Home Exam

Formalities.
Please return your solutions latest Friday, 18.10. at 15.00 either putting them into my mailbox in D5-175 or sending them by email. Don't forget to add your summary.
If you have already your "exam number" in FY3403, use it. Otherwise put your name.
You can use for the evaluatation of traces the usual theorems without proof. Alternatively, you can evaluate traces using REDUCE or a similar program. If you do so, add a print-out of the code you used.

Exercises.
The Large Electron-Positron Collider at CERN accelerated in its first phase electrons and positrons to a total energy of half the mass of the Z boson, such that using processes like $e^+e^- \to Z \to \bar{f}f$ the properties of the Z and, more generally, of electroweak interactions could be studied. We will use this process to become familiar with the evaluation of Feynman diagrams, decay widths and cross sections.

1. Z decays
a) Derive the completeness relation for a massive spin-1 particle as the Z boson,

$$\sum_{r=1}^{3} \varepsilon_\mu^{(r)} \varepsilon_\nu^{(r)*} = -\eta_{\mu\nu} + k_\mu k_\nu / m^2 \,,$$

as follows: Choose for simplicity the rest-frame, construct then three mutually orthogonal polarization vectors $\varepsilon_\mu^{(r)}$ with the property $k^\mu \varepsilon_\mu^{(r)} = 0$ and $\varepsilon_\mu^{(r)} \varepsilon^{\mu(r)} = -1$. Then generalise the result for an arbitrary frame.
b) Calculate the decay width Γ of the Z boson into fermion pairs $f\bar{f}$ using as interaction vertex

$$-\mathrm{i}\frac{g}{2\cos\vartheta_W}\gamma_\mu(g_V - g_A\gamma^5)$$

where g_V and g_A are real coupling constants. You may set $m_f = 0$. Determine the numerical value of the decay width $\Gamma_{\nu\nu} \equiv \Gamma(Z \to \bar{\nu}\nu)$ into neutrinos (of one type of flavour). [Hint: The values of $g_{A,V}$ depend on the fermion type and can be found in any book on particle physics or in the particle data booklet.]

a.) A massive vector field A^μ has four components in $d = 4$ space-time dimensions, while it has only $2s+1 = 3$ independent spin components. Correspondingly, a four-vector A^μ transforms under a rotation as $(A^0, \boldsymbol{A})$, i.e. it contains a scalar and a three-vector. Thus the physical components of a massive spin-1 field in its rest-frame are given by $(0, \boldsymbol{A})$. We can choose the three polarisation vectors in the rest frame e.g. as the Cartesian unit vectors, $\boldsymbol{\varepsilon}_i \propto \mathbf{e}_i$. They satisfy $\varepsilon_\mu^{(r)} \varepsilon^{\mu(r)} = -1$

and, since in the rest-frame $k^\mu = (m, \mathbf{0})$ also $k_\mu \varepsilon_\mu^{(r)} = 0$.
Next we evaluate

$$\sum_r \varepsilon^{\mu(r)} \varepsilon^{\nu(r)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Now we have to express this using the relevant tensors, here $\eta^{\mu\nu}$ and $k^\mu k^\nu / m^2$, where we divided by m^2 to get the right dimension. This gives for $k^\mu = (m, \mathbf{0})$

$$\sum_r \varepsilon^{\mu(r)} \varepsilon^{\nu(r)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -\eta^{\mu\nu} + k^\mu k^\nu / m^2 \tag{2}$$

If we are not able to guess this, we can derive it formally: Using the tensor method, we express the polarisation sum as linear combination of relevant tensors,

$$\sum_r \varepsilon^{\mu(r)} \varepsilon^{\nu(r)} = A\eta^{\mu\nu} + B k^\mu k^\nu / m^2$$

Asking then $k_\mu \varepsilon^{\mu(r)} = 0$, gives

$$0 = Ak^\nu + Bk^\nu$$

or $A = -B$. The normalisation condition requires $A = -1$.

b.) See the notes.

2. The process $e^+e^- \to Z \to \bar{f}f$.
The process $e^+e^- \to \bar{f}f$ can proceed via the exchange of a virtual photon and of a virtual Z boson. Neglect the photon exchange diagram as well as fermion masses and calculate the cross section $\sigma(e^+e^- \to \bar{f}f)$ using as propagator for the Z

$$\mathrm{i}D_{\mu\nu}(k) = \mathrm{i}\frac{-\eta_{\mu\nu} + k_\mu k_\nu / m_Z^2\,,}{k^2 - m_Z^2}.$$

This process is discussed by Griffiths in Example 9.5. For the evaluation of the traces, see the notes.

3. The Z resonance.
The cross section derived in 2. for $e^+e^- \to Z \to \bar{f}f$ diverges at $s = m_Z^2$, since we assumed that the Z boson is a stable particle. For an unstable particle with decat rate Γ at rest, the wave-function should be modified as

$$\psi \propto \mathrm{e}^{-\mathrm{i}mt} \to \psi \propto \mathrm{e}^{-\mathrm{i}mt} \mathrm{e}^{-\Gamma t/2}.$$

This corresponds to the change $m \to m - \mathrm{i}\Gamma/2$ or

$$m_Z^2 \to (m_Z - \mathrm{i}\Gamma_Z/2)^2 = m_Z^2 - \mathrm{i}m_Z\Gamma_Z - \Gamma_Z^2 \simeq m_Z^2 - \mathrm{i}m_Z\Gamma_Z$$

where we used $\Gamma_Z^2 \ll m_Z^2$ in the last step.
a.) Show that with this modification, one can express σ close to $s = M_Z^2$ as

$$\sigma = C\, \frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to \bar{f}f)}{(s-m_Z^2)^2 + m_Z^2\Gamma^2}\,.$$

Determine C.
b.) The decay width Γ of the Z boson can be expressed via its partial decay width Γ_{ii} into charged leptons, hadrons and neutrinos as

$$\Gamma = 3\Gamma_{ll} + \Gamma_{\rm had} + N_\nu \Gamma_{\nu\nu}$$

where we assumed "lepton universality", $\Gamma_{ll} = \Gamma_{ee} = \Gamma_{\mu\mu} = \Gamma_{\tau\tau}$ and denoted by N_ν the number of neutrino families coupled to the Z. At the Z resonance, the following cross sections were measured

$$\sigma(e^+e^- \to \mu^+\mu^-) = 1.9993\,\text{nb}, \qquad \sigma(e^+e^- \to \text{hadrons}) = 41.476\,\text{nb},$$

while the total decay width of the Z was found as $\Gamma = (2.4952 \pm 0.0023)\,\text{GeV}$. Determine Γ_{ll} and $\Gamma_{\rm had}$. Use your result for $\Gamma_{\nu\nu}$ from 1.b) to determine the number N_ν of neutrino families coupled to the Z.
c.) The Large Electron-Positron Collider was running from 1989 to 1995 at the Z resonance. The integrated luminosity recorded in this period by one of its four detectors, the DELPHI experiment, is given in Table 1. Determine the total number of Z's produced at LEP.

a.) For $s \sim M_Z^2$, we can approximate $|\mathcal{M}|^2 = |\mathcal{M}_\gamma + \mathcal{M}_Z|^2 \simeq |\mathcal{M}_Z|^2$ with

$$\mathcal{M}_Z = \frac{g_Z^2}{s - M_Z^2 + \mathrm{i}M_Z\Gamma_Z}\frac{1}{4}[\bar{v}(p_2)\gamma^\mu(g_V^e - g_A^e\gamma^5)u(p_1)][\bar{v}(p_4)\gamma^\mu(g_V^f - g_A^f\gamma^5)u(p_3)] \tag{3}$$

where $\mathrm{i}M_Z\Gamma_Z$ takes into account the finite life-time of the Z. This gives (cf. (9.111) in Griffiths)

$$\sigma(e^+e^- \to \bar{f}f) = \frac{g_Z^4\, s}{192\pi}\, \frac{[(g_V^e)^2 +) + (g_A^e)^2][(g_V^f)^2 + (g_A^f)^2]}{(s-M_Z^2)^2 + (M_Z\Gamma_Z)^2} \tag{4}$$

Recall now the decay width of the Z,

$$\Gamma(Z^0 \to \bar{f}f) = \frac{g_Z^2 m_Z}{48\pi}\,[(g_V^f)^2 + (g_A^f)^2] \tag{5}$$

Inserted into (4), it follows

$$\sigma(e^+e^- \to \bar{f}f) = \frac{12\pi s}{m_Z^2}\, \frac{\Gamma(Z^0 \to e^+e^-)\Gamma(Z^0 \to \bar{f}f)}{(s-M_Z^2)^2 + (M_Z\Gamma_Z)^2} \tag{6}$$

Year	1989	1990	1991	1992	1993	1994	1995
$\int \mathrm{d}t\mathscr{L}/\mathrm{pb}^{-1}$	0.7	5.8	17.2	24.1	36.3	46.3	31.7

Table 1: Integrated luminosity recorded by the DELPHI exeriment.

or $C = 12\pi s/m_Z^2 = 12\pi$.

Note: This "Breit-Wigner formula" holds for all resonances R,

$$\sigma(12 \to R \to 34) = \frac{4\pi s}{p_{\text{cms}}^2} \frac{(2s_r+1)}{(2s_1+1)(2s_2+1)} \frac{\Gamma(R \to 12)\Gamma(R \to 34)}{(s-M_R^2)^2+(M_R\Gamma_R)^2} \tag{7}$$

where Γ_R is the total decay width of the resonance with mass m_R and spin s_R, while s_1 and s_2 are the spins of the particles in the initial state.
In our case, the formula gives $C = 4\pi s/(s/4) \times 3/4 = 12\pi$.

b.) Inverting the cross section at the Z resonance,

$$\sigma_{ff}^0 = \frac{12\pi}{m_Z^2} \frac{\Gamma_{ee}\Gamma_{ff}}{\Gamma_Z^2}$$

gives

$$\Gamma_{ee}\Gamma_{ff} = \frac{\sigma_{ff}^0 \Gamma_Z^2 m_Z^2}{12\pi}.$$

Using lepton universality, $\Gamma_{\mu\mu} = \Gamma_{ll}$ and $\sigma(e^+e^- \to Z \to \mu^+\mu^-) = 1.99993\text{nb}$ or $5.152 \times 10^{-6}\text{GeV}^{-2}$ gives

$$\Gamma_{ee}^2 = 1.136 \times \Gamma_Z^2.$$

Proceeding in the same way for the hadronic ratio,

$$\Gamma_{ee}\Gamma_{had} = \frac{\sigma_{had}^0 \Gamma_Z^2 m_Z^2}{12\pi} = 2.357 \times 10^{-2}\Gamma_Z^2.$$

or $\Gamma_{had} = 0.69992\Gamma_Z$. Comparing next

$$N_\nu \Gamma_{\nu\nu} = \Gamma_Z - 3\Gamma_{ll} - \Gamma_{\text{had}} = 0.19997\Gamma_Z = 498\text{MeV}$$

with the width into one neutrino species, 167 MeV, it follows $N_\nu = 498/167 = 2.98$.

c.) The integrated luminosity of all 4 experiments is $\simeq 650$/pb. With $\sigma_{tot}^0 = 60$nb, the number of produced Z's is $60 \times 650 \times 1000 = 40$ millions. In reality, the number was smaller, since the numerical values of the cross sections given in the exercise are corrected for "initial state radiation".

The general formula for a cross section is

$$\mathrm{d}\sigma = \frac{1}{4I}\,(2\pi)^4\,\delta^{(4)}(P_i - P_f)|\mathcal{M}_{fi}|^2 \prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{2E_f(2\pi)^3} = \frac{1}{4I}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi^{(n)}\,, \tag{8}$$

where $\mathcal{M}$ is the Feynman amplitude, P_i and P_f are the total momentum of the initial and final state, $\Phi^{(n)}$ is the n-particle phase space,

$$\mathrm{d}\Phi^{(n)} = (2\pi)^4\,\delta^{(4)}(P_i - P_f)\,\prod_{f=1}^{n} \frac{\mathrm{d}^3 p_f}{2E_f(2\pi)^3}\,, \tag{9}$$

and I is the flux factor,

$$I = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}. \tag{10}$$

(p_1 and p_2 are four-momenta of the particles in the initial state, m_1 and m_2 their masses.)

The decay rate (or width) of a particle with energy E_i is given by

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2E_i}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi^{(n)}\,. \tag{11}$$