Executive Summary
Write a 3–4 page executive summary of the lectures (ending with spin 1/2). Omit derivations.

Higgs decay into fermions and the optical theorem.
In the Standard Model, the Higgs particle \( h \) is a scalar particle that interacts with all fermions via a Yukawa coupling \( y \) proportional to the fermion mass \( m \), \( y = \frac{1}{2} gm / m_W \),

\[ h \quad \longrightarrow \quad -\frac{1}{2} ig \frac{m}{m_W} \]

a.) Calculate the decay width \( \Gamma (h \rightarrow \bar{f} f) \) of a Higgs particle with mass \( M \) into a antifermion-fermion pair (at tree-level).
b.) Consider the following contribution of fermions to the self-energy \( \Sigma(p^2) \) of the Higgs,

\[ i\Sigma^{\bar{f}f} = \]

Use dimensional regularisation to calculate \( \Sigma^{\bar{f}f} \) and show that

\[ \Sigma^{\bar{f}f} = \frac{A}{\varepsilon} + B \left[ C + \int_0^1 dz a^2 \ln(a^2/\mu^2) \right] \]

with \( a^2 = m^2 - p^2 z (1-z) - i\varepsilon \). Note: In \( d \) spacetime dimensions, the Clifford algebra becomes \( \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu}I_d \) with \( I_d \) as the \( d \)-dimensional unit matrix. Thus contractions change to

\[ \gamma^\mu \gamma_\mu = dI_d, \quad \gamma^\mu \gamma_\nu = (2-d)\delta^\mu_\nu, \quad \gamma^\mu \gamma_\nu \gamma_\mu = 4a \cdot bI_d - (d-4)\delta^\mu_\nu. \quad (1) \]

However, it is standard to define \( \text{tr}(I_d) = 4 \), which has the advantage that trace relations like \( \text{tr}[\phi \bar{\phi}] = 4a \cdot b \) are unchanged.

c.) Determine the imaginary part \( \Im \Sigma^{\bar{f}f} \) of the self-energy and show that the optical theorem holds, i.e. that \( \Im \Sigma^{\bar{f}f} = M \Gamma (h \rightarrow \bar{f} f) \) for \( p^2 = M^2 \).

d.) Obtain \( \Im \Sigma^{\bar{f}f} \) directly by “cutting the self-energy”: Consider

\[ i\Sigma^{\bar{f}f}(p^2) = \int \frac{d^4q}{(2\pi)^4} \cdots \]

for \( p = (M, 0) \); find the poles and apply the identity

\[ \frac{1}{x \pm i\varepsilon} = P \left( \frac{1}{x} \right) \mp i\pi \delta(x) \]

to the \( q^0 \) integral in order to obtain the imaginary part.
a.) The Feynman amplitude is
\[ i\mathcal{A} = -iy \frac{m}{m_W} \tilde{u}(q_1, s_1)\bar{v}(q_2, s_2). \]
Squaring and summing over final spins gives
\[ \sum_{s_1, s_2} |i\mathcal{A}|^2 = y^2 \sum_{s_1, s_2} \tilde{u}(q_1, s_1)\bar{v}(q_2, s_2)\bar{u}(q_1, s_1)\bar{v}(q_2, s_2) = y^2 tr[(\hat{q}_1 + m)(\hat{q}_2 - m)] \]
with
\[ tr[(\hat{q}_1 + m)(\hat{q}_2 - m)] = 4(q_1 q_2 - m^2). \]
In the cms frame, the momenta are
\[ p^\mu = (M, 0), \quad q_1^\mu = (M/2, q_{\text{cms}}), \quad q_2^\mu = (M/2, -q_{\text{cms}}). \]
with
\[ q_{\text{cms}} = \frac{M}{2} \left(1 - \frac{4m^2}{M^2}\right)^{1/2} = \frac{M}{2} \beta^{1/2} \quad \text{and} \quad 4(q_1 q_2 - m^2) = 4 \left(\frac{M^2}{4} + q_{\text{cms}}^2 - m^2\right) = 2M^2 \beta. \]
Here, \( \beta \) denotes the velocity of the \( W \)'s in the cms frame. Since \( |\mathcal{A}|^2 \) is a scalar, the remaining angular integration in \( d\Phi^{(2)} \) gives a trivial factor \( 4\pi \). Assembling everything, the total decay width in one fermion type follows as
\[ \Gamma = N_c \frac{y^2}{8\pi} M \left(1 - \frac{4m^2}{M^2}\right)^{3/2} = N_e \frac{\alpha^2}{32\pi} \frac{M^3}{m_W^2} \beta^{3/2}, \]
where \( N_C = \{1, 3\} \) for leptons/quarks takes into account the 3 quark colors.

b.) Applying the Feynman rules and DR gives
\[ i\Sigma(p^2) = -(-iy)^2(\mu^2)^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{tr[(\hat{q} + m)i(\hat{q} - \hat{p} + m)]}{(q^2 - m^2 + i\epsilon)(p^2 - m^2 + i\epsilon)}, \]
where the first minus sign comes from the fermion loop. (We used also that the symmetry factor is 1, and forget for the moment \( N_C \).) Next use
\[ \frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1 - z)]^2} \quad (2) \]
to combine the two denominators, and eliminate the terms linear in \( q \) in the denominator \( \mathcal{D} \) by substituting \( q^2 = |q - \hat{p}(1 - z)|^2 \),
\[ \mathcal{D} = q^2 - m^2 + p^2 z(1 - z) = q^2 - a^2. \]
(3)
Since \( d^d q = d^d q' \), we can drop the primes. Shifting then the nominator and dropping terms linear in \( q \), we arrive at
\[ \Sigma^{ff} = -4y^2(\mu^2)^{4-d} \int dz \int \frac{d^d q}{(2\pi)^d} \frac{q^2 + a^2}{(q^2 - a^2)^2} = -4y^2(\mu^2)^{4-d} \int_0^1 dz [a^2 I_0(d, 2) + I_2(d, 2)]. \]
Looking up the $I$ functions, expanding in $\varepsilon$ and collecting terms gives

$$
\Sigma f_l = N_c \frac{12y^2}{(4\pi)^2} \left[ \mu^{2\varepsilon} \left( m^2 - \frac{q^2}{6} \right) \frac{1}{\varepsilon} + \left( -\frac{m^2}{3} + \frac{q^2}{18} + \int_0^1 dz a^2 \ln(a^2/\mu^2) \right) \right]
$$

where we added $N_c$ and absorbed the constants in $\tilde{\mu}$.

c.) Only the log term can generate an imaginary part for $q^2 \geq 4m^2$,

$$
F = \left[ m^2 - q^2 z(1 - z) \right] \int_0^1 dz \ln \left[ m^2 - q^2 z(1 - z) \right].
$$

(4)
The argument of the logarithm becomes negative for

$$
z_{1/2} = \frac{1}{2} \left[ 1 \pm \sqrt{1 - 4m^2/q^2} \right].
$$

(5) Using now $\Im[\ln(-x-i\varepsilon)] = -\pi$, the imaginary part follows as

$$
\Im(F) = -\pi \int_{z_1}^{z_2} dz \left[ m^2 - q^2 z(1 - z) \right] = \frac{1}{6} \sqrt{1 - \frac{4m^2}{q^2} \left( q^2 - 4m^2 \right)} = \frac{1}{6} \left( 1 - \frac{4m^2}{q^2} \right)^{3/2}.
$$

(6) Adding the prefactor $N_c \frac{12y^2}{(4\pi)^2}$, we find

$$
\Im(\Sigma) = N_c \frac{y^2}{8\pi} q^2 \left( 1 - \frac{4m^2}{q^2} \right)^{3/2},
$$

(7) what agrees for $q^2 = M^2$ with the prediction of the optical theorem, $M\Gamma = \Im\Sigma$.

d.) Consider in

$$
i\Sigma = -y^2 \int \frac{d^4q}{(2\pi)^4} \frac{4[q^2 - qp + m^2]}{(q^2 - m^2 + i\varepsilon)(q^2 - p^2 - m^2 + i\varepsilon)}
$$

the denominator. Setting $p = (M, 0)$ and $E_q = +\sqrt{q^2 + m^2}$, we find as poles of the integrand $q^0 = E_q - i\varepsilon$, $q^0 = -E_q + i\varepsilon$, $q^0 = M + E_q - i\varepsilon$, and $q^0 = M - E_q + i\varepsilon$. We can choose the integration contour either in the upper or lower half-plane. Choosing the lower one, we pick up the two residues at $q^0 = E_q - i\varepsilon$ and $q^0 = M + E_q - i\varepsilon$. Hence we obtain

$$
\Sigma = -y^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{2ME_q} \left( \frac{ME_q - 2m^2}{M - 2E_q + i\varepsilon} + \frac{A}{M + 2E_q - i\varepsilon} \right)
$$

The second denominator never vanishes and thus gives no contribution to the imaginary part. For the first one, we obtain using the given identity

$$
\Im\Sigma = 4y^2 \pi \int \frac{d^3q}{(2\pi)^3} \frac{ME_q - 2m^2}{2ME_q} \delta(M - 2E_q)
$$

As $E_q = +\sqrt{q^2 + m^2} \geq m$, the argument of the delta function is never zero for $M \leq 2m$ and the imaginary part of the amplitude vanishes thus. For $M > 2m$, we can set $E_q = M/2$ and perform then the integral,

$$
\Im\Sigma = \frac{y^2}{16\pi} \sqrt{1 - \frac{4m^2}{M^2}}
$$

Thus we confirmed again the relation $M\Gamma = \Im\Sigma$.  