Home Exam

Formalities.
Please return your solutions latest Friday, 18.10. at 15.00 either putting them into my mailbox in D5-175 or sending them by email. Don't forget to add your summary.
If you have already your "exam number" in FY3403, use it. Otherwise put your name.
You can use for the evaluatation of traces the usual theorems without proof. Alternatively, you can evaluate traces using REDUCE or a similar program. If you do so, add a print-out of the code you used.

Exercises.
The Large Electron-Positron Collider at CERN accelerated in its first phase electrons and positrons to a total energy of half the mass of the Z boson, such that using processes like $e^+e^- \to Z \to \bar{f}f$ the properties of the Z and, more generally, of electroweak interactions could be studied. We will use this process to become familiar with the evaluation of Feynman diagrams, decay widths and cross sections.

1. Z decays
a) Derive the completeness relation for a massive spin-1 particle as the Z boson,

$$\sum_{r=1}^{3} \varepsilon_\mu^{(r)} \varepsilon_\nu^{(r)*} = -\eta_{\mu\nu} + k_\mu k_\nu / m^2 \,,$$

as follows: Choose for simplicity the rest-frame, construct then three mutually orthogonal polarization vectors $\varepsilon_\mu^{(r)}$ with the property $k^\mu \varepsilon_\mu^{(r)} = 0$ and $\varepsilon_\mu^{(r)} \varepsilon^{\mu(r)} = -1$. Then generalise the result for an arbitrary frame.
b) Calculate the decay width Γ of the Z boson into fermion pairs $f\bar{f}$ using as interaction vertex

$$-\mathrm{i}\frac{g}{2\cos\vartheta_W}\gamma_\mu(g_V - g_A\gamma^5)$$

where g_V and g_A are real coupling constants. You may set $m_f = 0$. Determine the numerical value of the decay width $\Gamma_{\nu\nu} \equiv \Gamma(Z \to \bar{\nu}\nu)$ into neutrinos (of one type of flavour). [Hint: The values of $g_{A,V}$ depend on the fermion type and can be found in any book on particle physics or in the particle data booklet.]

2. The process $e^+e^- \to Z \to \bar{f}f$.
The process $e^+e^- \to \bar{f}f$ can proceed via the exchange of a virtual photon and of a virtual Z boson. Neglect the photon exchange diagram as well as fermion masses and calculate the cross section $\sigma(e^+e^- \to \bar{f}f)$ using as propagator for the Z

$$\mathrm{i}D_{\mu\nu}(k) = \mathrm{i}\frac{-\eta_{\mu\nu} + k_\mu k_\nu/m_Z^2\,,}{k^2 - m_Z^2}\,.$$

3. The Z resonance.

The cross section derived in 2. for $e^+e^- \to Z \to \bar{f}f$ diverges at $s = m_Z^2$, since we assumed that the Z boson is a stable particle. For an unstable particle with decat rate Γ at rest, the wave-function should be modified as

$$\psi \propto \mathrm{e}^{-imt} \to \psi \propto \mathrm{e}^{-imt}\mathrm{e}^{-\Gamma t/2}.$$

This corresponds to the change $m \to m - \mathrm{i}\Gamma/2$ or

$$m_Z^2 \to (m_Z - \mathrm{i}\Gamma_Z/2)^2 = m_Z^2 - \mathrm{i}m_Z\Gamma_Z - \Gamma_Z^2 \simeq m_Z^2 - \mathrm{i}m_Z\Gamma_Z$$

where we used $\Gamma_Z^2 \ll m_Z^2$ in the last step.

a.) Show that with this modification, one can express σ close to $s = M_Z^2$ as

$$\sigma = C\,\frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to \bar{f}f)}{(s - m_Z^2)^2 + m_Z^2\Gamma^2}\,.$$

Determine C.

b.) The decay width Γ of the Z boson can be expressed via its partial decay width Γ_{ii} into charged leptons, hadrons and neutrinos as

$$\Gamma = 3\Gamma_{ll} + \Gamma_{\text{had}} + N_\nu\Gamma_{\nu\nu}$$

where we assumed "lepton universality", $\Gamma_{ll} = \Gamma_{ee} = \Gamma_{\mu\mu} = \Gamma_{\tau\tau}$ and denoted by N_ν the number of neutrino families coupled to the Z. At the Z resonance, the following cross sections were measured

$$\sigma(e^+e^- \to \mu^+\mu^-) = 1.9993\,\text{nb}, \qquad \sigma(e^+e^- \to \text{hadrons}) = 41.476\,\text{nb},$$

while the total decay width of the Z was found as $\Gamma = (2.4952 \pm 0.0023)\,\text{GeV}$. Determine Γ_{ll} and Γ_{had}. Use your result for $\Gamma_{\nu\nu}$ from 1.b) to determine the number N_ν of neutrino families coupled to the Z.

c.) The Large Electron-Positron Collider was running from 1989 to 1995 at the Z resonance. The integrated luminosity recorded in this period by one of its four detectors, the DELPHI experiment, is given in Table 1. Determine the total number of Z's produced at LEP.

Year	1989	1990	1991	1992	1993	1994	1995
$\int \mathrm{d}t\mathscr{L}/\text{pb}^{-1}$	0.7	5.8	17.2	24.1	36.3	46.3	31.7

Table 1: Integrated luminosity recorded by the DELPHI exeriment.

The general formula for a cross section is

$$\mathrm{d}\sigma = \frac{1}{4I}\,(2\pi)^4\,\delta^{(4)}(P_i - P_f)|\mathcal{M}_{fi}|^2 \prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{2E_f(2\pi)^3} = \frac{1}{4I}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi^{(n)}\,, \tag{1}$$

where $\mathcal{M}$ is the Feynman amplitude, P_i and P_f are the total momentum of the initial and final state, $\Phi^{(n)}$ is the n-particle phase space,

$$\mathrm{d}\Phi^{(n)} = (2\pi)^4\,\delta^{(4)}(P_i - P_f)\,\prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{2E_f(2\pi)^3}\,, \tag{2}$$

and I is the flux factor,

$$I = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}. \tag{3}$$

(p_1 and p_2 are four-momenta of the particles in the initial state, m_1 and m_2 their masses.)

The decay rate (or width) of a particle with energy E_i is given by

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2E_i}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi^{(n)}\,. \tag{4}$$