Formalities.
Solutions should be submitted via Inspera Monday 18.03., latest 12.00. Only if adminstration fails to set up this in time, you can hand in your solutions into my mailbox (D5-166), by email or in the lectures.

Lense-Thirring effect.
In this home exam, you will derive the Lense-Thirring effect, i.e. the rate of precession of a gyroscope freely-falling towards the Earth induced by the rotation of the Earth.

a.) Consider the Kerr metric in Boyer-Lindquist coordinates. Verify that the metric satisfies the vacuum Einstein equations using a computer program of your choice. Determine the singularity(ies) of the metric. (Add the input/output you used.)

b.) Show that at lowest order in the angular momentum $J = aM$, the metric simplifies to

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\text{Schwarzschild}} + \frac{4GJ}{r}\sin^2\vartheta \mathrm{d}\phi \mathrm{d}t. \tag{1}$$

Introduce then the Newtonian gravitational potential, take the weak-field limit, change to Cartesian coordinates and go back to units keeping explicitly c and G.

c.) A gyroscope can be modelled as a spinning rigid body of negligible extension. Thus the gyroscope is decribed completely by its four-velocity u^α and spin vector s^α. The latter is in the rest-system of the gyroscope a space-like vector. Show that this implies that

$$\boldsymbol{u}(\tau) \cdot \boldsymbol{s}(\tau) = 0 \tag{2}$$

and

$$\frac{Ds^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}s^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha{}_{\mu\nu} s^\mu u^\nu = 0. \tag{3}$$

d.) Write out the last equation for a gyroscope falling along the z azis, i.e. along the rotation axis of the Earth. Calculate the non-zero Christoffel symbols using the “Lagrange approach”: In order to simplify calculation keep i) only the lowest order term (in $1/c$) required in $\mathrm{d}s^2$, ii) use that Christoffel symbols $\propto x, y$ vanish on the rotation axis.

e.) Assume the non-relativistic limit for the velocity of the gyroscope and keep again only the leading term. Show that the rotation velocity in the system of the gyroscope is

$$\omega_{\text{LT}} = \frac{2GJ}{c^2 z^3}.$$

Interpretation? What is the rotation velocity measured by an observer at Earth?