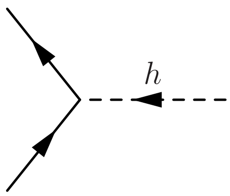
Formalities.
Solutions should be emailed latest Friday 27.03, at 15.00.

Executive summary.
Write a 3–4 page executive summary of the lectures (ending [and including] with spin 1/2). Omit derivations.

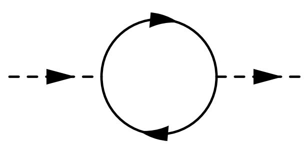
Higgs decay into fermions and the optical theorem.
In the Standard Model, the Higgs particle h is a scalar particle that interacts with all fermions via a Yukawa coupling y proportional to the fermion mass m, $y = \frac{1}{2}\, gm/m_W$,



$-\frac{1}{2}\,\mathrm{i}g\,\frac{m}{m_W}$

a.) Calculate the decay width $\Gamma(h \to \bar{f}f)$ of a Higgs particle with mass M into a antifermion-fermion pair (at tree-level).
b.) Show that a fermion loop leads to an additional minus sign in the Feynman amplitude.
c.) Consider the following contribution of fermions to the self-energy $\Sigma(p^2)$ of the Higgs,

$\mathrm{i}\Sigma^{\bar{f}f} =$



Use dimensional regularisation to calculate $\Sigma^{\bar{f}f}$ and show that

$$\Sigma^{\bar{f}f} = \frac{A}{\varepsilon} + B\left[C + \int_0^1 \mathrm{d}z a^2 \ln(a^2/\mu^2)\right]$$

with $a^2 = m^2 - p^2 z(1-z) - \mathrm{i}\varepsilon$. Note: In d spacetime dimensions, the Clifford algebra becomes $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_d$ with I_d as the d-dimensional unit matrix. Thus contractions change to

$$\gamma^\mu\gamma_\mu = dI_d, \qquad \gamma^\mu \not{a}\gamma_\mu = (2-d)\not{a}, \qquad \gamma^\mu \not{a}\not{b}\gamma_\mu = 4a\cdot b I_d - (d-4)\not{a}\not{b}. \qquad (1)$$

However, it is standard to define $\mathrm{tr}(I_d) = 4$, which has the advantage that trace relations like $\mathrm{tr}[\not{a}\not{b}] = 4a \cdot b$ are unchanged.
d.) Determine the imaginary part $\Im\Sigma^{\bar{f}f}$ of the self-energy and show that the optical theorem holds, i.e. that $\Im\Sigma^{\bar{f}f} = M\Gamma(h \to \bar{f}f)$ for $p^2 = M^2$.
e.) Obtain $\Im\Sigma^{\bar{f}f}$ directly by "cutting the self-energy": Consider

$$\mathrm{i}\Sigma^{\bar{f}f}(p^2) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \cdots$$

for $p = (M, \mathbf{0})$; find the poles and apply the identity

$$\frac{1}{x \pm \mathrm{i}\varepsilon} = P\left(\frac{1}{x}\right) \mp \mathrm{i}\pi\delta(x)$$

to the q^0 integral in order to obtain the imaginary part.

Good luck!

a.) The Feynman amplitude is

$$\mathrm{i}\mathcal{A} = -\mathrm{i}y\,\frac{m}{m_W}\,\bar{u}(q_1,s_1)v(q_2,s_2).$$

Squaring and summing over final spins gives

$$\sum_{s1,s2}|\mathcal{A}|^2 = y^2\sum_{s1,s2}\bar{u}(q_1,s_1)v(q_2,s_2)\bar{v}(q_2,s_2)u(q_1,s_1) = y^2\mathrm{tr}[(\not{q}_1+m)(\not{q}_2-m)]$$

with $\mathrm{tr}[(\not{q}_1+m)(\not{q}_2-m)] = 4(q_1q_2-m^2)$. In the cms frame, the momenta are

$$p^\mu = (M,\mathbf{0}),\quad q_1^\mu = (M/2,\boldsymbol{q}_{\rm cms}),\quad q_2^\mu = (M/2,-\boldsymbol{q}_{\rm cms}).$$

with

$$q_{\rm cms} = \frac{M}{2}\left(1-\frac{4m^2}{M^2}\right)^{1/2} = \frac{M}{2}\,\beta^{1/2}\quad\text{and}\quad 4(q_1q_2-m^2) = 4\left(\frac{M^2}{4}+q_{\rm cms}^2-m^2\right) = 2M^2\,\beta.$$

Here, β denotes the velocity of the fermions in the cms frame. Since $|\bar{\mathcal{A}}|^2$ is a scalar, the remaining angular integration in $\mathrm{d}\Phi^{(2)}$ gives a trivial factor 4π. Assembling everything, the total decay width in one fermion type follows as

$$\Gamma = N_c\frac{y^2}{8\pi}\,M\left(1-\frac{4m^2}{M^2}\right)^{3/2} = N_c\,\frac{g^2}{32\pi}\,\frac{M^3}{m_W^2}\,\beta^{3/2},$$

where $N_C = \{1,3\}$ for leptons/quarks takes into account the 3 quark colors.

b.) See the paragraph "fermion loops" in the notes.

c.) Applying the Feynman rules and DR gives

$$\mathrm{i}\Sigma(p^2) = -(-\mathrm{i}y)^2(\mu^2)^{4-d}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\,\frac{\mathrm{tr}[\mathrm{i}(\not{q}+m)\mathrm{i}(\not{q}-\not{p}+m)]}{(q^2-m^2+\mathrm{i}\varepsilon)[(q-p)^2-m^2+\mathrm{i}\varepsilon]},$$

where the first minus sign comes from the fermion loop. (We used also that the symmetry factor is 1, and forget for the moment N_c.) Next use

$$\frac{1}{ab} = \int_0^1\frac{\mathrm{d}z}{[az+b(1-z)]^2} \tag{2}$$

to combine the two denominators, and eliminate the terms linear in q in the denominator $\mathcal{D}$ by substituting $q'^2 = [q-p(1-z)]^2$,

$$\mathcal{D} = q'^2 - m^2 + p^2z(1-z) = q'^2 - a^2\,. \tag{3}$$

Since $\mathrm{d}^dq = \mathrm{d}^dq'$, we can drop the primes. Shifting then the nominator and dropping terms linear in q, we arrrive at

$$\Sigma^{\bar{f}f} = -4y^2(\mu^2)^{4-d}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^dq}{(2\pi)^d}\,\frac{q^2+a^2}{(q^2-a^2)^2} = -4y^2(\mu^2)^{4-d}\int_0^1\mathrm{d}z[a^2I_0(d,2)+I_2(d,2)]\,.$$

Looking up the I functions, expanding in ε and collecting terms gives

$$\Sigma^{\bar{f}f} = N_c \frac{12y^2}{(4\pi)^2}\left[\mu^{2\varepsilon}\left(m^2-\frac{q^2}{6}\right)\frac{1}{\varepsilon}+\left(-\frac{m^2}{3}+\frac{q^2}{18}+\int_0^1 \mathrm{d}z a^2 \ln(a^2/\tilde{\mu}^2)\right)\right]$$

where we added N_c and absorbed the constants in $\tilde{\mu}$.

d.) Only the log term can generate an imaginary part for $q^2 \geq 4m^2$,

$$F = \left[m^2 - q^2 z(1-z)\right]\int_0^1 \mathrm{d}z \, \ln\left[m^2 - q^2 z(1-z)\right]. \tag{4}$$

The argument of the logarithm becomes negative for

$$z_{1/2} = \frac{1}{2}\left[1 \pm \sqrt{1-4m^2/q^2}\right]. \tag{5}$$

Using now $\Im[\ln(-x-\mathrm{i}\varepsilon)] = -\pi$, the imaginary part follows as

$$\Im(F) = -\pi \int_{z_1}^{z_2} \mathrm{d}z \, \left[m^2 - q^2 z(1-z)\right] = \frac{1}{6}\sqrt{1-\frac{4m^2}{q^2}}\,(q^2-4m^2) = \frac{1}{6}\left(1-\frac{4m^2}{q^2}\right)^{3/2}. \tag{6}$$

Adding the prefactor $N_c\,12y^2/(4\pi)^2$, we find

$$\Im(\Sigma) = N_c\,\frac{y^2}{8\pi}\,q^2\left(1-\frac{4m^2}{q^2}\right)^{3/2}, \tag{7}$$

what agrees for $q^2 = M^2$ with the prediction of the optical theorem, $M\Gamma = \mathrm{Im}\Sigma$.

e.) Consider in

$$\mathrm{i}\Sigma = -y^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\,\frac{4[q^2 - qp + m^2)]}{(q^2-m^2+\mathrm{i}\varepsilon)[(q-p)^2-m^2+\mathrm{i}\varepsilon]}$$

the denominator. Setting $p=(M,0)$ and $E_q = +\sqrt{q^2+m^2}$, we find as poles of the integrand $q^0 = E_q - \mathrm{i}\varepsilon$, $q^0 = -E_q + \mathrm{i}\varepsilon$, $q^0 = M + E_q - \mathrm{i}\varepsilon$, and $q^0 = M - E_q + \mathrm{i}\varepsilon$. We can choose the integration contour either in the upper or lower half-plane. Choosing the lower one, we pick up the two residues at $q^0 = E_q - \mathrm{i}\varepsilon$ and $q^0 = M + E_q - \mathrm{i}\varepsilon$. Hence we obtain

$$\Sigma = -y^2 \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\,\frac{1}{2ME_q}\left(\frac{ME_q - 2m^2}{M - 2E_q + \mathrm{i}\varepsilon} + \frac{A}{M+2E_q-\mathrm{i}\varepsilon}\right)$$

The second denominator never vanishes and thus gives no contribution to the imaginary part. For the first one, we obtain using the given identity

$$\mathrm{Im}\Sigma = 4y^2\pi \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\,\frac{ME_q - 2m^2}{2ME_q}\delta(M-2E_q)$$

As $E_q = +\sqrt{q^2+m^2} \geq m$, the argument of the delta function is never zero for $M \leq 2m$ and the imaginary part of the amplitude vanishes thus. For $M > 2m$, we can set $E_q = M/2$ and perform then the integral,

$$\mathrm{Im}\Sigma = \frac{y^2}{16\pi}\sqrt{1-\frac{4m^2}{M^2}}$$

Thus we confirmed again the relation $M\Gamma = \mathrm{Im}\Sigma$.