2 Symmetries

Symmetries are central to our understanding of particle physics. They lead to conservation laws and allow us to see order principles in the "particle zoo" of meson and baryon states. Even more importantly, symmetry principles are also crucial to describe the interactions of particles.

We can divide symmetries into space-time symmetries and internal symmetries that mix a certain group of fields. Both types of symmetries can be discrete or continuous. Another important distinction is between global and local symmetries: In the former case, we perform at each space-time point the same symmetry transformation, e.g. a rotation by a fixed angle α. In the case of a local symmetry, the rotation angle could change from point to point, $\alpha = \alpha(x^\mu)$. A list of symmeties is shown in Table 2.1.

The symmetries of Minkowski space-time require that field equations respect Lorentz and translation symmetry. The interactions carried by spin-1 bosons are characterised by local symmetries: electromagnetic interactions by the symmetry group U(1), weak interactions by SU(2), and strong interactions by SU(3). Often the standard model (SM) of particle physics is therefore summarised as a U(1)$\otimes$ SU(2)$\otimes$ SU(3) gauge theory. We come latter back to this local symmetries, considering now global continuous ones.

2.1 Symmetries in Quantum Mechanics

Emmy Noether showed in 1917 that every global continuous symmetry of a system described by a Lagrangian leads classically to a (local) conservation law. Thus the experimental observation of a conserved quantity informs us that the interaction has to satisfy the corresponding symmetry. This helps us to constrain possible interaction terms.

In quantum mechanics, the relation between symmetries and conservation laws is similar:

conserved quantity	symmetry	non-observable
continuous		
energy E	time translation $t \to t + t_0$	absolute time
momentum $\boldsymbol{p}$	space translation $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{x}_0$	absolute space
angular momentum $\boldsymbol{L}$	rotation $x_i \to R_{ij}(\alpha) x_j$	absolute orienation
cm velocity	boost $x^\mu \to \Lambda^\mu{}_\nu(\eta) x^\nu$	absolute velocity
charges q_a	gauge transformation $\phi_i \to \mathrm{e}^{\mathrm{i}\vartheta^a T^a} \phi_i$	relative phases
discrete		
parity	spatial inversion	distinction left-right helix
time inversion invariance	time inversion	time arrow
charge parity	charge conjugation	absolute sign of charge

Table 2.1: Conservation laws, underlying global symmetries and corresponding "non-observables".

Let us assume that the two set of states $|\psi_n\rangle$ and $|\psi'_n\rangle = U|\psi_n\rangle$ predict the same measurements. Then

$$|\langle\psi_n|\,\psi_m\rangle|^2 = |\langle\psi'_n|\,\psi'_m\rangle|^2 = |\langle U\psi_n|\,U\psi_m\rangle|^2 = |\langle\psi_n|\,U^\dagger U\,|\psi_m\rangle|^2 \tag{2.1}$$

and thus U is unitary, $U^\dagger U = 1$ (or anti-unitary). If valid at the fixed time t_0, then it remains valid at arbitrary time t, if $[H, U] = 0$:

$$\begin{aligned} |\psi'(t)\rangle &= U\,|\psi(t)\rangle = U\exp[-\mathrm{i}H(t-t_0)]\,|\psi(t_0)\rangle && (2.2)\\ &= \exp[-\mathrm{i}H(t-t_0)]U\,|\psi(t_0)\rangle && (2.3)\\ &= \exp[-\mathrm{i}H(t-t_0)]\,|\psi'(t_0)\rangle\ . && (2.4)\end{aligned}$$

where we used $HU = UH$ in the second step[1]. Thus a symmetry implies the existence of an unitary operator which commutes with H.

For a continuous transformation, we can expand $U = 1 + \mathrm{i}\varepsilon G + \mathcal{O}(\varepsilon^2)$, keeping only the linear term. From

$$1 = UU^\dagger = (1 + \mathrm{i}\varepsilon G)(1 - \mathrm{i}\varepsilon G^\dagger) = 1 + \mathrm{i}\varepsilon(G - G^\dagger) + \mathcal{O}(\varepsilon^2)$$

we see that the generator G of the symmetry transformation is hermitian and thus corresponds to an observable. Moreover, $[H, U] = [H, 1 + \mathrm{i}\varepsilon G] = 0$ implies $[H, G] = 0$ and thus the expectation value of G is constant.

We conclude that a continuous symmetry transformation leads in QM to a conservation law for the corresponding generator which is an observable.

Ex.: Translation operator: Consider as example that a system is invariant under a translation $\boldsymbol{a}$ in space. Thus its wave function satisfies

$$T(\boldsymbol{a})\psi(\boldsymbol{x}) = \psi(\boldsymbol{x} + \boldsymbol{a}) = \psi(\boldsymbol{x}).$$

We can determine T performing a Taylor expansion of $\psi(\boldsymbol{x} + \boldsymbol{a})$,

$$\psi(\boldsymbol{x} + \boldsymbol{a}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{a}\cdot\boldsymbol{\nabla})^n\psi(\boldsymbol{x}) = \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{p})^n\psi(\boldsymbol{x}) = \exp(-\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{p})\psi(\boldsymbol{x}). \tag{2.5}$$

Thus the the momentum operator generates space translation, which in turn implies that momentum is conserved, if the system is invariant under translations. Finally, we show that the eigenfunctions of the translation operator are plane-waves,

$$T(\boldsymbol{a})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = (1 + \boldsymbol{a}\cdot\boldsymbol{\nabla} + \cdots)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = \mathrm{e}^{\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}+\cdot\boldsymbol{a})}. \tag{2.6}$$

Using a relativistic notation, translation a^μ in spacetime are generated by

$$T(a^\mu) = \exp(\mathrm{i}a^\mu p_\mu). \tag{2.7}$$

[1] We are only interested in time-independent H; moreover U is also time-independent, since we consider global symmetries.

Manifestations How do symmetries manifest themselves? First, we note that conservation laws lead to selection rules. Consider the example that the system is invariant under rotations, $[H, L^2] = [H, L_z] = 0$. Then

$$0 = \langle lm| [H, L^2] \left|l'm\right\rangle = [l'(l'+1) - l(l+1)] \langle lm| H \left|l'm\right\rangle \tag{2.8}$$

$$0 = \langle lm| [H, L_z] \left|lm'\right\rangle = [m' - m)] \langle lm| H \left|l'm\right\rangle \tag{2.9}$$

and thus transition elements are diagonal in the conserved quantum numbers,

$$\langle lm| H \left|l'm'\right\rangle = \delta_{ll'}\delta_{mm'} \langle lm| H |lm\rangle \,. \tag{2.10}$$

In other words: If certain processes do not occour, a corresponding symmetry should forbid them.

Second, a symmetry may manifest itself via degenerated states: Multiplying from the left the Schrödinger equation $H|\psi_\alpha\rangle = E_\alpha |\psi_\alpha\rangle$ with U and using $[H, U] = 0$ as well as $|\psi'\rangle = U|\psi\rangle$ gives $H|\psi'_\alpha\rangle = E_\alpha |\psi'_\alpha\rangle$. If $|\psi'_\alpha\rangle$ and $|\psi'_\alpha\rangle$ are linear independent, the energy E_α is degenerated. For free particles at rest, this implies that their masses are degenerated.

Often, the symmetry is hidden or slightly broken. In this case, we should look for a set of particles with approximately the same mass and similar interactions as sign for a symmetry.

2.2 Groups

A set of elements $\{a, b, c, \dots\}$ is called a group $\mathcal{G}$, if the following four properties are satisfied:

- For every pair a, b the product $ab = c \in \mathcal{G}$ is defined ("closure property").
- A unit element e exists in $\mathcal{G}$ such that for every a, $ae = ea = a$.
- The associative law holds: $(ab)c = a(bc)$.
- Each group element has an inverse, $aa^{-1} = a^{-1}a = e$.

Symmetry transformations in physics satisfy the group axioms. (As an example, think at translations $x \to x + a$.) Therefore symmetries can be described by groups, which may be discrete or continuous. An example for a discrete, finite group is the Z_2 symmetry $\phi \to -\phi$ which has only two elements. An example for a continuous symmetry is the phase (or gauge) transformation $\phi \to \exp(i\vartheta)\phi$ with the symmetry group U(1).

We consider mostly matrix representations of groups. Using then matrix multiplication as group operation, the associative law holds automatically. Note that there exist typically an infinite number of different representation of a given group.

The invertible $n \times n$ matrices satisfy all four conditions and form therefore a group, called $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ depending on if the matrix elements are real or complex. More specific examples are orthogonal ($O^T = O^{-1}$) and unitary ($U^\dagger = U^{-1}$) matrices. Here we have to check the closure property. If O_1 and O_2 are orthogonal, then their product $O = O_1O_2$ is orthogonal too, since

$$O^T O = (O_1O_2)^T O_1O_2 = O_2^T O_1^T O_1 O_2 = \mathbf{1} \,. \tag{2.11}$$

The corresponding group of n-dimensional orthogonal matrices is called $\mathrm{O}(n)$. In the same way, one shows that the product of unitary matrices is again a unitary matrix. The corresponding group of n-dimensional unitary matrices is called $\mathrm{U}(n)$. Adding the restriction that the determinant of the matrices is one, one obtains the special orthogonal groups $\mathrm{SO}(n)$ and the special unitary groups $\mathrm{SU}(n)$.

Lie groups A Lie group $\mathcal{G}$ is a continuous group which depends analytically on a finite number n of real parameters: The neighbourhood of any group element $g \in \mathcal{G}$ can be parametrised by these parameters, and a group element g can be expanded as a power series,

$$g(\vartheta) = 1 + \sum_{a=1}^{n} \mathrm{i}\vartheta^a T^a + \mathcal{O}(\vartheta^2) \equiv 1 + \mathrm{i}\vartheta^a T^a + \mathcal{O}(\vartheta^2) \,. \tag{2.12}$$

The linear transformation in the arbitrary direction ϑ^a is called an infinitesimal transformation, the T^a the (infinitesimal) generators of the transformation. The generators T^a can be obtained by differentiation, $T^a = -\mathrm{i}\, \mathrm{d}g(\vartheta)/\mathrm{d}\vartheta^a|_{\vartheta=0}$. Conversely, analyticity implies that the group element $g(\vartheta)$ can be obtained by exponentiation,

$$g(\vartheta) = \lim_{n\to\infty} [1 + \mathrm{i}\vartheta^a T^a/n]^n = \exp(\mathrm{i}\vartheta^a T^a) \,. \tag{2.13}$$

Note that a Lie group can consist of disconnected pieces. In this case, we can generate via Eq. (2.13) only the group elements in the piece containing the unit element.

The generators T^a form an algebra g called the Lie algebra. Three operations are defined in this algebra: Addition, multiplication by real numbers, and the Lie bracket $[A, B] = AB - BA$. One can express the commutator as a linear combination of generators,

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c \,, \tag{2.14}$$

where the real numbers f^{abc} are called structure constants. In general, it is easier to work out first the Lie algebra and then to construct finite transformation via Eq. (2.13).

Ex: Lorentz group SO(1,3): As an example, consider the Lorentz group SO(1,3). We can construct the Lie algebra of the Lorentz group differentiating the finite transformations, $T^a = -\mathrm{i}\, \mathrm{d}g(\vartheta)/\mathrm{d}\vartheta^a|_{\vartheta=0}$, in an arbitrary representation. Applied to the finite boost $B_x(\eta)$ along the x direction given in (1.6) we find as generator K_x

$$B_x(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad K_x = \frac{1}{\mathrm{i}} \left.\frac{\partial B_x(\eta)}{\partial\eta}\right|_{\eta=0} = -\mathrm{i} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.15}$$

and similiarly for the other two boosts. The 4-dim. generators of rotations are obtained simply by adding $(1, 0, 0, 0)$ as zeroth colum and raw to the known 3-dim. rotations, e.g.

$$R_z(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad J_z(\alpha) = -\mathrm{i} \left.\frac{\partial R_z(\alpha)}{\partial\alpha}\right|_{\alpha=0} = -\mathrm{i} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.16}$$

Calculating then their commutation relations, one finds

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk} J_k \,, \tag{2.17a}$$

$$[J_i, K_j] = \mathrm{i}\varepsilon_{ijk} K_k \,, \tag{2.17b}$$

$$[K_i, K_j] = -\mathrm{i}\varepsilon_{ijk} J_k \,. \tag{2.17c}$$

Note that the algebra of the rotation generators $\boldsymbol{J}$ is closed. Thus the rotations SO(3) form a subgroup of the Lorentz group. In contrast, the algebra of the boost generators is not closed (it contains J) and thus boost do not form a subgroup of the Lorentz group.

2.3 Groups U(n) and SU(n)

The $n \times n$ complex matrices satisfying $U^\dagger U = 1$ form a representation of the unitary group $\mathrm{U}(n)$. Setting $U = \exp(\mathrm{i}H)$, the matrix H is hermitian, $H = H^\dagger$ or $H^*_{ij} = H_{ji}$ for $i, j = 1, \ldots, n$. Thus unitarity implies n^2 constraints. In general, a $n \times n$ complex matrix has $2n^2$ real parameters. Accounting for the n^2 unitarity conditions, an element of $\mathrm{U}(n)$ is parametrised by n^2 real numbers. Since there are n^2 linear independent hermitian complex $n \times n$ matrices, we can choose them as generators.

From $U^\dagger U = 1$, we find

$$1 = \det(U)\det(U)^* = |\det(U)|^2 .$$

Thus $\det(U)$ corresponds to a phase $\mathrm{e}^{\mathrm{i}\vartheta_0}$. The matrices V of the special unitary group $\mathrm{SU}(n)$ satisfy $\det(V) = 1$. Extracting the phase $\mathrm{e}^{\mathrm{i}\vartheta_0}$, we can write

$$U = \mathrm{e}^{\mathrm{i}\vartheta_0} V = \mathrm{e}^{\mathrm{i}\vartheta_0} \exp(\mathrm{i}(\vartheta^a T^a)) .$$

Thus the unitary group is the product of a $\mathrm{U}(1)$ factor and the group $\mathrm{SU}(n)$, implying that $\mathrm{SU}(n)$ has $n^2 - 1$ parameters.

Next we use

$$1 = \det(V) = \exp \mathrm{tr} V = \exp(\mathrm{i}\,\mathrm{tr}(\vartheta^a T^a))$$

to find $\mathrm{tr}(\vartheta^a T^a) = 0$. Since the ϑ^a are arbitrary, the trace of each individual generator has to vanish, $\mathrm{tr}(T^a) = 0$. Thus the generators of $\mathrm{SU}(n)$ are $n^2 - 1$ traceless, hermitian matrices. By convention, physicists use as normalisation $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

SU(2) The three Pauli matrices σ^i provide a set of $n^2 - 1 = 3$ traceless, hermitian matrices. They satisfy $\sigma^i \sigma^j = \delta^{ij} + \mathrm{i}\varepsilon^{ijk}\sigma^k$. To comply with the normalisation convention, we have to set $T^a = \sigma^a/2$. Knowing the T^a, we can calculate the structure constants,

$$[T^a, T^b] = \mathrm{i}\varepsilon^{abc} T^c.$$

Note that they agree with those of the angular momentum operators given in Eq. (2.17a). Thus SO(3) and SU(2) agree locally, since their Lie algebra coincide. Physically, this corresponds to the fact that SU(2) generates rotations for two-spinors.

Finite transformations which we can apply on two-spinors are then given by $U = \exp(\mathrm{i}\boldsymbol{\alpha}\boldsymbol{\sigma}/2)$.

SU(3) The eight generators of SU(3) can be chosen as the Gell-Mann matrices, $T^a = \lambda^a/2$, with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}.$$

Note that the λ_i with $i = 1, 4, 6$ and $i = 2, 5, 7$ can be obtained from the Pauli matrices σ_1 and σ_2 adding a raw and a column with zeros. This reflects the fact that SU(3) contains as subgroups three SU(2) factors (of which only two are independent).

2.4 Flavor symmetries and the quark model

Isospin and SU(2)$_f$ see Griffiths 4.3.

Hypercharge and Deltas see Griffiths 4.3.

Remark 2.1: We use non-relativistic QM to descibe hadrons as bound-states of quarks. Since $m_{u,d} \ll m_\pi$, quarks inside a hadron are however relativistic. Therefore (virtual) processes like $q \to qg \to qq\bar{q} \to \ldots$ are possible. They create a "sea" of quark-antiquark pairs and gluons which total quantum numbers cancel to zero. This cancellation means that some aspects of hadrons can be described reasonbably well with the naive non-relativistic quark model we are using; however it fails in other aspects.

Experimental evidence for the colour Early evidence for colour was

- Decays like $\pi^0 \to 2\gamma$ are proportional to the squared number of light quarks, $\Gamma \propto N_c^2$.
- The cross section $e^+e^- \to$hadrons depends on the number of kinematically accessible quark states. It is convinient to consider the ratio

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sigma(e^+e^- \to \bar{q}q)}{\sigma(e^+e^- \to \mu^+\mu^-)}. \tag{2.18}$$

In $\sigma(e^+e^- \to \text{hadrons})$ we have to include all quarks with mass $s \geq 4m_q^2$. If we neglect m_f, the Feynman amplitude for the process $e^+e^- \to \bar{f}f$ with the exchange of a virtual photon depends on the type of fermion only via its electric charge. Thus the ratio R is

$$R = \frac{\sum q_i^2 e^2}{e^2} = N_c \sum_{\text{flavor}} q_i^2 \tag{2.19}$$

where q_i measures the electic charge of the quark with flavour i in umits of the electron charge e, and $N_c = 3$ counts the number of colors. If only the u,d,s quarks can be produced, then

$$R = 3\left[\frac{1}{9} + \frac{4}{9} + \frac{1}{9}\right] = 2.$$

If additionally c quarks can be produced, then

$$R = 3\left[\frac{1}{9} + \frac{4}{9} + \frac{4}{9} + \frac{1}{9}\right] = 10/3$$

etc. In Fig. 2.1, experimental data are compared with this prediction (and a more precise higher-order calculation). Several spikes can be seen when a new production channel opens, $s \gtrsim 4m_q^2$. They correspond to the mass of mesons containing this quark type. After that the cross section becomes smooth and approaches the predicition.

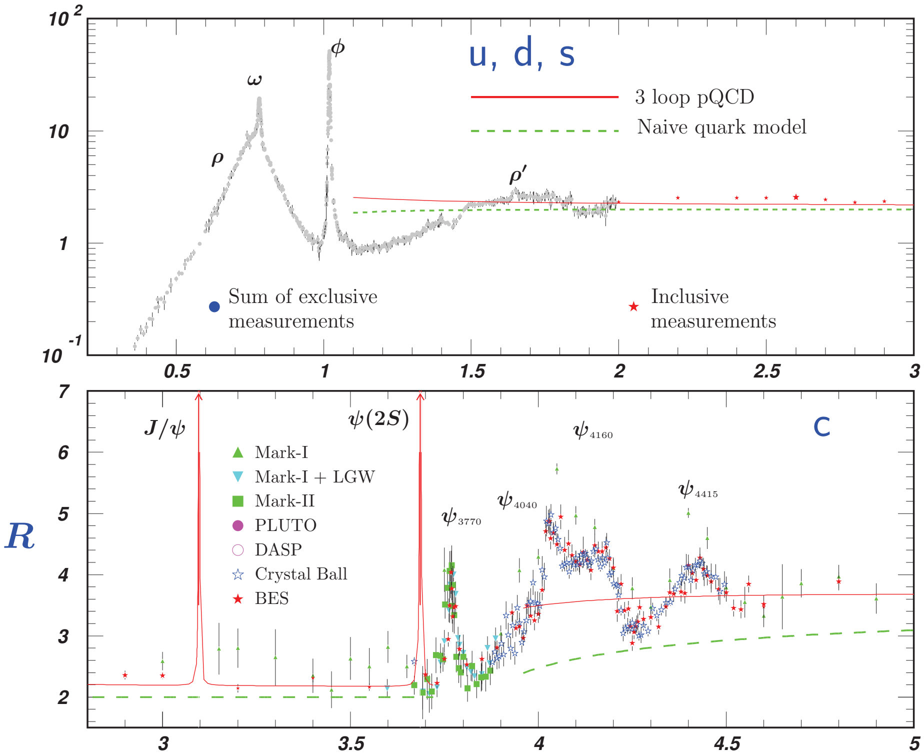

Figure 2.1: The ratio R as function of $\sqrt{s}$/GeV [PDG].

Quark model On a fundamental level, the SU(2) symmetry between light baryons and mesons corresponds to a SU(2) symmetry between the u and down quark. The latter is caused by the flavor symmetry of QCD: A gluon couples to color while the flavor of the quark does not enter at all. The symmetry is only approcximate, because of the very small mass difference of the u and down quark, $m_u \sim 2\,\mathrm{MeV}$ and $m_d \sim 5\,\mathrm{MeV}$.

Since also the s quark mass is comparable to the masses of light mesons, and much lighter than the one of baryons, an extension to a SU(3) symmetry connecting the u, d and s quark seems appropriate.

On the meson level, “strangeness” was introduced to explain the proprties of K mesons: they are produced via strong interactions, but decay much slower via e.g. $K^+ \to \mu^+ + \nu_\mu$. This suggests that the decay is induced by weak interactions, and thus the flavor of a quarks is changed.

SU(3)$_f$ Our construction of the Gell-Mann matrices suggests that SU(3) contains three SU(2) factors. First, we set $F_i \equiv \lambda_i/2$ to obtain correctly normalised generators. Then we introduce

$$T_\pm = F_1 \pm \mathrm{i}F_2\,, \qquad T_3 = F_3 \tag{2.20}$$

$$V_\pm = F_4 \pm \mathrm{i}F_5\,, \qquad Y = \frac{2}{\sqrt{3}}F_8 \tag{2.21}$$

$$U_\pm = F_6 \pm \mathrm{i}F_7\,. \tag{2.22}$$

The ladder operators $T_\pm$ and T_3 generate the standard SU(2) algebra,

$$[T_+, T_-] = 2T_3 \qquad [T_3, T_\pm] = \pm T_\pm \tag{2.23}$$

Calculating the corresponding comutation relations for $V_\pm$ and $U_\pm$, one finds,

$$[U_+, U_-] = \frac{3}{2}Y - T_3 \equiv 2U_3\,, \qquad [V_+, V_-] = \frac{3}{2}Y + T_3 \equiv 2V_3. \tag{2.24}$$

With these two definitions, we see that SU(3) contains indeed 3 closed SU(2) subgroups, generated by T_i, U_i and V_i. The only non-zero commutators are

$$[T_\pm, Y] = 0 \quad and \quad [T_3, Y] = 0$$

$$[T_\pm, V_\pm] = [T_\pm, U_\pm] = [U_\pm, V_\pm] = 0.$$

Thus there only two generators which we can diagonalise at the same time. Choosing them as Y and T_3, the remaining six generators are ladder operators which connect the states inside a multiplet. Thus we classify states by Y and T_3, calling them $|T_3, Y\rangle$.

Next we determine the action of $U_\pm$ and $V_\pm$ on these states. First we calculate

$$[T_3, U_\pm] = \mp\frac{1}{2}U_\pm\,, \qquad [T_3, V_\pm] = \pm\frac{1}{2}V_\pm. \tag{2.25}$$

Applying then e.g. $[T_3, V_\pm] = \pm\frac{1}{2}V_\pm$ on this ket, we obtain

$$(\hat{T}_3\hat{V}_\pm - \hat{V}_\pm\hat{T}_3)\,|T_3, Y\rangle = \pm\frac{1}{2}\hat{V}_\pm\,|T_3, Y\rangle\,. \tag{2.26}$$

Using $\hat{T}_3\,|T_3, Y\rangle = T_3\,|T_3, Y\rangle$, we arrive at

$$\hat{T}_3\hat{V}_\pm\,|T_3, Y\rangle = (T_3 \pm \frac{1}{2})\hat{V}_\pm\,|T_3, Y\rangle\,. \tag{2.27}$$

Hence $V_\pm$ changes T_3 by $\pm 1/2$. Since $[T_3, U_\pm] = \mp\frac{1}{2}U_\pm$, it follows immediately that $U_\pm$ changes T_3 by $\mp 1/2$. Finally, we have to determine how $V_\pm$ and $U_\pm$ change Y. Repeating the same argument for the commutator $[Y, U_\pm]$ and $[Y, V_\pm]$, you find that $V_\pm$ changes Y by ± 1 and $U_\pm$ changes Y by ∓ 1. Thus we can visualize the action of $V_\pm$ and $U_\pm$ in the $T_3 - Y$ plane as shown in Fig. 2.2.

Quarks should sit in the fundamental (i.e. the smallest non-trivial) representation. We assume that the Gell-Mann–Nishima relation $Q = \frac{1}{2}Y + T_3$ holds. Then Q is a conserved quantum number of the states $|T_3, Y\rangle$. The smallest representations are triangles, where one side is a $X = \{T_3, U_3, V_3\}$ doublet and the oppossite corner is as $X = \{T_3, U_3, V_3\}$ singlet, cf. with Fig. 2.3.

We now show as illustration how one can determine the quantum numbers of the states in this representation. The T_3 quantum numbers of the representation shown on the left are

$$T_3\,|q_1\rangle = +\frac{1}{2}\,|q_1\rangle\,, \qquad T_3\,|q_2\rangle = -\frac{1}{2}\,|q_2\rangle\,, \qquad T_3\,|q_3\rangle = 0. \tag{2.28}$$

We determine the Y quantum numbers as follows: q_1 is a U_3 singlet, $U_3\,|q_3\rangle = 0$. It is $U_3 = \frac{3}{4}Y - \frac{1}{2}T_3$ and thus

$$Y\,|q_1\rangle = \left(\frac{4}{3}U_3 + \frac{2}{3}T_3\right)|q_1\rangle = \frac{1}{3}\,|q_1\rangle\,.$$

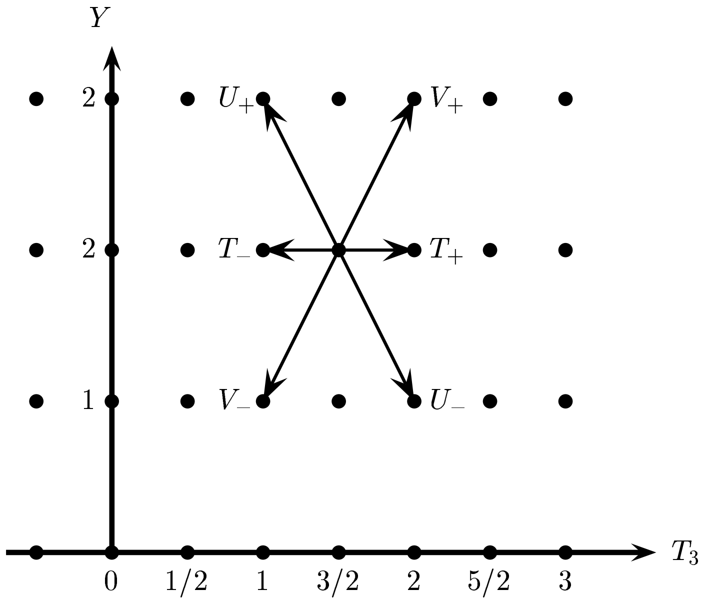

Figure 2.2: The action of the ladder operators on the state $|T_3, Y\rangle$.

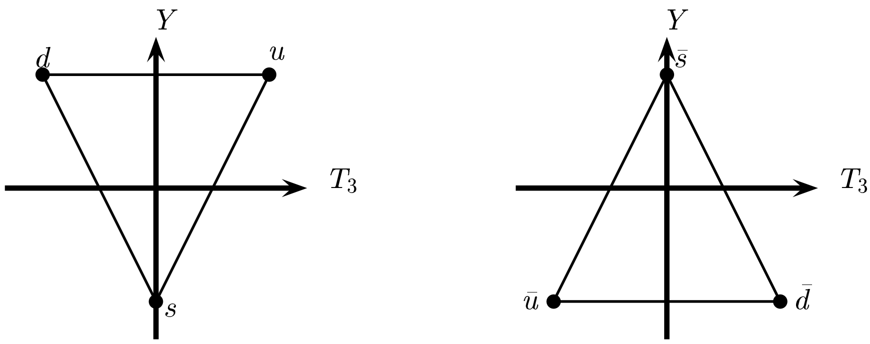

Figure 2.3: The fundamental represenation of SU(3) for quarks aand antiquarks.

Since $|q_1\rangle$ and $|q_2\rangle$ have the same quantum number Y, it is

$$Y|q_2\rangle = \left(\frac{4}{3}U_3 + \frac{2}{3}T_3\right)|q_2\rangle = \frac{1}{3}|q_2\rangle$$

and thus $U_3 = 1/2$. Next we find

$$Y|q_3\rangle = \left(\frac{4}{3}U_3 + \frac{2}{3}T_3\right)|q_2\rangle = \left(\frac{4}{3}(-1/2) + 0\right)|q_2\rangle = -\frac{2}{3}|q_2\rangle\,,$$

giving

$$Q|q_3\rangle = \left(\frac{1}{2}Y + T_3\right)|q_2\rangle] = \left(\frac{1}{2}(-2/3) + 0\right)|q_3\rangle = -\frac{1}{3}|q_3\rangle\,.$$

Similarly we find $q_1 = 2/3$, $q_2 = -1/3$, and $q_3 = 2/3$, corresponding to the u, d and s quark.

Larger multiplets can be analysed in the same way: The action of $T_\pm$, $V_\pm$ and $U_\pm$ generates the six sides of an hexagon. Thus larger multipletts are hexagons centered around $Y = T_3 = 0$. The states predicted by the smallest $q\bar{q}$ and qqq representations which correspond to meson and baryon states have been found. In principle, additional states like $q\bar{q}q\bar{q}$ or "penta-quarks" $q\bar{q}\,qqq$ might exist. Some evidence for the existence of such penta-quarks have been found in the last years[2].

[2] After a first "discovery" in 2003, the 2008 Review of Particle Physics wrote: " There are two or three recent experiments that find weak evidence for signals near the nominal masses, but there is simply no point in tabulating them in view of the overwhelming evidence that the claimed pentaquarks do not exist... The whole story—the discoveries themselves, the tidal wave of papers by theorists and phenomenologists that followed, and the eventual "undiscovery"—is a curious episode in the history of science." Finally, the LHCb experiment identified pentaquarks in 2015.