Formalities.
Solutions should be handed in Monday 13.03., latest 15.15, in my mailbox (D5-166), by email or in the lectures.

Relativistic Stars.
In this home exam, you will derive step-by-step the equations of stellar structure for a relativistic, compact star. Typical examples like neutron stars have a polytropic equation of state, $P = P(\rho(r))$, and are therefore described by only two equations: i) the continuity equation and ii) the hydrostatic equilibrium equation.

a.) Argue that a static, isotropic metric can be written as

$$\begin{aligned} \mathrm{d}s^2 &= A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2) && (1)\\ &= \mathrm{e}^{\sigma(r)}\mathrm{d}t^2 - \mathrm{e}^{\lambda(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2)\,. && (2)\end{aligned}$$

b.) Show that the non-zero components of the Ricci tensor in this metric are given by

$$\begin{aligned} R_{00} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, && (3)\\ R_{11} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, && (4)\\ R_{22} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), && (5)\\ R_{33} &= R_{22}\sin^2\vartheta, && (6)\end{aligned}$$

where we order coordinates as $x^\mu = (t, r, \vartheta, \phi)$ and primes denote differentiation w.r.t. r. You may use a program of your choice to do this calculation; if so, attach the code/output you used/produced. If you do the calculation "by hand", it is sufficient to calculate one of the 4 non-zero elements.

c.) Write down the Einstein equations for this metric and the case of an ideal fluid,

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta - Pg^{\alpha\beta}. \qquad (7)$$

[Note: You have two options, using either i) $G_{\mu\nu} = \ldots$ or ii) $R_{\mu\nu} = \ldots$. The first one requires the calculation of R, but has the advantage that ρ and P are decoupled. The second (recommended) option requires the calculation of the simpler trace T, but you have to find in e) a combination of the equations such that ρ decouples.]

d.) The conservation law for the stress tensor should be connected to the hydrostatic equilibrium condition. Evaluate therefore next $\nabla_\alpha T^{\alpha\beta} = 0$. Show that the "constraint"

$$P' + \frac{1}{2}(P + \rho)\sigma' = 0 \qquad (8)$$

for $\sigma' = A'/A$ follows.

e.) The metric (1) becomes equal to the Schwarzschild solution ar $r \geq R$. This is ensured by setting

$$B(r) = \mathrm{e}^{\lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1} \tag{9}$$

and $m(R) = M$. (At the moment, $m(r)$ is just a different way to label r.) If you used option 1), you can use this to rewrite the 00 equation as

$$\frac{2}{r^2}\frac{\mathrm{d}m}{\mathrm{d}r} = 8\pi\rho$$

Then

$$m(r) = 4\pi \int_0^r \mathrm{d}r'\, r'^2 \rho(r').$$

If you used option ii), show that the linear combination

$$\frac{R_{00}}{A} + \frac{R_{11}}{B} + \frac{2R_{22}}{r^2}$$

depends only on ρ and B, while it is independent of P and A. Using this expression, you can perform the same steps as for option i).

f.) Finally we note that in the 22 equation A and A' enter only via their ratio. Using the constraint to eliminate A'/A and (9) for B, you obtain an expression for the pressure gradient, the hydrostatic equilibrium equation ("TOV equation").

g.) Interpretation: Look up the corresponding non-relativistic equations and explain briefly differences/correction terms.