic sums will from now on be denoted by

$$\sum_{\{n\}}$$
$$\sum_{n}, \tag{5.13}$$

respectively.

In the limit $m_f \to 0$, this becomes

$$\ln(Z_0) = -\left[\sum_{n,\vec{p}} \ln(\omega_n^2 + \vec{p}^2) \right] + N_f \frac{4}{2} \sum_{\{n\},\vec{p}} \ln(\omega_n^2 + \vec{p}^2), \tag{5.14}$$

where the factor $\frac{4}{2}$ followed from the determinant $\det(\not{p}) = \sqrt{\det(\not{p}^2)}$. Since all the following functional integrals assumes one flavour of fermions for notational simplification, the partition function is

$$Z_0 = \prod_{n,\vec{p}} \frac{(\omega_n^2 + \vec{p}^2)}{\sqrt{\det(\mathbf{M}(n,\vec{p}))}} \det(\not{p} - m), \tag{5.15}$$

and will be used in the perturbative expansion later in this chapter.

5.2 Two-loop approximation to the QED partition function

Now, as stated previously γ^0 is now defined to be $i\gamma^0$. This means that the term involving A_0 in the interacting part of the Lagrangian does not require the substitution $A_0 \to iA_0$ as is neccessary in the non-interacting part, since A_0 is associated with γ^0.

By looking at Eq. (5.7), it is easy to se that if one adds the interaction term in the Lagrangian as described in Eq. (5.4), the complete partition function takes the form

$$\begin{aligned} Z &= \prod_x \int d\overline{C}(x) dC(x) d\overline{\Psi}(x) d\Psi(x) dA_\mu(x) \times \\ &\quad e^{\int_0^\beta d\tau \int d^3x (-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\sigma}(\partial_\mu A^\mu)^2 - \overline{C}(x)\Box C(x) + \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi)} \times \\ &\quad \exp\left[e\int_0^\beta d\tau \int d^3x \overline{\Psi}(x) \not{A} \Psi(x)\right]. \end{aligned} \tag{5.16}$$

By expanding the last part in terms its Taylor series, the above is equivalent to

$$\begin{aligned} Z &= \prod_x \int d\overline{C}(x) dC(x) d\overline{\Psi}(x) d\Psi(x) dA_\mu(x) \times \\ &\quad e^{\int_0^\beta d\tau \int d^3x (-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\sigma}(\partial_\mu A^\mu)^2 - \overline{C}(x)\Box C(x) + \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi)} \times \\ &\quad \sum_l e^l \left[\int_0^\beta d\tau \int d^3x \overline{\Psi}(x) \not{A} \Psi(x)\right]^l \frac{1}{l!} \\ &= \prod_{n,\vec{p}} \int d\overline{C}_n(\vec{p}) dC_n(\vec{p}) d\overline{\Psi}_n(\vec{p}) d\Psi_n(\vec{p}) dA_{n\mu}(\vec{p}) \left[e^{\int_0^\beta d\tau \int d^3x \mathcal{L}_0 + \mathcal{L}_I}\right] \\ &\Rightarrow \ln(Z) = \ln(Z_0) + \ln(Z_I), \\ \ln(Z_I) &= \ln\left(1 + \frac{1}{Z_0}\sum_l \frac{e^l}{l!} \prod_{n,\vec{p}} \int d\overline{C}_n(\vec{p}) dC_n(\vec{p}) d\overline{\Psi}_n(\vec{p}) d\Psi_n(\vec{p}) dA_{\mu n}(\vec{p}) e^{S_0} \cdot S_I^l\right), \end{aligned} \tag{5.17}$$

where $S = S_0 + S_I = \int_0^\beta d\tau \int d^3x[\mathcal{L}_0 + \mathcal{L}_I]$.

Since S_0 only contains second-order terms in the gauge field, i.e e^{S_0} is symmetric around $A_\mu = 0$, the only terms contributing to the correction are even powers of S_I. All other terms vanish by (anti-) symmetric integration. The lowest order contributions are then of second order in the coupling, e. Now, the logarithm in Eq. (5.17) is expanded in terms of its Taylor series. The second order correction to the partition then becomes

$$\ln(Z_I) = \frac{e^2}{2}\langle\left(\overline{\Psi}(x)\not{A}(x)\Psi(x)\right)^2\rangle. \tag{5.18}$$

The latter means the mean value of S_I^2 over the distribution $\frac{1}{Z_0}e^{S_0}$ generated by the unperturbed ensemble.

By inserting explicit expressions for all the terms contained in Eq. (5.18), it follows that

$$\begin{aligned}\ln(Z_I) &= \frac{1}{Z_0}\left[\prod_{n',\vec{p}'}(\omega_{n'}^2+\vec{p}'^2)\right]\int d\tau_1 d\tau_2\int d^3x_1 d^3x_2\times \\ &\prod_{n,\vec{p}}\int d\overline{\Psi}_n(\vec{p})d\Psi_n(\vec{p})dA_{\mu n}(\vec{p})\times \\ &e^{S_0}\overline{\Psi}_\alpha(x_1)\not{A}_{\alpha,\beta}(x_1)\Psi_\beta(x_1)\overline{\Psi}_\delta(x_2)\not{A}_{\delta,\rho}(x_2)\Psi_\rho(x_2).\end{aligned} \tag{5.19}$$

The indices on the Ψs and $\not{A}$s are spinor indices and are as previously stated left out on the differentials, due to notational simplifications. They are, however, explicitly written on the fields in the integrand to obtain correct matrix multiplication. The integrations over the ghosts were made directly, since they are not involved in the interaction term.

By integrating over the space-time coordinates and once again change integration variables from space-time to the energy-momentum representation of the fields, this becomes

$$\begin{aligned}\ln(Z_I) &= \frac{1}{Z_0\beta V}\left[\prod_{n',\vec{p}'}(\omega_{n'}^2+\vec{p}'^2)\right]\cdot\sum_{n_i,\vec{p}_i}\prod_{n,\vec{p}}\int d\overline{\Psi}_n(\vec{p})d\Psi_n(\vec{p})dA_{\mu\ n}(\vec{p})e^{S_0}\times \\ &\overline{\Psi}_{\alpha n_1}(\vec{p_1})\not{A}_{\alpha\beta n_2}(\vec{p_2})\Psi_{\beta n_3}(\vec{p_3})\overline{\Psi}_{\delta n_4}(\vec{p_4})\not{A}_{\delta\rho n_5}(\vec{p_5})\Psi_{\rho n_6}(\vec{p_6}).\end{aligned} \tag{5.20}$$

The sum in the expression above goes over all n and $\vec{p}$ carrying indices. Moreover, it is not indicated whether the sums are fermionic or bosonic due to notational convenience.

From the integration over the space-time coordinates, some constraints in form of delta-functions arise. These constraints are $n_1+n_2=n_3$, $\vec{p_1}+\vec{p_2}=\vec{p_3}$, $n_4+n_5=n_6$, $\vec{p_4}+\vec{p_5}=\vec{p_6}$. The physical interpretation of these constraints is *conservation of four momentum*.

Firstly the factor $\frac{1}{\beta V}$ comes from the normalization constants of the fields and the delta-functions obtained when integrating over all space-time coordinates.

Moreover, to reduce this problem into smaller pieces, the integrals over all $A_\mu(\vec{p})$ are calculated first. Since all $A_\mu(x)$s are real fields, $A_{\mu -n}(-\vec{p}) = A_{\mu n}(\vec{p})^*$. This means that the integral over A_μ only contributes if $n_2=-n_5$ and $\vec{p_2}=-\vec{p_5}$ in Eq. (5.20).

A useful integral to help evaluate Eq. (5.20) is

$$\sum_{n_1,\vec{p_1}}\int dA_{\mu n}(\vec{p})e^{-\frac{1}{2}A_{\mu -n}(-\vec{p})M^{\mu\nu}A_{\nu n}(\vec{p})}A_{\lambda -n_1}(-\vec{p_1})A_{\zeta n_1}(\vec{p_1}) \tag{5.21}$$

In the previous chapter, it was shown that by applying an orthogonal transformation to the gauge field, one could diagonalize the exponent in the integral above. It was shown that $A_\lambda = O^{\gamma\lambda}A'_\gamma$, diagonalized the problem given that the matrix $\mathbf{O}$ is chosen to transform $\mathbf{A}$ in terms of the eigenvectors of the matrix $\mathbf{M}$.

By using this, the expression above becomes

$$\begin{aligned}
&\sum_{n_1,\vec{p_1}} O^{\gamma\lambda}O^{\delta\zeta}\int dA'_{\mu n}(\vec{p})e^{-\frac{1}{2}D^{\nu\nu}|A'_{\nu n}(\vec{p})|^2}A_{\gamma -n_1}(-\vec{p}_1)A'_{\delta n_1}(\vec{p}_1) \\
= &\sum_{n_1,\vec{p_1}} \delta^{\delta\gamma}O^{\gamma\lambda}O^{\delta\zeta}\left(-2\frac{\partial}{\partial D^{\gamma\gamma}(n_1,\vec{p_1})}\right)\prod_{\nu n\vec{p}}\frac{1}{\sqrt{D^{\nu\nu}(n,\vec{p})}} \\
= &\ O^{\gamma\lambda}O^{\gamma\zeta}[D^{-1}(n_1\vec{p_1})]^{\gamma\gamma}\prod_{\nu n\vec{p}}\frac{1}{\sqrt{D^{\nu\nu}(n,\vec{p})}} \\
= &\sum_{n_1,\vec{p_1}}\left[\mathbf{O}^T\mathbf{D^{-1}}(\omega_{n_1},\vec{p}_1)\mathbf{O}\right]^{\zeta\lambda}\cdot\prod_{n,\vec{p}}\frac{1}{\sqrt{\det(\mathbf{M}(\omega_n,\vec{p}))}} \\
= &\sum_{n_1,\vec{p_1}} M^{-1\zeta\lambda}\prod_{n,\vec{p}}\frac{1}{\sqrt{\det\mathbf{M}(\omega_n,\vec{p})}}. \qquad (5.22)
\end{aligned}$$

One can next evaluate the fermionic part of the expression in Eq. (5.20). By making the substitution described in Eq. (5.10), one can define a useful integral for solving the Grassmann integral over the Ψs in Eq. (5.20). By considering the integral: (This time the spinor indices are written on both the differentials and fields, to make the mathematics easier to handle. The spinor indices are always the left most index.)

$$\sum_{\{n'\},\vec{p}'}\prod_{n,\vec{p},\alpha}\int d\overline{\Psi}_{\alpha,n}d\Psi_{\alpha,n}(\vec{p})e^{\overline{\Psi}_{\alpha,n}(\vec{p})(\not{p}-m)^{\alpha\beta}\Psi_{\beta,n}(\vec{p})}\overline{\Psi}_{\rho,n}(\vec{p}')\Psi_{\lambda,n}(\vec{p}'), \qquad (5.23)$$

one can in general find the contributions from the fermionic part of Eq. (5.20). These integrals are the only contributions from the fermionic part, since the rules for Grassmann integration stated that integration over constants or higher powers of one spinor components vanish. The only way for the spinors to contribute is then when they appear on the form $\overline{\Psi}_n(\vec{p})\Psi_n(\vec{p})$.

By substituting

$$\begin{aligned}
\overline{\Psi}_n(\vec{p}) &\rightarrow \overline{\eta}_n(\vec{p}) \\
\Psi_{\rho n}(\vec{p}) &\rightarrow -(\not{p}-m)^{-1\rho\delta}\eta_{\delta n}(\vec{p}), \qquad (5.24)
\end{aligned}$$

Eq. (5.23) becomes

$$\begin{aligned}
&-\sum_{\{n'\},\vec{p}'}\prod_{\alpha,n,\vec{p}}\det(\not{p}-m)\int d\overline{\eta}_{\alpha n}d\eta_{\alpha n}(\vec{p})e^{-\overline{\eta}_{\alpha n}(\vec{p})\eta_\alpha(\vec{p})}\overline{\eta}_{\rho n'}(\vec{p}')(\not{p}'-m)^{-1\lambda,\delta}\eta_{\delta n'}(\vec{p}') \\
= &\sum_{\{n'\},\vec{p}'}\left[\prod_{n,\vec{p}}\det(\not{p}-m))\right](\not{p}'-m)^{-1\lambda\delta}\delta^{\delta\rho} \\
= &\sum_{\{n'\},\vec{p}'}\left[\prod_{n,\vec{p}}\det(\not{p}-m))\right]\left[\frac{1}{(\not{p}'-m)}\right]^{\lambda\rho}, \qquad (5.25)
\end{aligned}$$

where the negative sign vanishes due to permutations of $\overline{\eta}_{\rho n'}(\vec{p'})$ and $\eta_{\delta n'}(\vec{p'})$ to get them into the proper integration order with the variable missing a bar first.

Now, Eq. (5.20) can be evaluated by using the Eqs. (5.22), (5.15) and (5.25). By normalizing with Eq. (5.15) as required by Eq. (5.20), most of the terms appearing in Eqs. (5.22) and (5.25) cancel from Eq. (5.20).

Now, two expressions for the correction can be found. When $\vec{p_1} = \vec{p_3}$ and $\vec{p_4} = \vec{p_6}$, one obtains

$$\sum_{\{n_1\},\{n_4\}} \sum_{\vec{p}_1,\vec{p}_4} \frac{e^2}{2} (\not{p}_1 - m)^{\beta\alpha} \gamma^{\lambda}_{\alpha\beta} (\not{p}_4 - m)^{\rho\delta} \gamma^{\alpha}_{\delta\rho} \left(M^{-1}\right)^{\lambda\sigma} (0,0). \tag{5.26}$$

$(M^{-1})^{\lambda\sigma}$ is recognized as the imaginary-time photon propagator, and $((\not{p} - m)^{-1})^{\rho\delta}$ as the fermion propagator. This can be written as

$$\ln(Z_I)_1 = \frac{e^2 V}{2\beta} \left[\sum_{\{n_1\},\{n_2\}} \int \frac{d^3p_1}{(2\pi)^3} \mathrm{Tr}\Big\{\gamma^{\lambda}(\not{p}_1 - m)^{-1}\Big\} \int \frac{d^3p_2}{(2\pi)^3} \mathrm{Tr}\Big\{\gamma^{\sigma}(\not{p}_2 - m)^{-1}\Big\} \right] \left(M^{-1}\right)^{\lambda\sigma} (0,0), \tag{5.27}$$

However, when $\vec{p_1} = \vec{p_6}$ and $\vec{p_3} = \vec{p_5}$, one obtains by proper permutation of the spinors, the diagram

$$\ln(Z_I)_2 = -\frac{e^2 V}{2\beta} \sum_{\{n_1\},\{n_2\}} \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \mathrm{Tr}\Big\{\gamma^{\mu}(\not{p}_1 - m)^{-1}\gamma^{\nu}(\not{p}_2 - m)^{-1}\Big\} \left(M^{-1}\right)^{\mu\nu} (\omega_{n_1} - \omega_{n_2}, \vec{p}_1 - \vec{p}_2), \tag{5.28}$$

and the complete second order correction is

$$\ln(Z_I) = \ln(Z_I)_1 + \ln(Z_I)_2. \tag{5.29}$$

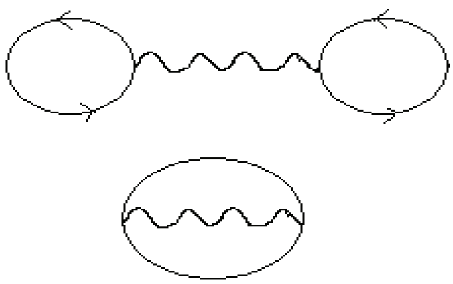

Figure 5.1: The two contributions to the second order correction can be illustrated by these two Feynman diagrams.

Now, one must recall from earlier in this chapter that the notation implied that γ^0 was renamed $i\gamma^0$, and $p_0 = \omega_n$. Furthermore, the four vector product $\not{p} = i\gamma^0\omega_n - \gamma^i p_i$[1].

[1] Several articles let $p_0 = -\omega_n$ to get the product Euclidian. However, this does not change the physics because ω_n is summed over both positive and negative frequencies.

When looking at the components inside $\mathbf{M}$, one can see that all the diagonal elements are in fact the Euclidian length of the four-vector $P = (\omega_n, \vec{p})$. Consequently, since the new anti-commutation relations for the gamma matrices are

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}, \tag{5.30}$$

the metric is also Euclidean.

The theory is now established and here comes the justification of the previously used Wick rotation, to get from the real-time Feynman diagrams to the corresponding imaginary-time diagrams.

If one wants to express the diagrams in terms of the real-time gamma-matrices, one can remove the imaginary is from the γ^0s representing the vertices in Eq. (5.28). These can be absorbed by $\mathbf{M}^{-1}$. By looking at the diagonal elements of $\mathbf{M}$, which are the only elements in Feynman gauge, one can see that this absoption leads to $M^{-1\ 00} \to -M^{-1\ 00}$. Furthermore, the imaginary γ^0s in the fermion propagator can be converted back by renaming $p_0 \to ip_0$. This however changes the matrix $\mathbf{M}$ and does not leave the expression invariant. This is fixed by writing the propagator as $M^{-1} \to \frac{\eta_{\mu\nu}}{p^2}$ where p is a Minkowski vector. When $p^0 \to ip^0$, it reproduduces $\mathbf{M}^{-1}$ when $M^{-1\ 00} \to -M^{-1\ 00}$.

Now, only the imaginary is in the real-time propagators and vertices must be accounted for. There is always one i associated with each vertex in the real-time QED diagrams. Each vertex is in vaccuum diagrams always associated with one fermion propagator. In the sunset diagram this gives no i^4 and no sign change. However, there is an i associated with the Wick rotation for each indefinite four momentum. For the sunset diagram this gives a factor -1. This factor is simply removed by placing $-i$ in front of the matrix with which the imaginary-time propagator was replaced and hence, it becomes the real-time photon propagator. When calculating real-time Feynman diagrams, one always find iF where F is the quantity the diagram represents. This removes the additional i emerging from the photon propagator.

This means that the Feynman diagrams can be worked out in Minkowski space with the real-time representations of the propagators, and then transformed back to the imaginary-time formalism by a Wick-rotation.

The first diagram in Fig. 5.1, represents $\ln(Z_I)_1$ and does not contribute according to Furry's theorem [10]. This means that the only contribution to the second-order correction of the partition function is minus one half the second diagram, i.e

$$\ln(Z_I)_2 = -\frac{e^2 V}{\beta} \sum_{\{n_1\},\{n_2\}} \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} Tr\{\gamma^\mu(\not{p}_1 - m)^{-1}\gamma^\nu(\not{p}_2 - m)^{-1}\} \left(M^{-1}\right)^{\mu\nu} (\omega_{n_1} - \omega_{n_2}, \vec{p}_1 - \vec{p}_2), \tag{5.31}$$

in the imaginary-time representation.

From the information stated from the beginning of this chapter, until now, one can draw the conclusion that the corrections to the partition function, represented by the series in

Eq. (5.17), are nothing but a perturbative expansion represented by the sum of all possible connected Feynman diagrams, since the Eqs. (5.22) and (5.25) represent contractions of gauge fields and fermion spinors respectively when they are divided (normalized) by Eq. (5.15).

Returning to Eq. (5.31) to finish the calculations of the two-loop approximation. Eq. (5.31) will be rewritten in terms of the real-time matrices and operators, as stated previously.

By using the relations in Eq. (3.3), remembering the volume factors from the periodic boundary conditions of the spatial momentum, this becomes

$$\begin{aligned} i\ln(Z_I) &= i\frac{(ie)^2V^2}{2\beta V}\int\frac{d^4p_1d^4p_2}{(2\pi)^8}Tr\{\gamma^\mu(\not{p}_1-m)^{-1}\gamma^\nu(\not{p}_2-m)^{-1}\}\times \\ & \frac{1}{(p_1-p_2)^2}\left[\eta_{\mu\nu}-(1-\alpha)\frac{(p_1-p_2)_\mu(p_1-p_2)_\nu}{(p_1-p_2)^2}\right] \\ \ln(Z_I) &\equiv -\frac{e^2V^2}{2\beta V}\int\frac{d^4p_1d^4p_2}{(2\pi)^6}A(p_{0i},\vec{p}_i). \end{aligned} \tag{5.32}$$

Here, $A(p_{0i},\vec{p}_i)$ contains four expressions. The trace over two gamma matrices proportional to m^2, and the trace over four gamma matrices, both combined with the two factors within the photon propagator. In a hot plasma, the fermion masses, $m_f \ll T$ so the fermion mass is negligible. This reduces $A(p_{0i},\vec{p}_i)$ down to

$$A = \mathrm{Tr}\{\gamma^\mu(\not{p}_1)^{-1}\gamma^\nu(\not{p}_2)^{-1}\}\frac{1}{q^2}(\eta_{\mu\nu}-(1-\alpha)\frac{q_\mu q_\nu}{q^2}), \tag{5.33}$$

where $q = p_1 - p_2$. The gauge-dependent term is

$$R(p_1,p_2) = -(1-\alpha)\mathrm{Tr}\{\gamma^\mu(\not{p}_1)^{-1}\gamma^\nu(\not{p}_2)^{-1}\frac{q_\mu q_\nu}{q^4}, \tag{5.34}$$

where $q = (p_1 - p_2)$ and hence q^0 is bosonic since the difference between two half integers is an integer. It will now be shown that this term vanishes, thus the correction to the partition function is gauge invariant within the class of Lorentz gauges. Since the partition function is an observable physical quantity, it is important that it is gauge invariant just like other physical observables.

By using the relations stated in Eq. (3.3), Eq. (5.34) becomes

$$R(p_1,p_2) = -4(1-\alpha)\frac{1}{p_1^2p_2^2q^4}\left(2(q\cdot p_1)(q\cdot p_2)-q^2(p_1\cdot p_2)\right). \tag{5.35}$$

To find the gauge-dependent contribution to the correction, this term is Wick rotated, summed over all P_1^0, P_2^0 and integrated over all $(\vec{p}_1,\ \vec{p}_2)$ as previously done. The operator meaning sum over zero'th components and integration over the vector components will from now on be denoted by

$$T^2\sum_{p^0,\ q^0}\int\frac{d^{D-1}q}{(2\pi)^{D-1}}\frac{d^{D-1}p}{(2\pi)^{D-1}} \equiv \sum\!\!\!\!\!\!\int_{p,q} \tag{5.36}$$

with subscripts pointing out the variables over which the "sum-integral" is taken. Furthermore, all Euclidean vectors will be denoted by large letters, i.e $p_1 \to P_1$. This means that the gauge-dependent part of the correction is written as follows

$$\beta V \sum\!\!\!\!\!\!\!\int_{\{P_1\},\{P_2\}} 4(1-\alpha)\frac{1}{P_1^2 P_2^2 Q^4}\left(2(Q\cdot P_1)(Q\cdot P_2) - Q^2(P_1\cdot P_2)\right). \tag{5.37}$$

One can now change integration and summation variables from $P_1,\ P_2$ to P_1, Q. In dim-reg, which the definition of the sumintegral indicates, this is allowed without changing the limits because there are no cut-off parameters limiting the integrals. From now on, if there is no explicit notation indicating the dimension of the problem, it is underlying that $D \to 4$. By making the substitutions, one obtains

$$\beta V(1-\alpha)\sum\!\!\!\!\!\!\!\int_{\{P_1\},Q} 8\frac{(Q\cdot P_1)^2}{P_1^2(P_1-Q)^2Q^4} - 4\frac{Q\cdot P_1}{P_1^2(P_1-Q)^2Q^2} - \frac{4}{(P_1-Q)^2Q^2}. \tag{5.38}$$

This expression is not very convenient to work with, but by exploiting the relation

$$Q\cdot P_1 = -\frac{1}{2}\left[(P_1-Q)^2 - P_1^2 - Q^2\right], \tag{5.39}$$

it becomes

$$\beta V(1-\alpha)\sum\!\!\!\!\!\!\!\int_{\{P_1\},Q} 2\frac{P_1^2}{(P_1-Q)^2Q^4} + 2\frac{(P_1-Q)^2}{P_1^2Q^4} - \frac{4}{Q^4} - \frac{2}{Q^2P_1^2} - \frac{2}{(P_1-Q)^2Q^2}. \tag{5.40}$$

The first and the last term in Eq. (5.40) are converted by substituting $P_1' = P_1 - Q$. Since the sums over Q^0 and P_1^0 goes from $-\infty$ to ∞, $P_1 - Q$ can take all values for each Q thus the variables P_1' and Q can be treated as independent integration variables and the limits are again unchanged due to dim-reg. The same argument is used to convert the integral. Eq. (5.40) can then be written as follows

$$\beta V(1-\alpha)\left[\sum\!\!\!\!\!\!\!\int_{\{P_1'\},Q}\left[2\frac{(P_1'+Q)^2}{P_1'^2Q^4}\right] + \sum\!\!\!\!\!\!\!\int_{\{P_1\},Q}\left[2\frac{(P_1-Q)^2}{P_1^2Q^4} - \frac{4}{Q^4} - \frac{2}{Q^2P_1^2}\right] - \sum\!\!\!\!\!\!\!\int_{\{P_1'\},Q}\left[\frac{2}{P_1'^2Q^2}\right]\right]. \tag{5.41}$$

Now, by renaming $P_1' \to P_1$ this becomes

$$\beta V(1-\alpha)\sum\!\!\!\!\!\!\!\int_{\{P_1\},Q}\left[4\frac{(P_1^2+Q^2)}{P_1^2Q^4} - \frac{4}{Q^4} - \frac{2}{Q^2P_1^2} - \frac{2}{P_1^2Q^2}\right] = 0. \tag{5.42}$$

The terms over which the sum-integral is taken cancel algebraically and hence the two-loop correction to the partition function is gauge-invariant within the class of Lorentz gauges.

The only thing remaining is to explicitly work out the gauge-independent terms of the correction in Eq. (5.32). By starting with

$$\ln(Z_I) = -\frac{e^2}{2V\beta}V^2\int d^4p_1 d^4p_2 \mathrm{Tr}\{\gamma^\mu \not{p}_1^{-1}\gamma^\nu \not{p}_2^{-1}\}\frac{\eta_{\mu\nu}}{q^2}, \tag{5.43}$$

Wick rotating and using the relations in Eq. (3.3), one obtains

$$\ln(Z_I) = \frac{4\beta V}{2}e^2\sum\!\!\!\!\!\!\!\int_{\{P_1\},\{P_2\}} \frac{1}{(P_1-P_2)^2P_2^2} + \frac{1}{(P_1-P_2)^2P_1^2} - \frac{1}{P_1^2P_2^2}, \tag{5.44}$$

where the notation implies that both P_1 and P_2 are fermionic and Euclidian. By making the substitution $Q = (P_1 - P_2)$ in the first two terms, this becomes

$$\ln(Z_I) = 2e^2\beta V \left[\sum\!\!\!\!\!\!\!\int_{\{P_1\},Q} \left[\frac{1}{Q^2 P_1^2}\right] + \sum\!\!\!\!\!\!\!\int_{\{P_2\},Q} \left[\frac{1}{Q^2 P_2^2}\right] - \sum\!\!\!\!\!\!\!\int_{\{P_1\},\{P_2\}} \left[\frac{1}{P_1^2 P_2^2}\right]\right]. \tag{5.45}$$

Now there are two integrals of particular interest, these are

$$\begin{aligned} \chi(T) &= \sum\!\!\!\!\!\!\!\int_{\{P_1\}} \left[\frac{1}{P_1^2}\right] = \lim_{D\to 4} \int \frac{d^{D-1}P}{(2\pi)^{D-1}} \frac{1}{2\left|\vec{P}\right|} \tanh \frac{\beta}{2}\left|\vec{P}\right| = E_0 - \frac{1}{24}T^2 \\ \Omega(T) &= \sum\!\!\!\!\!\!\!\int_{Q} \left[\frac{1}{Q^2}\right] = \lim_{D\to 4} \int \frac{d^{D-1}Q}{(2\pi)^{D-1}} \frac{1}{2|\vec{Q}|} \coth \frac{\beta}{2}|\vec{Q}| = E_0 + \frac{1}{12}T^2 \end{aligned} \tag{5.46}$$

where $E_0 = V\int \frac{d^3P}{(2\pi)^3}\frac{1}{2|\vec{P}|}$. This UV-divergent quantity is set to zero, because it only contributes by shifting the vacuum energy. Since the vacuum energy is a non-observable quantity, it can be redefined without changing the physics of the problem under consideration.

This means that the final expression for Eq. (5.45) is

$$\ln(Z_I) = 2\beta V e^2 \left[2\chi(T)\Omega(T) - \chi(T)^2\right] = -\frac{5V}{288}e^2T^3. \tag{5.47}$$

As stated previously, one has to multiply the above with N_f number of fermion flavours to get the total contribution. From this, and Eq. (5.14) one obtains the final expression

$$\begin{aligned} \ln(Z) &= -\sum_{n,\vec{p}} \ln(\omega_n^2 + \vec{p}^2) + N_f\frac{4}{2}\sum_{\{n\},\vec{p}} \ln(\omega_n^2 + \vec{p}^2) - \frac{5N_fV}{288}e^2T^3. \\ &= 2V\frac{\pi^2}{90}T^3 + 4N_fV\frac{7}{8}\frac{\pi^2}{90}T^3 - \frac{5V}{288}e^2T^3. \end{aligned} \tag{5.48}$$

5.3 The complete two loop partition function

In this chapter, the bare propagator has been used to obtain an expression for the partition function up to a two-loop approximation. The rest of this chapter will be made in the Euclidean space, since some of the calculations are considerably simpler in the imaginary-time representation.

When expanding the partition function to higher loop-order, one gets contributions going like

$$\propto \sum\!\!\!\!\!\!\!\int_{Q} \left[\frac{1}{(Q^2)^N}\right] \tag{5.49}$$

from the bare photonic propagators. These contributions are clearly IR-divergent for all $N > 1$, i.e they are divergent in the region where the spatial momentum goes to zero. These divergences come from the zero bosonic frequency mode $\omega_n = 0$. When the logarithm in Eq. (5.17) was expanded in terms of its taylor series, only the first order contribution was taken into account. This leads to the fact that only connected Feynman diagrams contributes to $\ln(Z)$, since all the others would be cancelled by equal terms with opposite signs in the higher order expansion of the logarithm [4].

First one can consider the ring diagrams in Fig. 5.2. They can be obtained by connecting several of the one-loop diagrams in Fig. 3.1. They give rise to IR-divergences, due to conservation of four momentum at each vertex. This means that the photon propagators give a factor $\frac{1}{Q^2}$ each. Furthermore, the fermion bubbles can be expressed in terms of the polarization tensor $\Pi^{\mu\nu}$, derived in chapter three. The cofactor in the first diagram

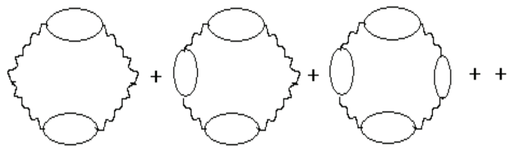

Figure 5.2: The IR divergent terms, where the bubbles represent fermion anti-fermion propagators.

recieves a factor of $1\frac{1}{4!}$ from the denominator of $\frac{\langle S_i^4\rangle}{4!}$, and there are 3! ways to connect four fermion-photon vertices to get equivalent diagrams. This means that the cofactor is $\frac{3!}{4!}$. The next cofactors are $\frac{5!}{6!} = \frac{1}{6}$ and $\frac{1}{8}$ etc etc. The leading contributions from the zero bosonic Matsubara frequency modes of these diagrams come from the divergeces where the photon momentum $\vec{Q} \to 0$. Assuming Feynman gauge, the sum of all these divergences is

$$\begin{aligned}\Delta\ln(Z) &= \lim_{D\to 4}\lim_{\vec{Q}\to 0} V\int\frac{d^{D-1}Q}{(2\pi)^{D-1}}\Big[\frac{1}{4}(M^{-1})^{\lambda\nu}\Pi^{\nu\mu}M^{-1\mu\delta}\Pi^{\delta\lambda}+ \\ &+ \frac{1}{6}(M^{-1})^{\lambda\nu}\Pi^{\nu\mu}(M^{-1})^{\mu\delta}\Pi^{\delta\rho}(M^{-1})^{\rho\lambda}+++\Big]_{Q_0=0} \\ &= \lim_{\vec{Q}\to 0}\int\frac{d^3Q}{(2\pi)^3}\frac{1}{4}\frac{\Pi^{\lambda\mu}\Pi^{\mu\lambda}}{(\vec{Q}^2)^2}+\frac{1}{6}\frac{\Pi^{\lambda\mu}\Pi^{\mu\rho}\Pi^{\rho\lambda}}{(\vec{Q}^2)^3}+\frac{1}{8}\frac{\Pi^{\lambda\mu}\Pi^{\mu\rho}\Pi^{\rho\eta}\Pi^{\eta\lambda}}{(\vec{Q}^2)^4}+++\end{aligned} \tag{5.50}$$

According to Eq. (3.42), only one component of $\Pi^{\mu\nu}$ is nonzero in this limit, and that is the longitudinal mode $\Pi^{00} = -m^2$, as $\vec{Q} \to 0$. By inspection, Eq. (5.50) becomes

$$\begin{aligned}\Delta\ln(Z) &= \lim_{D\to 4} V\int\frac{d^{D-1}Q}{(2\pi)^{D-1}}\sum_{n=2}^{\infty}\frac{1}{2n}\frac{\Pi^{00n}}{\vec{Q}^{2n}} \\ &= V\int\frac{d^3Q}{(2\pi)^3}\sum_{n=2}^{\infty}\frac{1}{2n}(-1)^n\frac{m^{2n}}{\vec{Q}^{2n}} \\ &= V\int\frac{d^3Q}{(2\pi)^3}\left[-\frac{1}{2}\ln(1+\frac{m^2}{\vec{Q}^2})+\frac{1}{2}\frac{m^2}{\vec{Q}^2}\right].\end{aligned} \tag{5.51}$$

The last term is UV-divergent and vanishes by dim-reg, thus it does not contribute to the correction. The first integral is convergent in dim-reg. Now one has summed an infinite series of divergent terms, and found a convergent answer. Furthermore, by adopting Eq. (5.14), the complete partition function, taking into account the contributions from these leading order IR-divergences, is

$$\frac{\ln(Z)}{V} = -\frac{1}{2}\sum_{n,\vec{Q}}\left[\ln(Q^2)+\ln(Q^2+\delta_{\omega_n,0}m^2)\right]+\frac{4N_f}{2}\sum_{\{n\},\vec{P}}\left[\ln(P^2)\right]-\frac{5}{288}e^2T^3. \tag{5.52}$$

By comparing the contribution from the vector bosons in Eq. (5.52) to Eq. (4.82), this is exactly the same result as one obtains when adding m^2 to the zeroth component of the inverse imaginary-time propagator $\mathbf{M}^{00}$. After this modification, the new propagator is referred to as the dressed propagator. This however, changes the system and one is forced to subtract the same quantity in some way. It can be obtained by modifying the Lagrangian defining system by giving A_0 an effective thermal mass, adding this term to the free part of the Lagrangian, and subtracting it from the interacting part as follows

$$\begin{aligned} \mathcal{L}_0 &\to \mathcal{L}_0 + \frac{1}{2}m^2 A_0 A^0 \delta_{\omega_n,0} \\ \mathcal{L}_{int} &\to \mathcal{L}_{int} - \frac{1}{2}m^2 A_0 A^0 \delta_{\omega_n,0}. \end{aligned} \tag{5.53}$$

This leads to a modified real-time propagator in Feynman gauge, $\alpha = 1$

$$iD^{\mu\nu}(q) = -\frac{i}{q^2}(\eta^{\mu\nu})(1 - \delta^{\omega_n,0}\delta^{\mu 0}\delta^{\nu 0}) - \frac{i}{q^2 - m^2}\delta^{\omega_n,0}\delta^{\mu 0}\delta^{\nu 0}, \tag{5.54}$$

and the modified imaginary-time propagator

$$M^{\mu\nu} = \frac{\delta^{\mu 0}\delta^{\nu 0}}{P^2 + m^2\delta_{\omega_n 0}} + \frac{\delta^{i\mu}\delta^{i\nu}}{P^2}. \tag{5.55}$$

Now, one must recall that the free Lagrangian required $A_0 \to iA_0$, which means that the signs in front of the mass terms in Eq. (5.53) are changed as one goes from the real-time to the imaginary-time Lagrangian. The new free partition function now reproduces the first sum in Eq. (5.52). This is a method constructed to resum all the IR-divergent terms expressable by Π^{00} an insert them into the propagator as a thermal mass. Hence, the longitudinal photons have now become dressed due to medium effects, i.e they are treated as quasiparticles, and the propagator can be referred to as the quasiparticle propagator.

The perturbation series one obtains from modifying the theory is

$$\sum_n \langle S^n_{int}\rangle \frac{1}{n!} = \sum_n \frac{1}{n!}\Big\langle \Big(\frac{m^2}{2}A_0 A_0 \delta_{\omega_n 0} + e\overline{\Psi}\not{A}\Psi\Big)^n \Big\rangle. \tag{5.56}$$

Note that $\delta_{\omega_n 0}$ only fixes the zeroth component of the *bosonic* four momenta related to A^0. By expanding Eq. (5.56), the first diagram appearing is of the order e^3. It is

$$\frac{m^2}{2}\langle A_0 A_0 \delta_{\omega_n,0}\rangle = \lim_{D\to 4}\frac{1}{2}\int \frac{d^{D-1}Q}{(2\pi)^{D-1}}\frac{m^2}{(\vec{Q}^2 + m^2)}, \tag{5.57}$$

and is the counterterm illustrated as the left most Feynman diagram in Fig. 5.3.

The second contribution comes from the second diagram in Fig. 5.3. It is the diagram previously referred to as the sunset diagram, only now the photon propagator has been modified. It gives explicitly

$$\frac{1}{2}\langle (e\overline{\Psi}\not{A}\Psi)^2\rangle = -\frac{5}{288}e^2T^3 - \lim_{D\to 4}\frac{1}{2}\int \frac{d^{D-1}Q}{(2\pi)^{D-1}}\frac{m^2}{(\vec{Q}^2 + m^2)} + \text{higher order}, \tag{5.58}$$

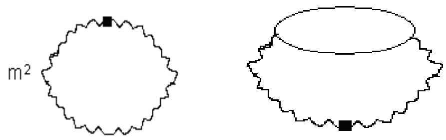

Figure 5.3: The resummed propagator connected with a coupling strength m^2, and the sunset diagram with a resummed propagator. Resummed propagators are characterized by a hard dark square.

where the fermion bubble has been Taylor expanded with respect to the external momentum $\vec{Q}$ when $Q_0 = 0$. Since the zero matsubara frequency component of the dressed photon propagators give most of their contributions when $\vec{Q} \sim eT \ll T$ when $Q_0 = 0$, and the fermion momentum is always associated with $T \gg 1$ in the fermion propagators, integrals involving both these momenta can be Taylor expanded in $\frac{|\vec{Q}|}{|\vec{P}|}$ and $\frac{\vec{Q}}{T}$, where $\vec{P}$ is the fermion momentum. Only the lowest order contributions are accounted for in these considerations, i.e where $\vec{P} - \vec{Q} \simeq \vec{P}$ and $P - \vec{Q} \simeq P$ when $Q_0 = 0$. Mathematically, this can be seen by considering the integral

$$\int d^3P d^3Q \left[\frac{1}{(P_0^2 + \vec{P}^2)(P_0^2 + (\vec{P} - \vec{Q})^2)}\right] \left[\frac{1}{(\vec{Q}^2 + m^2)}\right]. \tag{5.59}$$

As P_0 is always proportional to T, the first fraction is suppressed by the temperature. In other words, it is small, but gives approximately the same values for all $\vec{P} \ll T$, since the value of the denominator is mostly fixed by T in this region. The last fraction gives most of its contribution for small $\vec{Q}$ since $m \propto eT \ll T$, and contributes much more to the total value of the integral than the first fraction. Here, when both $\vec{Q}$ and $\vec{P} \ll T$, the first fraction remains close to unchanged since P_0 is the dominating term, and one might as well neglect $\vec{Q}$ from the first fraction, and the integrals over the two momenta decouple. When $\vec{Q}$ is of the order T, which it must be to affect the first fraction, the second fraction, thus the integrand, is a factor e^2 smaller than when $\vec{Q} \sim eT$, and the contribution from this region is disappearingly small. Moreover, for $\vec{P}$ to affect the first fraction, it must be of the order T. Given the above, the total value of the integral remains close to unchanged if one makes the approximation $\vec{P} - \vec{Q} \simeq \vec{P}$ in the region where both $\vec{P}$ and $\vec{Q} \geq T$, since the second fraction suppresses the total contribution in this region. When $\vec{Q} \sim eT$ and $\vec{P} \geq T$, it is trivial that $\vec{Q}$ can be neglected from the first fraction. This all means that a good first approximation to such integrals, is to assume that $\vec{P} - \vec{Q} \simeq \vec{P}$ and $P_0 - \vec{Q} \simeq P_0$ *not* only for small $\vec{Q}$, but for all $\vec{P}$ and $\vec{Q}$. It was previously done for the sunset diagram in Eq. (5.58). When Taylor expanding the first fraction in Eq. (5.59) around $\vec{Q} = 0$, the integral over $\vec{Q}$ gives higher order contributions in m. Since $m \propto eT$, this gives higher order expansions in the coupling.

From the above, one can draw the conclusion that to lowest order in the coupling e, the fermion bubbles decouple from the external momentum from the photon propagators when the photon propagators' Matsubara frequency is zero, thus they can be represented by the polarization tensor $\Pi^{\mu\nu}(Q_0 = 0, \vec{Q} \to 0)$, from chapter three.

The first contribution in Eq. (5.58) is equivalent to the previously calculated sunset diagram without dressed propagators. This is because the zero frequency mode had before mass insertion been set to zero by dim-reg, thus this term emerges from the diagram when $Q_0 \neq 0$. The second contribution from Eq. (5.58) is exactly canceled by the counterterm in Eq. (5.57), and is due to the dressing of the zero Matsubara frequency mode of the propagators. This contribution comes from the $Q_0 = 0$ mode, and is the lowest order contribution from the Taylor expansion of the fermion bubble. This means that the next order contribution from these diagrams are higher order than e^3 as implied by Eq. (5.58) thus omitted. Hence, the complete two-loop partition function to the order e^3 can be obtained from the above.

First, one can see what happens to the previously mentioned three-loop ring diagram [2] which was in the case without dressed propagators, IR divergent. It was the zero Matsubara frequency mode of this diagram, in the limit where the photon propagator momentum went to zero, that caused problems.

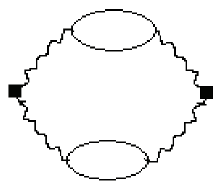

Figure 5.4: The three loop diagram with resummed propagators.

With dressed propagators, it is the contraction illustrated in Fig. 5.4. Due to the combinatorics of the problem, it has as previously stated a factor of $\frac{1}{4}$ in front. The contribution now, when $Q_0 = 0$, is

$$\lim_{D\to 4} \frac{1}{4} \int \frac{d^{D-1}Q}{(2\pi)^{D-1}} \frac{(\Pi^{00})^2}{(\vec{Q}^2 + m^2)^2} + \text{higher order} = \lim_{D\to 4} \frac{1}{4} \int \frac{d^{D-1}Q}{(2\pi)^{D-1}} \frac{(m^2)^2}{(\vec{Q}^2 + m^2)^2} + \text{higher order}, \tag{5.60}$$

hence this term is no longer IR divergent. It simply results in a term proportional to e^3 for the leading order contribution.

[2] Only the previously IR divergent terms are considered here, to show that the divergences disappear. The three loop discussed is not at all the only three loop diagram, but it is the only one giving rise to IR divergences without dressed propagators.

By going to second and third order in the action in Eq. (5.56), the contractions cracking the leading order term in Eq. (5.60) are

$$\frac{2}{8}\delta_{\omega_n,0}\langle m^4(A_0A_0)^2\rangle = \lim_{D\to 4}\frac{1}{4}\int\frac{d^{D-1}Q}{(2\pi)^{D-1}}\frac{m^4}{(\vec{Q}^2+m^2)^2} \tag{5.61}$$

$$\begin{aligned}\frac{1}{3!}\frac{3\cdot 2}{2}\delta_{\omega_n,0}\langle A_0A_0m^2(\overline{\Psi}\not{A}\Psi)^2\rangle &= \lim_{D\to 4}\frac{1}{2}\int\frac{d^{D-1}Q}{(2\pi)^{D-1}}\frac{m^2\Pi^{00}}{(\vec{Q}^2+m^2)^2} + \text{higher order}\\ &= -\lim_{D\to 4}\frac{1}{2}\int\frac{d^{D-1}Q}{(2\pi)^{D-1}}\frac{m^4}{(\vec{Q}^2+m^2)^2} + \text{higher order,}\end{aligned} \tag{5.62}$$

where the prefactors are the combinatorical factors and the factors $\frac{1}{n!}$ emerging from Eq. (5.56). Since $\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$ the e^3 contribution from these two counterterms cancel exactly the lowest order (e^3) contribution from the three-loop diagram. Higher order expansions in the external momentum for the fermion bubbles in the three-loop diagram when $Q_0=0$ give higher order contributions in the coupling as implied by Eq. (5.60). This is also stated below Eq. (5.59). Moreover, when $Q_0\neq 0$ the three-loop diagram is of the order e^4.

All the contributions from higher loop-order diagrams, proportional to e^3, get cracked one by one by corresponding counterterms, and hence the only contribution to order e^3 comes from the modified free Lagrangian.

Now, from Eq. (5.53) one can see that the only thing one has done when introducing the thermal mass is to add zero to the Lagrangian in a clever way. This means that the theory should remain unchanged, and one can draw the conclusion that no other e^3 contributions should arise since the IR divergences emerging from the ring diagrams in Fig. 5.2 are already summed up inside the modified free Lagrangian. After this recapitulation on higher loop-orders, one can return to the two-loop and lower diagrams with dressed propagators. It is now time to find the complete two-loop partition function to order e^3 in the coupling, which was the main purpose of this chapter.

The only diagrams contributing to two-loop are the resummed sunset diagram and the corresponding counterterm. Both are illustrated in Fig. 5.3. Moreover, the contractions are calculated in the Eqs. (5.57) and (5.58). By taking into account the modified zeroth order partition function, and these two diagrams, the two-loop partition function to order e^3 becomes[3]

$$\begin{aligned}
\ln(Z) &= \sum_n \int V \frac{d^3P}{(2\pi)^3}\left[-\frac{1}{2}\ln(P^2+\delta_{\omega_n,0}m^2)-\frac{1}{2}\ln P^2\right] \\
&\quad -\frac{5V}{288}e^2T^3 + 2N_f\sum_{\{n\}} V\int \frac{d^3P}{(2\pi)^3}\left[\ln(P^2)\right] \\
&= -\frac{1}{2}\sum_n \int \frac{d^3P}{(2\pi)^3}\left[2\ln(P^2)-\delta_{\omega_n,0}\ln(\vec{P}^2)+\delta_{\omega_n,0}\ln(\vec{P}^2+m^2)\right]+ \\
&\quad +2VN_f\sum_{\{n\}}\int\left[\frac{d^3P}{(2\pi)^3}\ln(P^2)\right]-\frac{5V}{288}e^2T^3 \\
&= V\int \frac{d^3P}{(2\pi)^3}-\left[2\ln(1-e^{-\beta|\vec{P}|})+\frac{1}{2}\ln(\vec{P}^2+m^2)\right]-\frac{5V}{288}e^2T^3+ \\
&\quad +2V\int\frac{d^3P}{(2\pi)^3}2\ln(1+e^{-\beta|\vec{P}|}) \\
&= 2V\frac{\pi^2}{90}T^3+4N_fV\frac{7}{8}\frac{\pi^2}{90}T^3+\frac{V}{12\pi}m^3-\frac{5V\cdot N_f}{288}e^2T^3. \qquad (5.63)
\end{aligned}$$

From the partition function, one can obtain the pressure [5]. It simply becomes,

$$\boxed{P=\frac{T}{V}\ln(Z)=\frac{\pi^2}{45}T^4(1+2N_f\frac{7}{8})+\frac{1}{12\pi}m^3T-\frac{5N_f}{288}e^2T^4,} \qquad (5.64)$$

where $m^2=\frac{N_f}{3}e^2T^2$. The next contributions to the pressure are of the order e^4.

[3] d^3P here implies $d^{D-1}P$ as $D\to 4$.

Chapter 6

QCD

Now, to transfer the already known expression for the partition function in QED to QCD there are for the free theory just a few group-theoretical constants one needs to consider. These appear because one goes from a $U(1)$ gauge theory to an $SU(3)$ gauge theory. There are also some extra corrections due to the fact that the colour mediators, the gluons[1], unlike the neutral gauge boson, the photon, also carry colour. They are said to be colour charged. This makes the gluons able to interact with each other. In a pure $SU(3)$ gauge theory, which is the theory that the rest of this thesis is about, the gluons do not only interact with themselves, but also with the ghosts introduced to explicitly write out the meaning of the determinant appearing in Eq. (4.53). The partition function will also here be found up to two-loop.

In QCD, there is a diagram with one vertex (with four legs) and two loops that contribute, in addition to the two-vertex diagrams, analogous to the sunset diagram from the previous chapter. All these diagrams are illustrated in Figs. 6.4 through 6.6. They will be found both with and without dressed propagators, neglecting all terms of higher order than g^3. Furthermore, the last section of the previous chapter told that the divergent terms of all the higher-order diagrams could be summed using an effective mass, m. This will also be done here, but this time an ansatz will be made instead of working out the polarization tensor. The ansatz will be that the thermal mass goes like

$$m = m_o g T. \tag{6.1}$$

Now, to find Z_0 for a gas consisting of gluons and fermions, the Lagrangian defining a non-abelian Gauge theory in presence of fermions, is the Yang-Mills Lagrangian [4]

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \overline{\Psi}(i\not{D} - m)\Psi \tag{6.2}$$

where a is summed over the states within the colour octet. Consequently,

$$\begin{aligned} D_\mu &= \partial_\mu - igA^a_\mu t^a_3 \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu, \end{aligned} \tag{6.3}$$

and t^a_3 are the matrices of the fundamental matrix representation of $SU(3)$ (The Gell-Mann matrices), and will from now on be denoted by t^a. The spinors do not only have

[1]The gluons are the mediating bosons of the strong force.

components in the regular spin and coordinate space, but also in colour space in which the $SU(3)$ matrices represent the operators defining the basis. Moreover, f^{abc} are the structure constants of the representation [4], just as the levi-civita tensor in $SU(2)$. It is defined by

$$\left[t^a, t^b\right] = if^{abc}t^c, \tag{6.4}$$

and all its properties used througout this chapter will be stated when they are used for the first time. For now it is sufficient to see from the commutation relation between two matrices that this tensor is totally antisymmetric.

When there are no fermions present, one can completely neglect the terms involving the Dirac spinors. This turns the Yang-Mills Lagrangian into the following expression:

$$\mathcal{L} = \mathcal{L}_0 - gf^{abc}\partial_\mu A^a_\lambda A^{\mu b}A^{\lambda c} - \frac{1}{4}g^2\left(f^{eab}A^a_\mu A^b_\lambda\right)\left(f^{ecd}A^{\mu c}A^{\lambda d}\right) \text{ , where}$$
$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}). \tag{6.5}$$

$\mathcal{L}_0$ is very similar to the one in the $U(1)$ gauge theory. The only difference is that they carry an additional index, a. The upper left-most latin index on the vector field components A^a_ν is the the index related to the Gell-Mann matrices thus it is a colour-space configuration index. The action can from the above be written as follows

$$S = S_0 + gS_1 + g^2S_2. \tag{6.6}$$

To make the Lagrangian invariant under local $SU(3)$ transformations, i.e local rotations of the spinors in colour space, the relation

$$\overline{\Psi}e^{-i\alpha^r t^r}(i\not{D}')e^{i\alpha^p t^p}\Psi = \overline{\Psi}(i\not{D})\Psi \tag{6.7}$$

must be valid for all $\alpha^i(x)$ [4]. For this to be satisfied, the transformation neccessary on the gauge field is

$$A^{a\prime}_\mu t^a = e^{\alpha^r t^r}(A^a t^a + \frac{i}{g}\partial_\mu)e^{-\alpha^c t^c}. \tag{6.8}$$

Summed up, this means that the fields, when both fermions and gauge bosons are present, must transform according to [4]

$$\begin{aligned}
\Psi \to \Psi' &= e^{i\alpha^b(x)t^b}\Psi \\
A^a_\mu \to A^{a\prime}_\mu &= A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c. \\
&= A^a_\mu + \frac{1}{g}\left[\partial_\mu\delta^{ac} + gf^{abc}A^b_\mu\right]\alpha^c,
\end{aligned} \tag{6.9}$$

for the system to be invariant under local $SU(3)$ phase transformations.

From the previous chapters, the partition function is essentially known to be the an exponential distribution of the action integral integrated over the volume extended by all the independent fields. However, there are some constraints in $SU(3)$ gauge theory similar to the ones in $U(1)$, related to the gauge conditions. These constraints appear again, because all the components of the gauge field are not independent. Just as in QED, all the fields related to one-another by a gaugetransformation should only be counted once, and the total should be multiplied with the gauge volume. Moreover, the components within the integrand must obey the gauge condition and the periodicity in β. This means that if one integrates over the volume extended by all $A^a_\mu(x)$, and restrited by the constraints, one obtains for the pure-glue partition function, an expression on the form

$$Z = N \prod_a \int_{constraints} DA^a e^S, \tag{6.10}$$

where the measure $DA^a = \prod_{x,\mu} dA^a_\mu(x)$. Notational simplification is crucial in QCD, and the reader is assumed to be familiar with the notation from the previous chapters on QED. All additional indices that do not appear in QED will be written explicitly.

Now, $A^a_0 = iA^a_0$ for the same reasons as explained in QED. For the covariant Lorentz gauge condition $G = \partial_\mu A^{a\mu} - \omega^a(x) = 0$ to be valid under the gauge transformation described in Eq. (6.9), $G \to G' = \partial_\mu A^{a\mu} + \partial_\mu \frac{1}{g}\left[\partial_\mu \delta^{ac} + gf^{abc}A^b_\mu\right]\alpha^c(x) - \omega^a(x) = 0$ for all functions $\alpha^c(x)$. Now, one can do as before and evaluate the integral

$$\prod_c \int D\alpha^c \delta(\partial_\mu A^{a\mu} + \partial_\mu \frac{1}{g}\left[\partial^\mu \delta^{ac} + gf^{abc}A^{b\mu}\right]\alpha^c - \omega^a(x)). \tag{6.11}$$

First, one makes a change of integration variables from α to α' by indentifying the matrix

$$R^{ac} = \frac{1}{g}\partial_\mu(\partial^\mu \delta^{ac} + gf^{abc}A^{\mu b})$$

$$\alpha'^a = R^{ac}\alpha^c$$

$$\frac{1}{\det(\mathbf{R})}\prod_a D\alpha'^a = \prod_a D\alpha^a. \tag{6.12}$$

Now, Eq. (6.11) can be rewritten as follows

$$\prod_a \int D\alpha'^a \det(\mathbf{R}^{-1})\delta(\partial_\mu A^{\mu a} + \alpha'^a - \omega^a(x)) = \det(\mathbf{R}^{-1}), \tag{6.13}$$

from the definition of the deltafunction. To obtain a unitary integral, one can simply multiply each side with $\det \mathbf{R}$. This means that Eq. (6.11) can be written as

$$\prod_a \int D\alpha'^a \delta(\partial_\mu A^{\mu a} + \alpha'^a - \omega(x))\det(\mathbf{R}) = 1. \tag{6.14}$$

By inserting this into the partition function, it becomes

$$Z = N \prod_a \int DA^a D\alpha^a \delta(\partial_\mu A^{\mu a} + R^{ac}\alpha^c(x) - \omega^a(x))\det(\mathbf{R})e^S. \tag{6.15}$$

To factor out the gauge volume, one can see that if the substitution $\partial_\mu A^{\mu a} \to \partial_\mu A^{\mu a\prime} = \partial_\mu A^{\mu a} + R^{ac}\alpha^c(x)$ is made, one has not done anything but to give the gauge field a linear shift related to α and rotated it in colour space by a unitary rotation [4]. Now, all the fields related to one-another by a gauge transformation are renamed A'_μ. The justification for why the measure is unchanged is because the transformation

$$e^{i\alpha^p t^p} A^a t^a e^{-i\alpha^r t^r} \tag{6.16}$$

represents a unitary, orthogonal rotation of the matrices t^a. It leaves their determinant unchanged. Since the set of $SU(3)$ matrices defining the operators in colour space are the generators for all possible rotations in this space, an arbitrary rotation matrix can be constructed from a linear combination of these matrices. When rotating such a matrix, the resulting matrix can be written as a linear combination of components of the set defining the basis. This means that a rotation of an arbitrary Gell-mann matrix t^a results in a linear combination on the form

$$t^a \to \sum_b c^b t^b, \tag{6.17}$$

where c^a are some constants chosen to leave the determinant of the resulting matrix invariant under the transformation.

Now, by defining a vector $\mathbf{A}_\mu$ in colour space consisting of $\mathbf{A}_\mu = (A^1, A^2.....A^8)_\mu$, and a vector consisting of the 3×3 Gell-Mann matrices $\mathbf{T} = (t^1, t^2, ...t^8)^T$, one can see that it is possible to perform a unitary transformation on the matrix $\mathbf{A}_\mu$ instead of rotating each t^a internally. The transformation leaves the determinant of $\mathbf{AT}$ unchanged and represents a rotation of the field components in colour space. Furthermore, this matrix must be an 8×8 matrix, due to the dimension of $\mathbf{A}_\mu$ thus it can be paramterized by the eight parameters per field component. According to Eqs. (6.16) and (6.17) such a transformation must exist. By applying a unitary rotation to the vector representing the gauge fields, the Jacobian is unitary and the measure is unchanged. It is analogous to rotating a vector in three dimensions, which clearly preserve the measure. Finally, the measure

$$\prod_{x,a,\mu} DA^a_\mu(x) = \prod_{x,a,\mu} DA^{a\prime}_\mu(x). \tag{6.18}$$

The Lagrangian is also invariant under the transformation $A^a_\mu \to A^{a\prime}_\mu$ according to Eq. (6.7) and Eq.(6.9), since this is the exactly transformation one had to do to make the Lagrangian invariant under local $SU(3)$ rotations. This can also be seen from Eq. (6.8). Finally, the partition function becomes

$$\begin{aligned} Z &= N\prod_a \int DA'^a D\alpha^a \delta(\partial_\mu A'^{a\mu} - \omega^a(x)) \det(\mathbf{R}) e^S \\ &= N\prod_a \int DA'^a \delta(\partial_\mu A'^{a\mu} - \omega^a(x)) \det(\mathbf{R}) e^S. \end{aligned} \tag{6.19}$$

The integrals over all α^a now give the gauge volume, represented by an infinite normalization constant and is as previously stated irrelevant for the physics contained in Z, and simply absorbed by N. The deltafunction now preserves the gauge condition. This expression is identical to Eq. (4.44), but was rederived here to identify in which space the determinant of the matrix defined by $\mathbf{R}$ is taken.

The complete expression for the partition function can now be written as follows

$$Z = N\prod_a \int DA^a \det(R) e^{S_0 - \frac{1}{2\alpha}(\partial_\mu A^{\mu a})^2 + gS_1 + g^2 S_2}, \tag{6.20}$$

where the unknown function $\omega(x)$ has been integrated over all possible functions assuming that the possible functions are gaussian distributed with a weight $e^{-\frac{\omega(x)^2}{2\alpha}}$ around $\omega(x) = 0$ for all x. By remembering the meaning of the determinant of the matrix $\mathbf{R}$, and recalling the rules for Grassmann integration, the determinant can be expressed as an integral over the Faddeev-Popov ghost fields $\overline{C}^a$ and C^a via Eq. (4.80) [4],

$$\det(\mathbf{R}) = \prod_a \int D\overline{C}^a DC^a e^{-\int_0^\beta d\tau \int d^3x \overline{C}^a R^{ac} C^c}. \tag{6.21}$$

This turns the partition function into

$$Z = N \prod_a \int DA^a D\overline{C}^a DC^a e^{-\int_0^\beta d\tau \int d^3x \overline{C}^a (\partial_\mu \partial^\mu \delta^{ac} + g f^{abc} A^b_\mu) C^c} \cdot e^{S - \frac{1}{2\alpha}(\partial_\mu A^{p\mu a})^2}. \tag{6.22}$$

The factor $\frac{1}{g}$ in front of the argument in the matrix $\mathbf{R}$ is absorbed by N, hence only contributes with an irrelevant normalization constant. From the above, it follows that the effective Lagrangian, also known as the Faddeev-Popov Lagrangian [4], can be written as

$$\mathcal{L} = \mathcal{L}_0 - g f^{abc} \partial_\mu A^a_\lambda A^{b\mu} A^{c\lambda} - \frac{1}{4} g^2 \left(f^{eab} A^a_\mu A^b_\lambda \right) \left(f^{ecd} A^{c\mu} A^{d\lambda} \right) - \frac{1}{2\alpha} (\partial_\mu A^{a\mu})^2 - \overline{C}^a R^{ac} C^c. \tag{6.23}$$

Now, the zeroth order partition function can be found by first going from space-time to the energy-momentum representation of the fields, then redefining the measure $DA^a \equiv \prod_{n,\vec{p},\mu} dA^a_{n\mu}(\vec{p})$ and adopting the matrix $\mathbf{M}$ from the QED partition function. It simply becomes

$$\begin{aligned} Z_0 &= \prod_a \int DA^a D\overline{C}^a DC^a e^{-\frac{1}{2} A^a_\mu M^{a\mu\nu} A^a_\nu - \overline{C}^a [R]^{ac}_{g=0} C^c} \\ &= \left[\prod_{n',\vec{k}} \det \mathbf{R}(\omega'_n, \vec{k}) \right] \left[\prod_{a,n,\vec{p}} \frac{1}{\sqrt{\det \mathbf{M}^a(\omega_n, \vec{p})}} \right] \\ \ln(Z_0) &= \sum_{n,\vec{p}} \left[8 \ln(\omega_n^2 + \vec{p}^2) - 16 \ln(\omega_n^2 + \vec{p}^2) \right] = -8 \sum_{n,\vec{p}} \ln \left[\omega_n^2 + \vec{p}^2 \right], \end{aligned} \tag{6.24}$$

where $\mathbf{M}^a(\omega_n, \vec{p})$ is the matrix defined earlier in QED and $\mathbf{R}$ is the 8×8 matrix defined in Eq. (6.12), dropping the first g since it is absorbed in N, and let the internal coupling $g \to 0$. The only difference between $\mathbf{M}^a(\omega_n, \vec{p})$ and $\mathbf{M}(\omega_n, \vec{p})$ is that there are eight of these matrices, and they are all identical according to the definition of $\mathcal{L}_0$ in Eq. (6.5). This is implied by the colour index, a, which runs over all $a = [1, N_c^2 - 1]$ where N_c is the number of colours. The dimension of the colour space in the adjoint representation of $SU(N_c)$ is $d = N_c^2 - 1$ [4].

This tells us that the only difference between a noninteracting gas of gluons, and that of a gas of photons are the internal degrees of freedom. It comes from the fact that there are eight different gluons, but only one photon. More generally, the answer can be written as

$$\ln(Z_0) = -\sum_{n,\vec{p}} (N_c^2 - 1) \ln \left[\omega_n^2 + \vec{p}^2 \right], \tag{6.25}$$

for a $SU(N_c)$ gauge theory. In QCD $N_c = 3$ since there are three colours, thus Eq. (6.25) becomes Eq. (6.24) when $N_c = 3$.

6.1 Interaction terms

From Eq. (6.23) one can see that there are three different interaction terms in the Faddeev-Popov Lagrangian. Two of them are proportional to g, and one is proportional to g^2. Since S_0 is quadratic in the fields $A^a_{n\mu}(\vec{p})$, only the terms proportional to even powers of A contribute to the expansion of the partition function due to (anti) symmetric integration. As already shown in QED, the corrections can be expressed as vacuum Feynman diagrams, similar to the S-matrix expansion. The corrections to the partition function simply describes the vacuum-vacuum transition amplitude, modified by introducing imaginary-time Feynman rules.

Now, by looking closer at Eq. (6.23), the first corrections are of the order $O(g^2)$.

$$\mathcal{L} = \mathcal{L}_0 - gf^{abc}\partial_\mu A^a_\lambda A^{b\mu} A^{c\lambda} - \frac{1}{4}g^2\left(f^{eab}A^a_\mu A^b_\lambda\right)\left(f^{ecd}A^{c\mu}A^{d\lambda}\right) - \frac{1}{2\alpha}(\partial_\mu A^{a\mu})^2 - \overline{C}^a R^{ac} C^c. \tag{6.26}$$

By redefining $\mathcal{L}_0 \to \mathcal{L}_0 - \overline{C}^a\partial_\mu\partial^\mu C^a - \frac{1}{2\alpha}(\partial_\mu A^{a\mu})^2$, the Lagrangian can be rewritten as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 - g\underbrace{f^{abc}\partial_\mu A^a_\lambda A^{b\mu}A^{c\lambda}}_{3G} - \underbrace{\frac{1}{4}g^2\left(f^{eab}A^a_\mu A^b_\lambda\right)\left(f^{ecd}A^{c\mu}A^{d\lambda}\right)}_{4G} - g\underbrace{\overline{C}^a\partial^\mu f^{abc}A^b_\mu C^c}_{GG} \\ &= \mathcal{L}_0 + \mathcal{L}_{int}.\end{aligned} \tag{6.27}$$

The terms are given the names $3G$, $4G$ and GG which stands for three-gluon, four-gluon and ghost-gluon vertices. The names are due to the way the terms contribute to the partition function. In terms of Feynman diagrams, they are associated with a $3G$ vertex with coupling strength g, a $4G$ vertex with strength g^2 and a ghost-gluon vertex with strength g respectively. Why these are the contributions can be seen directly from Eq. (6.27) by remembering that the corrections to the partition function is nothing but all possible contractions of the fields contained in the interacting part of the Lagrangian to different orders, n in S^n_{int}. This was shown explicitly in the previous chapter. The vertices are illustrated in Figs. 6.1 through 6.3.

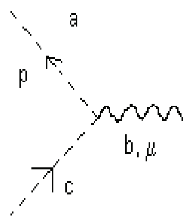

Figure 6.1: The ghost gluon vertex.

Once more one can write the partition function on the form

$$\begin{aligned} Z &= \prod_a \int DA^a D\overline{C^a} DC^a e^{S_0 + S_{int}} \\ &= \sum_{l=0}^{\infty}\prod_a \int DA^a D\overline{C^a} DC^a e^{S_0}\frac{(S_{int})^l}{l!}\end{aligned}$$

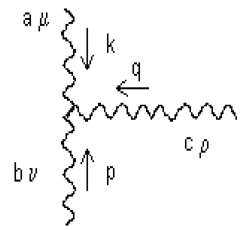

Figure 6.2: The three-gluon vertex.

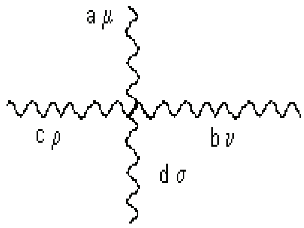

Figure 6.3: The four-gluon vertex.

$$\begin{aligned} \ln(Z) &= \ln\left[Z_0(1+\frac{1}{Z_0}\sum_{l=1}^{\infty}\prod_a \int DA^a D\overline{C^a} DC^a e^{S_0}\frac{S^l_{int}}{l!})\right] \\ &\simeq \ln(Z_0)+\sum_{l=1}^{\infty}\frac{1}{l!}\langle S^l_{int}\rangle. \end{aligned} \tag{6.28}$$

From this it can be seen that when $l = 1$ only one of the terms contained in S_{int} contribute, and that is the four-gluon vertex term. This is because it is the only term giving rise to an even number of gauge-field operators at first order in S_{int}. The four gluon vertex gives us the double-bubble diagram illustrated in Fig. 6.4. This is a four-gluon vertex with two

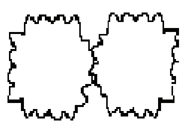

Figure 6.4: The double-bubble diagram.

gluon propagators. To find the explicit expression for this diagram, the four gluon vertex in Fig. 6.3 has a factor of

$$\begin{aligned} -ig^2\Big[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}) \\ f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}) \\ f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma})\Big], \end{aligned} \tag{6.29}$$

in the real-time formalism [4]. The gluon-propagator in the imaginary-time representation follows from the derivations in the previous chapter. It is simply

$$\langle A_n^{\lambda a'}(\vec{p})A_n^{\rho b'}(\vec{p})\rangle = \frac{1}{\int DA^b e^{-\frac{1}{2}A_\mu^b M^{\mu\nu b}A_\nu^b}}\int DA^b A_{-n}^{\rho b'}(-\vec{p})A_n(\vec{p})^{\lambda a'}e^{-\frac{1}{2}A_\mu^b M^{\mu\nu b}A_\nu^b} = M^{-1}(\omega_n,\vec{p})^{\lambda\rho}\delta^{a'b'}. \tag{6.30}$$

Since most of the calculations in this thesis are made in the real-time formalism before Wick rotating, the real-time propagator can be found directly from this by substituting $\delta^{\mu\nu} \to -i\eta^{\mu\nu}$, as was shown in QED. This substitution was first made by inspecting the real-time vs the imaginary-time photon propagator. In Feynman gauge, $\alpha = 1$ and the matrix $\mathbf{M}^a = \frac{\delta^{\mu\nu}}{P^2}$ is diagonal. This simply turns the real-time propagator into

$$-i\eta^{\lambda\rho}\delta^{a'b'}\frac{1}{p^2}. \tag{6.31}$$

To find the correct symmetryfactor, $\frac{1}{S}$, one has to recall that the meaning of this vertex and its cofactors is: when two gluons interact with each other, they interact with a coupling strength g^2 and two other gluons propagate from that point[2]. Since there are 4! ways to couple the four legs with four external particle lines, the four-gluon vertex has to be multiplied with a factor of $\frac{1}{4!}$ times the number of ways one can construct equivalent diagrams corresponding to the process under consideration. By doing this, one can see that the process where two gluons interact and two "'new"' gluons come out has a factor $\frac{1}{4!}\cdot 4! = 1$. Equivalently, the three gluon vertex has a factor of $\frac{1}{3!}$ in front. The number of equivalent diagrams corresponding to the process in Fig. 6.4 is 3!. The symmetryfactor is then found to be $\frac{1}{S} = \frac{3!}{4!} = \frac{1}{8}$. The contribution to the partition function from the process in Fig. 6.4, follows from the above

$$\begin{aligned} i\ln(Z_I)^{4G} &= -\frac{1}{8}\int\frac{d^Dp}{(2\pi)^D}\int\frac{d^Dq}{(2\pi)^D}ig^2\Big[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}) \\ &+ f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}) \\ &+ f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}\Big]\left(-i\frac{\eta_{\mu\nu}\delta^{ab}}{p^2}\right)\left(-i\frac{\eta_{\rho\sigma}\delta^{cd}}{q^2}\right) \\ &= i\frac{1}{8}g^2\underbrace{f^{abe}f^{abe}}_{d(g)C_2(G)}(N_c^2-1)N_c2D(D-1)\left(\int\frac{d^Dp}{(2\pi)^4}\frac{1}{p^2}\right)^2. \end{aligned} \tag{6.32}$$

Then by performing a Wick rotation and a discretization of p_0, and using the identity [4]

$$f^{abc}f^{abc} = d(G)C_2(G), \tag{6.33}$$

where $d(g)$ is the dimension of the adjoint representation of $SU(N_c)$, and $C_2(G) = N_c$ [4].

$$\ln(Z_I)^{4G} = -\frac{1}{4}g^2(N_c^2-1)N_cD(D-1)\left[\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2}\right]^2. \tag{6.34}$$

Large letters for the four vectors still represent Euclidian vectors.

[2]The process with the highest number of symmetries related to the vertex, defines the coupling.

Now, by returning to Eq. (6.28) one can see that when $l = 2$ one gets two more contributions. First one can consider the $3G$ term. Two of these multiplied together give an even number of A^a_μs. Since the two $3G$ terms are associated with different independent points in space, for example x_1 and x_2, they give two separated vertices. Furthermore, two fields related to the same structure constant cannot be contracted due to the antisymmetric properties of f^{abc}. From this, it follows that the $3G$ term cannot couple with the GG term to second order in S_{int}. Another reason for the $3G$-GG coupling not to contribute is due to the fact that there are no diagonal elements in the GG term resulting in terms proportional to $\overline{C^a}C^a$. Once again, the structure constant is the reason for this coupling not to contribute to the second-order correction. This means that the only contribution related to the $3G$ vertex of second order in S_{int}, is the sunset diagram in Fig. 6.5, and there are 3! ways to contract the fields to get equivalent diagrams. This means that the symmetry factor $\frac{1}{S} = \frac{3!}{3!3!} = \frac{1}{6}$. In addition there is a factor $\frac{1}{2!}$ from the expansion the exponential.

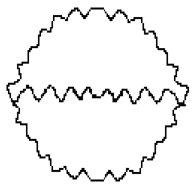

Figure 6.5: The sunset diagram.

There is a factor

$$g f^{abc}\Big[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\Big], \tag{6.35}$$

for each $3G$ vertex corresponding to Fig. 6.2 [4]. This follows from contracting the two $3G$ terms in the Faddeev-Popov Lagrangian. Furthermore, there are three gluon propagators connecting the two vertices. These can as previously implied be connected in 3! ways. The diagram is drawn using two of the vertices in Fig. 6.2 and changing the sign on q. This gives a contribution

$$\begin{aligned}
i\ln(Z_I)^{3G} &= -\frac{1}{12}\int \frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D} g f^{abc}\Big[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p+q)^\mu - \eta^{\rho\mu}(q+k)^\nu\Big]\times \\
&\quad \times g f^{a'b'c'}\Big[\eta^{\mu'\nu'}(k-p)^{\rho'} + \eta^{\nu'\rho'}(p+q)^{\mu'} - \eta^{\rho'\mu'}(q+k)^{\nu'}\Big]\times \\
&\quad \times \frac{(-i\eta_{\mu\mu'})}{k^2}\delta^{aa'}\frac{(-i\eta_{\nu\nu'})}{p^2}\delta^{bb'}\frac{(-i\eta_{\rho\rho'})}{q^2}\delta^{cc'} \\
&= -i\frac{1}{12}g^2 f^{abc}f^{abc}\int \frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{p^2q^2k^2}\Big[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p+q)^\mu - \eta^{\rho\mu}(q+k)^\nu\Big]\times \\
&\quad \Big[\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p+q)_\mu - \eta_{\rho\mu}(q+k)_\nu\Big] \\
&= -i\frac{1}{12}g^2 f^{abc}f^{abc}\int \frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^4}\frac{1}{p^2q^2k^2}\Big[D\left((k-p)^2 + (p+q)^2 + (q+k)^2\right) + \\
&\quad +2(k-p)(p+q) - 2(k-p)(q+k) - 2(p+q)(q+k)\Big],
\end{aligned} \tag{6.36}$$

where $q = (k+p)$. Moreover, the relation $2kp = (p+k)^2 - p^2 - k^2$ will be exploited frequently throughout the following calculation

$$\begin{aligned} i\ln(Z_I)^{3G} &= -i\frac{1}{12}g^2 f^{abc}f^{abc}\int \frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{p^2(p+k)^2k^2}\Big[D\left((k-p)^2+(p+q)^2+(q+k)^2\right) \\ &\quad -2(p+q)^2-(k-p)(2k+p)\Big] \\ &= -i\frac{1}{12}g^2 f^{abc}f^{abc}\int \frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{p^2(p+k)^2k^2}\Big[3D\left((p+k)^2+p^2+k^2\right) \\ &\quad -3(p+k)^2-3p^2-3k^2\Big] \\ &= -i\frac{1}{12}g^2 f^{abc}f^{abc}\int \frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{p^2(p+k)^2k^2}\Big[3(D-1)\left((p+k)^2+p^2+k^2\right)\Big]. \end{aligned} \tag{6.37}$$

By performing a Wick rotation, this becomes

$$i\ln(Z_I)^{3G} = i\frac{1}{12}g^2 f^{abc}f^{abc}\sum\!\!\!\!\!\!\int_{P,K}\left[3(D-1)\left(\frac{1}{(P+K)^2P^2}+\frac{1}{(P+K)^2K^2}+\frac{1}{K^2P^2}\right)\right]. \tag{6.38}$$

Now, by making the substitutions $K' = (P+K)^2$, $P' = (P+K)^2$, in the first and second integral respectively, this becomes

$$i\ln(Z_I)^{3G} = i\frac{1}{12}g^2 f^{abc}f^{abc}3(D-1)\left(\sum\!\!\!\!\!\!\int_{P,K'}\left[\frac{1}{K'^2P^2}\right]+\sum\!\!\!\!\!\!\int_{P',K}\left[\frac{1}{P'^2K^2}\right]+\sum\!\!\!\!\!\!\int_{P,K}\left[\frac{1}{K^2P^2}\right]\right). \tag{6.39}$$

Since all the parameters are bosonic, the resulting K', P' are also bosonic. From Eq. (6.33), the final expression becomes

$$\ln(Z_I)^{3G} = \frac{3}{4}g^2N_c(N_c^2-1)(D-1)\left[\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\right]^2. \tag{6.40}$$

Now, one more diagram which is topologically distinct from the two others, remain when neglecting all terms of higher order than $\langle S_{int}^2\rangle$. This contribution is also a second order contribution in S_{int}, and it comes from the GG term. As previously implied, the only contribution from this term comes from the double-GG vertex diagram shown in Fig. 6.6. In QED, it was shown that the sunset diagram recieved a negative cofactor. This was due to the Grassmann nature of the fermion fields. The ghost fields are also Grassmann fields, hence the diagram in Fig. 6.6 has a negative cofactor. There is only one way to connect the two vertices to one-another thus the symmetryfactor is 1. Since Eq. (6.28) gives a factor of $\frac{1}{2!}$ in front just as the previous sunset diagram, the cofactor must be $-\frac{1}{2!}$. The ghost-gluon vertex in Fig. 6.1 gives a factor [4]

$$-gf^{abc}p^\mu. \tag{6.41}$$

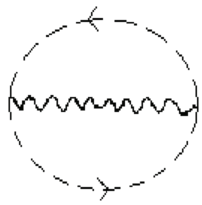

Figure 6.6: The ghost-gluon sunset diagram.

Since one has two of these vertices connected with two ghost propagators and a virtual gluon propagating from one vertex to the other, the contribution becomes

$$\begin{aligned}
i\ln(Z_I)^{GG} &= -g^2\frac{1}{2}f^{abc}f^{c'b'a'}\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}p^\mu k^\nu\frac{i\delta^{aa'}}{p^2}\frac{-i\eta_{\mu\nu}\delta^{bb'}}{(k-p)^2}\frac{i\delta^{cc'}}{k^2} \\
&= -ig^2\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{2}f^{abc}f^{cba}\frac{p\cdot k}{p^2(k-p)^2k^2} \\
&= i\frac{1}{2}g^2f^{abc}f^{abc}\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{p^2(k-p)^2k^2}\left[-\frac{1}{2}(p-k)^2+\frac{p^2+k^2}{2}\right]
\end{aligned} \tag{6.42}$$

Again, by performing a Wick rotation and discretisize k_0, p_0 as before, this becomes

$$i\ln(Z_I)^{GG} = -i\frac{1}{4}g^2f^{abc}f^{abc}\sum\!\!\!\!\!\!\!\int_{P,K}\left[-\frac{1}{P^2K^2}+\frac{1}{K^2(K-P)^2}+\frac{1}{P^2(K-P)^2}\right]. \tag{6.43}$$

Now, by choosing the proper substitutions, this becomes

$$\begin{aligned}
i\ln(Z_I)^{GG} &= -i\frac{1}{4}g^2f^{abc}f^{abc}\left[\sum\!\!\!\!\!\!\!\int_P\frac{1}{P^2}\right]^2 \\
\ln(Z_I)^{GG} &= -\frac{1}{4}g^2(N_c^2-1)N_c\left[\sum\!\!\!\!\!\!\!\int_P\frac{1}{P^2}\right]^2
\end{aligned} \tag{6.44}$$

Finally, the partition function can be found. By taking into account the corrections $\ln(Z_I)^{4G}, \ln(Z_I)^{3G}, \ln(Z_I)^{GG}$, multiplying them with βV for reasons that were worked out in QED[3] and adopting the zeroth order contribution from Eq. (6.25), the final partition function after neglecting all terms higher than $\mathcal{O}(g^2)$ becomes

$$\begin{aligned}
\ln(Z) &= -V(N_c^2-1)\sum_n\int\frac{d^3P}{(2\pi)^3}\ln(P^2)-g^2\beta V\Big[\frac{1}{4}(N_c^2-1)N_c-\frac{3}{4}N_c(N_c^2-1)(D-1)+ \\
&+ \frac{1}{4}(N_c^2-1)N_cD(D-1)\Big]\left[\sum\!\!\!\!\!\!\!\int_P\frac{1}{P^2}\right]^2 \\
\frac{\ln(Z)}{V} &= 16\frac{\pi^2}{90}T^3-\frac{1}{6}g^2T^3,
\end{aligned} \tag{6.45}$$

where the last line has been obtained by letting $D\to 4$ and $N_c = 3$. This is on the expected form. There are eight gluons, each having two independent spin polarizations, hence their

[3]This can also be seen from dimensional analysis. In QED it was shown that this came from the normalization constants of the field expansions, and the periodic constraints of the spatial momentum.

zeroth order contribution are 16 times the ideal Bose gas contribution [2]. Due to interactions between the gluons, a negative contribution is given by the coupling-dependent term.

6.2 The complete two-loop partition function

By inserting the thermal mass into the theory in the same way as was done Eq. (5.53) and substituting $A^0 \to A^{a0}$, one can see that each M^{a00} recieves a correction due to the mass insertion.

The polarization tensor is somehow different in pure-glue since the gluons interact with both eachother and the ghosts. The diagrams contributing to the one-loop self-energy, thus the polarization tensor are the Feynman diagrams in Fig. 6.7. Anyway, as stated in the introduction to this chapter, this contribution will not be found explicitly.

After inserting the thermal mass, the gluon propagator becomes modified. This again, changes the contributions from the Feynman diagrams due to the presence of dressed propagators. All the two-loop diagrams have to be recalculated. In addition a one-loop diagram will appear due to the mass insertion. The two-loop diagrams will be Taylor expanded in the thermal mass since it is assumed to be proportional to the coupling constant. Moreover, all terms higher than the order of g^3 will be neglected. Diagrams higher than two-loop will also be neglected. The partition function can then be written on the form

$$\ln(Z) = 16V\frac{\pi^2}{90}T^3 + \frac{2V}{3\pi}m^3 + g^2\Delta_1 + g^3\Delta_2, \tag{6.46}$$

where all terms proportional to g^3 are due to the thermal mass in the dressed propagator. First, the one-loop diagram with dressed propagator in Fig. 6.8 will be found. Since one now has become aquainted with the imaginary-time formalism, all the following calculations will be made in Euclidean space just as in the last section of the previous chapter. The one loop diagram is simply

$$W^{1G} = \frac{m^2}{2}\langle A_0^a A_0^a \delta_{P_0 0}\rangle = \frac{m^2}{2}\delta^{aa}\sum\!\!\!\!\!\!\int_P \frac{\delta_{P_0,0}}{P^2+m^2} = -\frac{m^3 T}{\pi}. \tag{6.47}$$

The last line followed from dimensional regularization. In fact, one can find it by defining the integral

$$\begin{aligned} {J'_N}^D &= \int \frac{d^D P}{(2\pi)^D}\frac{1}{P+m^2} = \frac{1}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(N-\frac{D}{2})}{\Gamma(N)}(m^2)^{\frac{d}{2}-N} \\ {J'_{N-1}}^D &= m^2\frac{N-1}{N-\frac{D}{2}-1}{J'_N}^D \\ \sum\!\!\!\!\!\!\int_P \frac{\delta_{P_0,0}}{P^2+m^2} &= T\int\frac{d^3P}{(2\pi)^3}\frac{1}{\vec{P}^2+m^2} = -T\lim_{D\to 4} m^2 2{J'_2}^{D-1} = -\frac{mT}{(4\pi)}. \end{aligned} \tag{6.48}$$

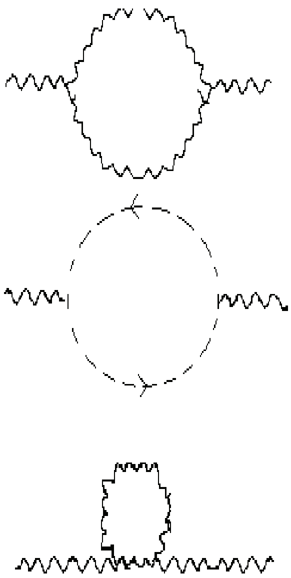

Figure 6.7: The one-loop contributions to the polarization tensor in a gluon plasma.

The next diagram is the two loop diagram related to the four-gluon vertex, with dressed propagators. It is

$$\begin{aligned} W^{4G} &= -\frac{g^2}{8} f^{ace} f^{ace} \sum\!\!\!\!\!\!\int_{P,Q} (2\delta^{\mu\nu}\delta^{\rho\sigma} - 2\delta^{\mu\sigma}\delta^{\nu\rho}) \\ &\quad \times \left[\frac{\delta^{\mu 0}\delta^{\nu 0}}{P^2 + m^2\delta_{P_0 0}} + \frac{\delta^{\mu i}\delta^{\nu i}}{P^2}\right] \left[\frac{\delta^{\rho 0}\delta^{\sigma 0}}{Q^2 + m^2\delta_{Q_0 0}} + \frac{\delta^{\rho j}\delta^{\sigma j}}{Q^2}\right] \\ &= -\frac{g^2}{4} f^{ace} f^{ace} \sum\!\!\!\!\!\!\int_{P,Q} 2\frac{(D-1)}{Q^2(P^2 + m^2\delta_{P_0 0})} + \frac{(D-1)(D-2)}{P^2 Q^2}. \end{aligned} \tag{6.49}$$

The integral

$$\sum\!\!\!\!\!\!\int_{P,Q} 2\frac{(D-1)}{Q^2(P^2 + m^2\delta_{P_0 0})} \tag{6.50}$$

can be rewritten, since only the $P_0 = 0$ mode recieves a mass correction. Due to dimentional regularization, all integrals $\int \frac{d^3P}{\vec{P}^2} = 0$. This is also obtainable from Eq. (6.48). This allows one to rewrite Eq. (6.50) as follows

$$\sum\!\!\!\!\!\!\int_{P,Q} 2\frac{(D-1)}{Q^2(P^2 + m^2\delta_{P_0 0})} = \sum\!\!\!\!\!\!\int_{P,Q} 2\frac{(D-1)}{Q^2 P^2} + 2\frac{(D-1)\delta_{P_0,0}}{Q^2(P^2 + m^2)}. \tag{6.51}$$

Now, this two-loop correction becomes

$$W^{4G} = -\frac{g^2}{4} f^{ace} f^{ace} D(D-1) \left[\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\right]^2 - \frac{g^2}{2} f^{ace} f^{ace} (D-1) \sum\!\!\!\!\!\!\int_{P,Q} \frac{\delta_{P_0,0}}{Q^2(P^2 + m^2)}. \tag{6.52}$$

The next contribution comes from the ghost-gluon sunset diagram with a dressed gluon

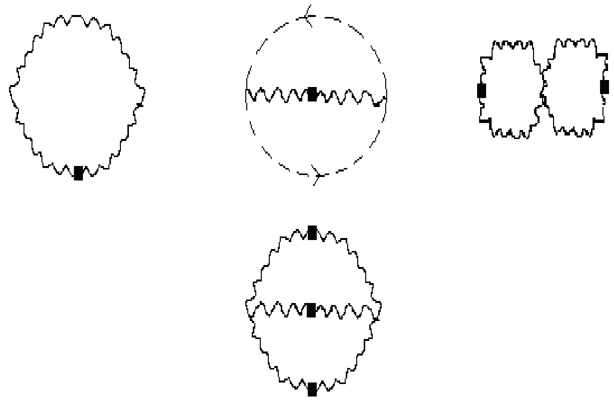

Figure 6.8: The resummed diagrams contributing to the partition function, thus the pressure at two-loop or lower. Resummed propagators are again denoted by a hard dark square just as in QED.

propagator. This correction can be written as follows

$$W^{GG} = -\frac{1}{2}g^2 f^{abc}f^{abc}\sum\!\!\!\!\!\!\!\int_{PQ}\frac{P^\mu(P-Q)^{\mu'}}{P^2(P-Q)^2}\left[\frac{\delta^{\mu 0}\delta^{\mu' 0}}{Q^2+m^2\delta_{Q_0 0}}+\frac{\delta^{\mu j}\delta^{\mu' j}}{Q^2}\right]$$
$$= -\frac{1}{2}g^2 f^{abc}f^{abc}\sum\!\!\!\!\!\!\!\int_{P,Q}\frac{P^0(P-Q)^0}{(Q^2+m^2\delta_{Q_0 0})P^2(P-Q)^2}+\frac{\vec{P}(\vec{P}-\vec{Q})}{Q^2P^2(P-Q)^2}. \quad (6.53)$$

Again due to dimentional regularization, and the fact that the integral over $\frac{1}{\vec{Q}^2+m^2}$ when $Q_0=0$ gives most of its contribution when $\vec{Q}\sim m$, the first term of the integral above can be Taylor expanded in $\vec{Q}$ for $P_0\neq 0$. This follows from the considerations below Eq. (5.59). If both $P_0=0$ and $Q_0=0$, this term vanishes. This means that the first integral above can be expanded as follows

$$\sum\!\!\!\!\!\!\!\int_{P,Q}\frac{P^0(P-Q)^0}{(Q^2+m^2\delta_{Q_0 0})P^2(P-Q)^2}=\sum\!\!\!\!\!\!\!\int_{P,Q}\frac{(P^0)^2\delta_{Q_0,0}}{(Q^2+m^2)P^4}+\sum\!\!\!\!\!\!\!\int_{P,Q}\frac{P^0(P-Q)^0}{Q^2P^2(P-Q)^2}+\mathcal{O}(g^2). \quad (6.54)$$

Furthermore the sum integral

$$\sum\!\!\!\!\!\!\!\int_{P}\frac{\vec{P}^2}{P^4}=\frac{3}{2}\sum\!\!\!\!\!\!\!\int_{P}\frac{1}{P^2}, \quad (6.55)$$

is found from integration by parts. The contribution from the ghost-gluon sunset diagram neglecting terms of the order $\mathcal{O}(g^4)$ and higher, is

$$W^{GG} = -\frac{1}{2}g^2 f^{abc}f^{abc}\sum\!\!\!\!\!\!\!\int_{P,Q}\left[\frac{(P^0)^2\delta_{Q_0,0}}{(Q^2+m^2)P^4}+\frac{P(P-Q)}{Q^2P^2(P-Q)^2}+\mathcal{O}(g^2)\right]$$
$$= -\frac{1}{4}g^2 f^{abc}f^{abc}\left[\sum\!\!\!\!\!\!\!\int_{P,Q}\frac{1}{P^2}\right]^2+\frac{1}{4}g^2 f^{abc}f^{abc}\sum\!\!\!\!\!\!\!\int_{P,Q}\frac{\delta_{Q_0,0}}{P^2(Q^2+m^2)}. \quad (6.56)$$

The only diagram remaining now, is the slightly more complicated gluon-gluon sunset diagram. It is

$$\begin{aligned} W^{3G} &= \frac{1}{12} g^2 f^{abc} f^{abc} \sum\!\!\!\!\!\!\!\int_{K,P} \left(\delta^{\mu\nu}(K-P)^{\rho} + \delta^{\nu\rho}(2P+K)^{\mu} - \delta^{\rho\mu}(2K+P)^{\nu}\right) \\ &\times \left(\delta^{\mu'\nu'}(K-P)^{\rho'} + \delta^{\nu'\rho'}(2P+K)^{\mu'} - \delta^{\rho'\mu'}(2K+P)^{\nu'}\right) \\ &\times \left[\frac{\delta^{\mu 0}\delta^{\mu' 0}}{K^2 + m^2\delta_{K_0 0}} + \frac{\delta^{\mu i}\delta^{\mu' i}}{K^2}\right] \\ &\times \left[\frac{\delta^{\nu 0}\delta^{\nu' 0}}{P^2 + m^2\delta_{P_0 0}} + \frac{\delta^{\nu j}\delta^{\nu' j}}{P^2}\right] \\ &\times \left[\frac{\delta^{\rho 0}\delta^{\rho' 0}}{(P+K)^2 + m^2\delta_{(P+K)_0 0}} + \frac{\delta^{\rho l}\delta^{\rho' l}}{(P+K)^2}\right]. \end{aligned} \tag{6.57}$$

By using the Eqs. (6.48), (6.51), (6.55) and (6.54) and neglecting integrals such as

$$g^2 \sum\!\!\!\!\!\!\!\int_{K,P} \frac{\delta_{K_0,0}\delta_{P_0,0}}{(P^2+m^2)(K^2+m^2)} \tag{6.58}$$

since they contribute to corrections of order $\mathcal{O}(g^4)$, the correction becomes

$$\begin{aligned} W^{3G} &= \frac{1}{4} g^2 f^{abc} f^{abc} \sum\!\!\!\!\!\!\!\int_{K,P} \left[\frac{(K-P)^2 + (2P+K)^2 + (2K+P)^2}{P^2K^2(K+P)^2}\right] \\ &+ \frac{1}{12} g^2 f^{abc} f^{abc} \sum\!\!\!\!\!\!\!\int_{K,P} \left[\frac{6\vec{P}^2\delta_{K_0,0}}{P^4(K^2+m^2)} + \frac{36(P^0)^2\delta_{K_0,0}}{P^4(K^2+m^2)}\right] \\ &= \frac{1}{4} g^2 f^{abc} f^{abc} \sum\!\!\!\!\!\!\!\int_{K,P} \left[\frac{(K-P)^2 + (2P+K)^2 + (2K+P)^2}{P^2K^2(K+P)^2}\right] \\ &+ \frac{1}{12} g^2 f^{abc} f^{abc} \sum\!\!\!\!\!\!\!\int_{K,P} \left[\frac{36\delta_{K_0,0}}{P^2(K^2+m^2)} - 30\frac{3}{2}\frac{\delta_{K_0,0}}{P^2(K^2+m^2)}\right] \\ &= \frac{3}{4} g^2 f^{abc} f^{abc} 3 \left[\sum\!\!\!\!\!\!\!\int_{P} \frac{1}{P^2}\right]^2 - \frac{3}{4} g^2 f^{abc} f^{abc} \sum\!\!\!\!\!\!\!\int_{K,P} \frac{\delta_{K_0,0}}{P^2(K^2+m^2)}. \end{aligned} \tag{6.59}$$

Finally by multiplying W^{1G}, W^{3G}, W^{4G} and W^{GG} with βV, let $D \to 4$, $N_c = 3$, using Eq. (6.48) and insert them into Eq. (6.46), the total two loop partition function can be found to the order of g^3 in the coupling. It becomes

$$\boxed{\frac{\ln(Z)}{V} = 16\frac{\pi^2}{90}T^3 - \frac{1}{3\pi}m^3 + g^2\frac{mT^2}{\pi} - \frac{g^2}{6}T^3,} \tag{6.60}$$

where m is defined in Eq. (6.1).

Chapter 7

Pressure in pure $SU(3)$ gauge theory

7.1 The complete two-loop pressure

The pressure in a pure $SU(3)$ gauge theory can be obtained directly from the pure-glue partition function in the previous chapter. From [2], the correct expression for the pressure in a gas with zero chemical potential[1] is

$$P = \frac{T}{V}\ln(Z), \tag{7.1}$$

where Z was obtained in Eq. (6.46). This gives the pressure as a function of the temperature and the thermal mass stated in Eq. (6.1). It is given by

$$P = 16\frac{\pi^2}{90}T^4 - \frac{1}{3\pi}m_0^3 g^3 T^4 - g^2\frac{1}{6}T^4 + g^3\frac{m_0}{\pi}T^4. \tag{7.2}$$

7.2 A brief recapitulation

The gluons are as mentioned in the previous chapter, the mediating vector bosons of the strong force. These particles behave quite differently from their QED "lookalikes" the photons. This is mostly due to the fact that the gauge fields carry an additional index which in this thesis is referred to as the colour index. This extra index makes the gluons able to interact with each other unlike the photon. It represents the gluon colour state. Since they are charged, they are also confined due to spatial confinement. The gluons form a colour octet[2], thus they must be confined at low temperatures. This means that they are trapped in what the MIT bag model describes as small bags representing the particles in the standard model. However, at very high temperatures, asymptotic freedom is a fact. It means that the gluons can be said to propagate as nearly freely propagating particles (they still interact with each other) just as the photons are at all temperatures. This phase is called the deconfinement phase. In this phase, at sufficiently high temperatures, perturbation theory is applicable since the coupling goes to zero.

[1] The gauge fields are real vector fields, thus the gluons are their own anti-particles

[2] The concept of colour confinement requires that only colour singlets can exist as free particles.

Now, to discuss some of these ideas in context of Eq. (7.2), one can see that the pressure should go as an ideal bose gas with a degeneracy of 16 at high temperatures since the coupling goes to zero. The 16 degrees of freedom comes from the eight different gluons with two different polarization states.

In context of the above, it is much more convenient to introduce a relative pressure, i.e the pressure divided by the ideal-gas pressure, to see how it behaves relatively. By dividing Eq. (7.2), by the ideal gas contribution, one obtains

$$\frac{P}{P_0} = 1 + \frac{15}{16\pi^2}\left[\left(\frac{6m_0 - 3m_0^3}{\pi}\right)g^3 - g^2\right]. \tag{7.3}$$

As stated above, the pressure should decrease relative to the ideal bose gas pressure as the temperature decreases. This means that the coupling-constant dependent term in Eq. (7.3) should give a negative contribution when the temperature decreases. In fact, with the proper values of the coupling constant, Eq. (7.3) should reproduce the graph in Fig. 7.1, given the substitution $T \to \frac{T}{T_c}$. The critical temperature T_c is defined to be the temperature where one finds the deconfinement phase transition. It was found to be 150 MeV at the time the lattice data was made. When taking a closer look at Fig. 7.1 one can see that the slope has a little kink just below $T = T_c$. This indicates that there are some non-perturbative effects affecting the system close to T_c, which again result in a phase transition between the confined and the deconfined phase.

To fit the expression in Eq. (7.3) to the lattice data, the coupling must be a function of $\frac{T}{T_c}$. If it was not, $\frac{P}{P_0}$ would be temperature independent and the whole theory would've broken down.

In this quasiparticle description, Eq. (7.3) is therefore modified by introducing a temperature-dependent coupling. This means that the particles in this model interact with a temperature-dependent coupling. One then obtains the expression for the relative pressure in a gas of pure-glue,

$$\frac{P}{P_0} = 1 + \frac{15}{16\pi^2}\left[\frac{(6m_0 - 3m_0^3)}{\pi}g^3\left(\frac{T}{T_c}\right) - g^2\left(\frac{T}{T_c}\right)\right]. \tag{7.4}$$

7.3 Data fitting

Now, MatLab is used to fit the running coupling as a function of $\left(\frac{T}{T_c}\right)$ for a well suited value of m_0. To make the fitting as easy as possible to handle, a new scaled coupling $g' = \frac{1}{\sqrt{24}}g$, will be introduced together with the constant $m_0 = \frac{1}{60}$. This constant is chosen at random within the region where it allows the coupling-dependent term to take on negative values. Now, Eq. (7.4) becomes

$$\frac{P}{P_0} = 1 + \frac{15}{4\pi^2}\left[\frac{3\sqrt{6}}{5\pi}\frac{7197}{3600}g'^3\left(\frac{T}{T_c}\right) - 6g'^2\left(\frac{T}{T_c}\right)\right]. \tag{7.5}$$

This function is fitted to the lattice data given by Fig. 7.1, and the result for the running coupling $g'\left(\frac{T}{T_c}\right)$ is given by Fig. 7.2.

The error, $\frac{\Delta P}{P}$ at each value of $\frac{T}{T_c}$ is found by inserting the coupling in Fig. 7.2 into Eq. (7.5), and compare it to the pressure given by the lattice data in Fig. 7.1 at each point. The result is given by Fig. 7.3. From Fig. 7.3 one can see that the accuracy is high in the deconfined phase, i.e above $T = T_c$. However, the fitted pressure cannot reproduce the lattice data below the critical temperature.

Now, from Fig. 7.2 one can see that the coupling decreases as the temperature increases just as predicted.

The coupling in Fig. 7.2 is now fitted to an analytic expression, using the ansatz

$$\frac{1}{g'} = c + a\ln\left(b\frac{T}{T_c}\right), \tag{7.6}$$

where the constants a,b and c are found by using a non-linear least square fit. They are found to be

$$\begin{aligned} a &\simeq \ 1,9557 \\ b &\simeq 0,95141 \\ c &\simeq 1,4250, \end{aligned} \tag{7.7}$$

and the plot in Fig. 7.4 shows the fit to the numerically estimated inverse coupling given by Eq. (7.6).

Now, the coupling calculated from lattice data given by Fig. 7.2, is compared to the coupling given by Eqs. (7.6) and (7.7) in Fig. 7.5 for $T \geq T_c$.

By remembering that the relative pressure is defined by Eq. (7.5), and inserting Eq. (7.6) given the constants in Eq. (7.7), the pressure can be found. It is plotted for $T \geq T_c$ in Fig. 7.6.

An analytic expression for the pressure is now obtained. This function is then used to give a prediction of how the pressure behaves at higher temperatures than $T = 4.5T_c$. It should increase towards the ideal gas or $\frac{P}{P_0} = 1$. This extrapolation gives the plot in Fig. 7.7 for $T \geq T_c$, which clearly indicates that the pressure increases towards the ideal gas pressure from below as the temperature increases.

The coupling in Eq. (7.2) is given by

$$g = \sqrt{24}g'\left(\frac{T}{T_c}\right). \tag{7.8}$$

By using this, the Eqs. (7.6) and (7.7), can be used to find an estimate of the temperature region in which g can be treated as a perturbation parameter. The running "finestructure" is found from

$$\alpha_g = \frac{g^2\left(\frac{T}{T_c}\right)}{4\pi}. \tag{7.9}$$

This is compared to the finestructure constant in QED, in Fig. 7.8. From Fig. 7.8, one can see that for temperatures around $50T_c$, the strong coupling $\alpha_g \simeq 3\alpha_e = \frac{3}{137}$. The QED coupling is also temperature dependent as was shown in chapter three. It was shown that it was momentum dependent, hence it will also be temperature dependent. It is only kept constant here to see when the strong coupling reaches low enough values for perturbation theory to become applicable. One knows that low-energy QED can be treated perturbatively, hence the coupling g can be treated as a perturbation parameter in pure-glue at temperatures around $50T_c$. The slope of the coupling levels out around $25T_c$. This means that one could probably apply perturbation theory to pure-glue at these temperatures. However, to be certain that the coupling is small enough to give satisfying results at low-order perturbation theory, temperatures $T \geq 50T_c$ would be recommended by this model.

7.4 Figures

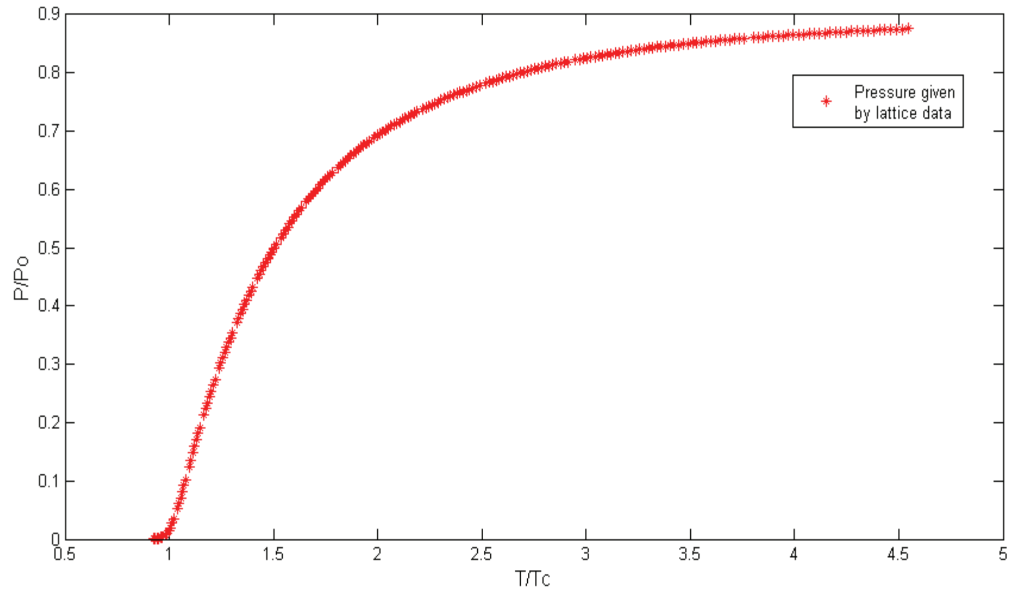

Figure 7.1: Lattice data for pressure vs ideal bose-gas pressure in pure glue.

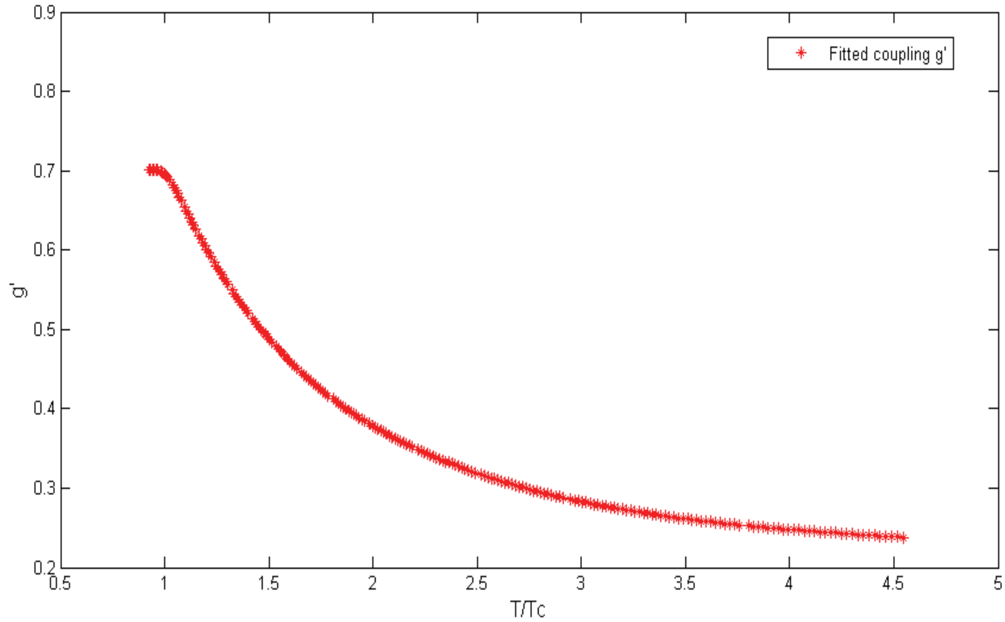

Figure 7.2: Fitted coupling in pure glue as a function of temperature.

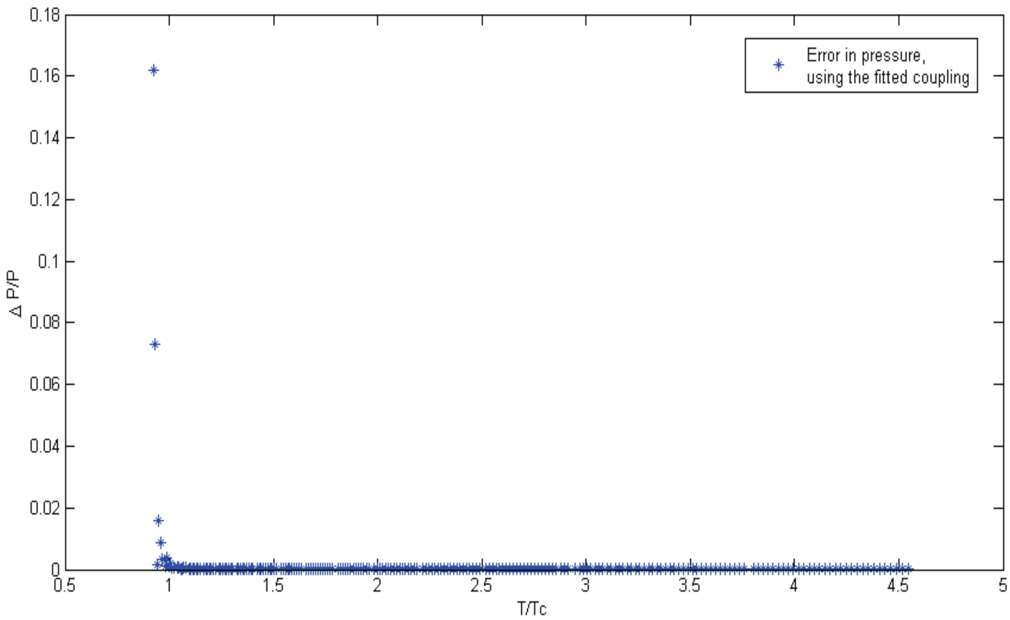

Figure 7.3: Estimated error in fitted pressure at each lattice point.

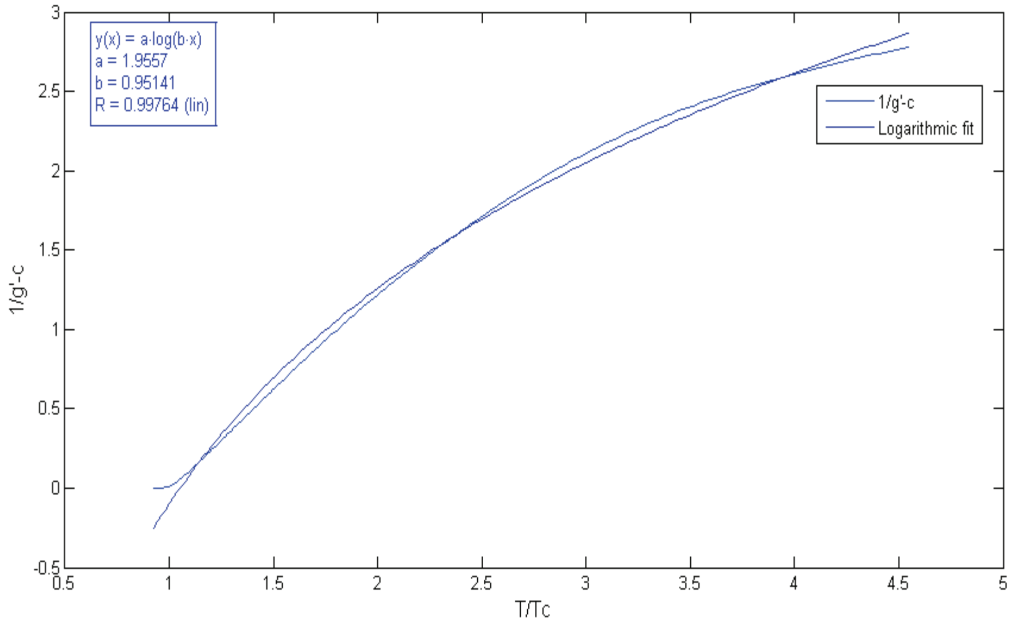

Figure 7.4: The inverse coupling (minus c) and its logarithmic best fit.

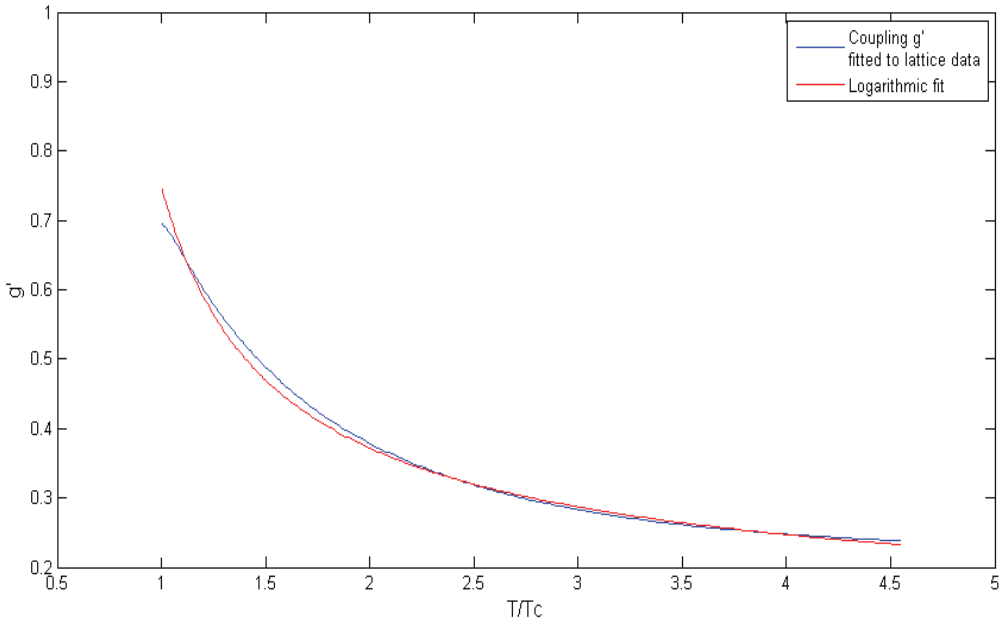

Figure 7.5: The coupling calculated from lattice data, and the estimated logarithmic function in Eq. (7.6).

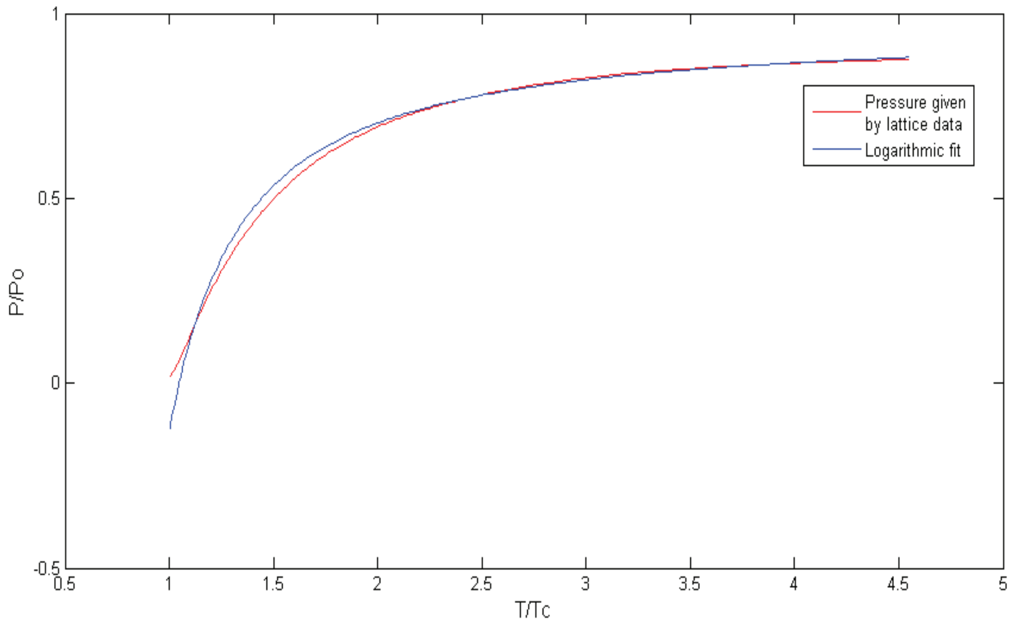

Figure 7.6: Pressure from the lattice data, and the pressure when inserting Eq. (7.6) into Eq. (7.5).

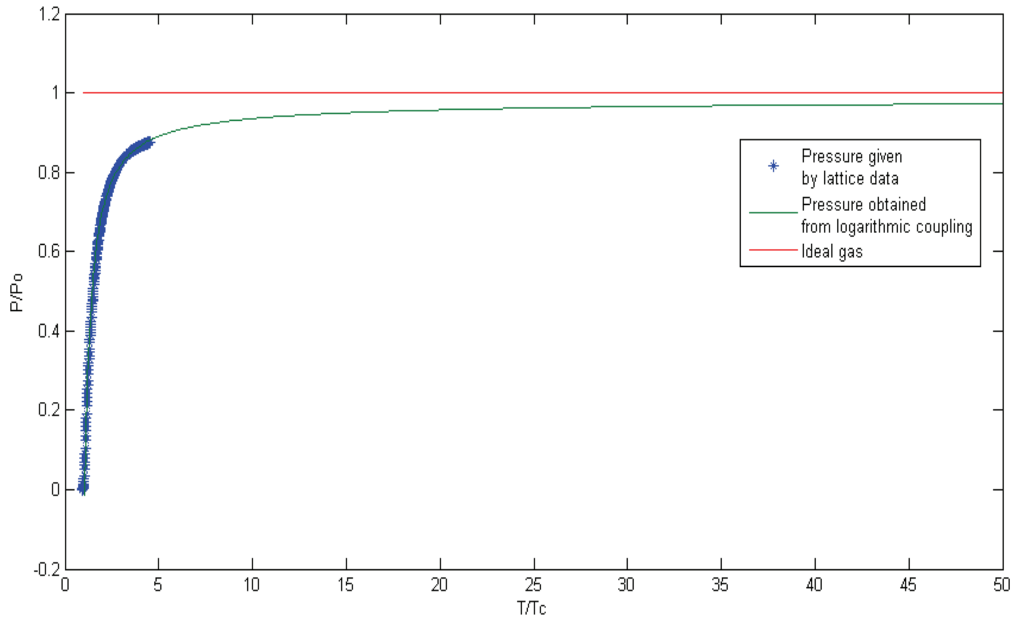

Figure 7.7: Pressure for high T by inserting Eq. (7.6) into (7.5).

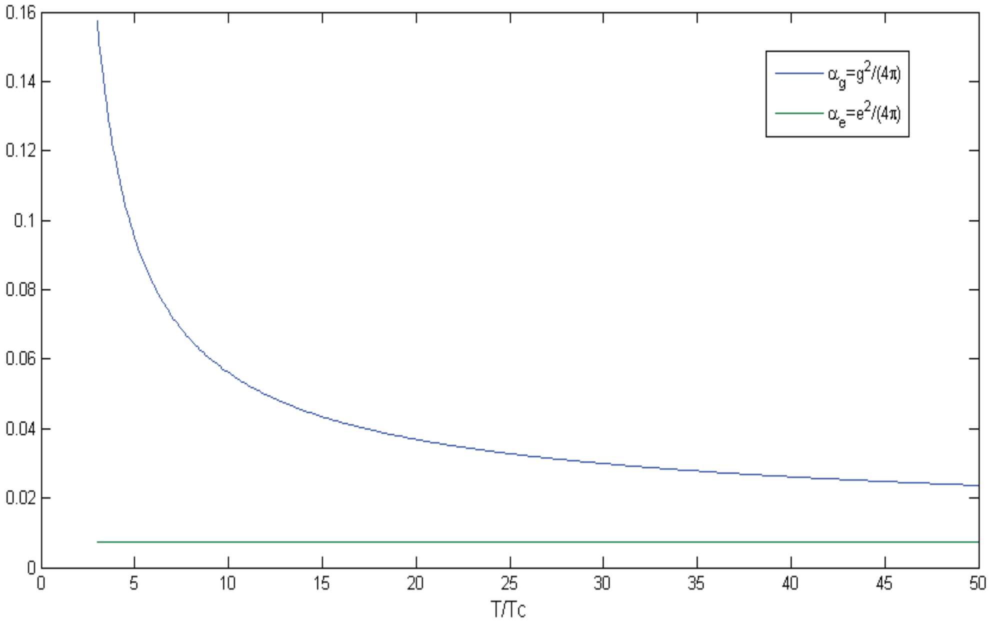

Figure 7.8: The finestructure constant, $\alpha_g = (\frac{(\sqrt{24}g')^2}{4\pi})$, versus the finestructure constant in QED.

Chapter 8

Summary and conclusion

In my project report [2], ordinary quantum statistics was used to obtain the equations of state for ideal Fermi and Bose-gases. In this formalism it is very hard to implement corrections due to interactions, and they simply do not seem to follow naturally from the theory itself. A way to include weak couplings between the particles was in chapter one found to be to include thermal masses generated by medium effects.

In the chapters four and five, I gave a description of a more modern approach to the area of quantum statistics, namely the functional representation of the partition function. It was obtained by starting with the diagonal transition amplitude, comparing it to the expression for the partition function, and making the substitutions necessary for them to be equivalent. From this approach, corrections from the ideal Bose and Fermi gases followed naturally, if not beautifully, by expansions in terms of Feynman diagrams. It must be the ultimate way to calculate thermodynamical quantities in quantum systems at weak coupling.

In chapter three, it was found that the longitudinal photons in QED acquire effective thermal masses as they propagate through a medium. This was consistent with the introduction to the first chapter which was about quasiparticle models. The rest of the first chapter, which was a summary of the article found in reference [1], implied that systems could be described in terms of these effective particles. It was also in chapter three found that the coupling in QED was momentum dependent due to vacuum polarization at zero temperature.

In the aftermath of chapter one through four, both a two-loop approximation to the pressure in high temperature QED and pure-glue (QCD without fermions) of order g^3 in the coupling were obtained in the last part of the chapters five and six. The latter coincides with the corrections listed in reference [13]. The two-loop approximation was in QED found to be gauge invariant within the class of Lorentz gauges. The pressure in the two gases were obtained by using effective propagators, meaning the propagators of the quasiparticles. They were found to be

$$P_{QED} = \frac{\pi^2}{45}T^4\left(1 + 2N_f\frac{7}{8}\right) + \frac{1}{12\pi}m^3 - \frac{5N_f}{288}e^2T^4, \tag{8.1}$$

and

$$P_{PG} = 16\frac{\pi^2}{90}T^4 - \frac{1}{3\pi}m^3T + g^2\frac{mT^3}{\pi} - \frac{g^2}{6}T^4, \tag{8.2}$$

where the masses denoted by m in P_{QED} and P_{PG} are the thermal masses of the longitudinal photons and gluons, respectively. The reason for why they acquire such effective masses was found to be due to medium effects.

The pressure in QED is simply the ideal gas contribution plus some small corrections due to interactions between the fermions and the photons.

In pure-glue, the pressure was used to fit a running temperature-dependent coupling to lattice data. This approach to data fitting of the pressure has, to the best of my knowledge, never been done before. It gave tremendously good results. This fitted pressure was again extrapolated to higher temperatures than $4.5T_c$ which was the highest values for the lattice data. It was found that the pressure in such a gas slowly increased towards the ideal gas pressure, as one would expect due to asymptotic freedom. It showed that weak- coupling expansion of the partition function could not successfully be applied to pure-glue below approximately $50T_c = 7500MeV$. The critical temperature at the time the lattice data was computed was 150MeV.

In the region below $4.5T_c$, the fitted pressure fitted nicely to the lattice data except from close to and below T_c, i.e the fitted pressure could not describe the deconfinement phase-transition.

Unfortunately, as i only had lattice data for the pressure, the data fitting is not made in a thermodynamically consistent way as the models described in chapter one.

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