Formalities.
Solutions should be submitted via Inspera Monday 18.03., latest 12.00. Only if adminstration fails to set up this in time, you can hand in your solutions into my mailbox (D5-166), by email or in the lectures.

Lense-Thirring effect.
In this home exam, you will derive the Lense-Thirring effect, i.e. the rate of precession of a gyroscope freely-falling towards the Earth induced by the rotation of the Earth.

a.) Consider the Kerr metric in Boyer-Lindquist coordinates. Verify that the metric satisfies the vacuum Einstein equations using a computer program of your choice. Determine the singularity(ies) of the metric. (Add the input/output you used.)

b.) Show that at lowest order in the angular momentum $J = aM$, the metric simplifies to

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\text{Schwarzschild}} + \frac{4GJ}{r}\sin^2\vartheta \mathrm{d}\phi \mathrm{d}t. \tag{1}$$

Introduce then the Newtonian gravitational potential, take the weak-field limit, change to Cartesian coordinates and go back to units keeping explicitly c and G.

c.) A gyroscope can be modelled as a spinning rigid body of negligible extension. Thus the gyroscope is decribed completely by its four-velocity u^α and spin vector s^α. The latter is in the rest-system of the gyroscope a space-like vector. Show that this implies that

$$\boldsymbol{u}(\tau) \cdot \boldsymbol{s}(\tau) = 0 \tag{2}$$

and

$$\frac{Ds^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}s^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha{}_{\mu\nu} s^\mu u^\nu = 0. \tag{3}$$

d.) Write out the last equation for a gyroscope falling along the z azis, i.e. along the rotation axis of the Earth. Calculate the non-zero Christoffel symbols using the "Lagrange approach": In order to simplify calculation keep i) only the lowest order term (in $1/c$) required in $\mathrm{d}s^2$, ii) use that Christoffel symbols $\propto x, y$ vanish on the rotation axis.

e.) Assume the non-relativistic limit for the velocity of the gyroscope and keep again only the leading term. Show that the rotation velocity in the system of the gyroscope is

$$\omega_{\text{LT}} = \frac{2GJ}{c^2 z^3}.$$

Interpretation? What is the rotation velocity measured by an observer at Earth?

a.) A vacuum solution has to satisfy $R_{\mu\nu} = 0$ for $r > 0$. The singularities can be determined from $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty$.

b.) The Kerr metric in Boyer-Lindquist coordinates is given by

$$\begin{aligned} \mathrm{d}s^2 = \left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{4Mar\sin^2\vartheta}{\rho^2}\mathrm{d}\phi\mathrm{d}t - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\vartheta^2 \\ - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\vartheta}{\rho^2}\right)\sin^2\vartheta\mathrm{d}\phi^2, \end{aligned} \tag{4}$$

$$a = J/M\,, \quad \rho^2 = r^2 + a^2\cos^2\vartheta = r^2 + \mathcal{O}(a^2), \quad \Delta = r^2 - 2Mr + a^2 = r^2(1 - 2M/r) + \mathcal{O}(a^2).$$

A short inspection shows that all corrections are of $\mathcal{O}(a^2)$ except for the linear term in

$$g_{t\phi} = \frac{4J\sin^2\vartheta}{r}\mathrm{d}\phi\mathrm{d}t + \mathcal{O}(a^2).$$

Moreover, neglecting the $\mathcal{O}(a^2)$ terms, all terms except $g_{t\phi}$ reduce to the ones in the Schwarzschild metric.

We will consider the movement of the gyroscope along the rotation (i.e. the z axis), where spherical coordinates are ill-defined. Therefore, we change to Cartesian coordinates. forming first the differential of $\phi = \arctan(y/x)$, obtaining

$$\mathrm{d}\phi = -\frac{y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y.$$

With $r^2\sin^2\vartheta = x^2 + y^2$, it follows

$$\sin^2\vartheta\mathrm{d}\phi = (x\mathrm{d}y - y\mathrm{d}x)/r^2.$$

Introducing the Newtonian potential $\Phi = GM/r$ and taking the limit $\Phi/c^2 \ll 1$, the metric becomes

$$\mathrm{d}s^2 = \left(1 - \Phi/c^2\right)(c\mathrm{d}t)^2 - \left(1 + \Phi/c^2\right)\mathrm{d}l^2 + \frac{4GJ}{(cr)^3}(x\mathrm{d}y - y\mathrm{d}x)c\mathrm{d}t, \tag{5}$$

where r is implicitely defined by $r = \sqrt{x^2 + y^2 + z^2}$.

c.) In the rest-frame of the gyroscope, it is $u^\alpha = (u^t, \mathbf{0})$ and $s^a = (0, \boldsymbol{s})$. Thus $\boldsymbol{u} \cdot \boldsymbol{s} = 0$ holds in the rest-frame and, since $\boldsymbol{u} \cdot \boldsymbol{s}$ is a scalar, in all frames.

The four-velocity is parallel transported along the geodesics of the gyroscope, $Du^\alpha/\mathrm{d}\tau = 0$. In order that $\boldsymbol{u} \cdot \boldsymbol{s} = 0$ continues to hold along the geodesics, also s^α has to be parallel transported. Thus

$$\frac{Ds^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}s^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha{}_{\mu\nu}s^\mu u^\nu = 0. \tag{6}$$

d.) With $u^\alpha = (u^t, 0, 0, u^z)$ and $s^z = 0$, it is

$$\frac{\mathrm{d}s^x}{\mathrm{d}\tau} + \Gamma^x{}_{xt}s^x u^t \Gamma^x{}_{xz}s^x u^z + \Gamma^x{}_{yt}s^y u^t + \Gamma^x{}_{yz}s^y u^z = 0 \tag{7}$$

and

$$\frac{\mathrm{d}s^y}{\mathrm{d}\tau} + \Gamma^y{}_{xt}s^x u^t \Gamma^y{}_{xz}s^x u^z + \Gamma^y{}_{yt}s^y u^t + \Gamma^y{}_{yz}s^y u^z = 0. \tag{8}$$

The Lagrangian on the z-axis, where $r = z$ and $\Phi = \Phi(z)$, is

$$L = \left(1 - \Phi/c^2\right)(c\dot{t})^2 - \left(1 + \Phi/c^2\right)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{4GJ}{(cz)^3}(x\dot{y} - y\dot{x})c\dot{t}. \tag{9}$$

The Φ terms would lead to $\mathcal{O}(1/c^2)$ corrections relative to the result neglecting them. Evaluating the x and y Lagrange equations for

$$L = (c\dot{t})^2 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{4GJ}{c^2 z^3}(x\dot{y} - y\dot{x})\dot{t} \tag{10}$$

gives then as the non-vanishing Christoffel symbols

$$\Gamma^x{}_{xt} = \frac{2GJ}{c^2 z^3} \quad \text{and} \quad \Gamma^y{}_{xt} = -\frac{2GJ}{c^2 z^3}. \tag{11}$$

e.) Inserting these Christoffel symbols gives

$$\frac{\mathrm{d}s^x}{\mathrm{d}\tau} = -\frac{2GJ}{c^2 z^3} s^x u^t \quad \text{and} \quad \frac{\mathrm{d}s^y}{\mathrm{d}\tau} = +\frac{2GJ}{c^2 z^3} s^x u^t. \tag{12}$$

Using then $u^t = \mathrm{d}t/\mathrm{d}\tau$, it follows

$$\frac{\mathrm{d}s^x}{\mathrm{d}t} = -\frac{2GJ}{c^2 z^3} s^x \quad \text{and} \quad \frac{\mathrm{d}s^y}{\mathrm{d}t} = +\frac{2GJ}{c^2 z^3} s^x. \tag{13}$$

Thus the gyroscope precesses on circle in the xy plane with the angular velocity

$$\omega_{\mathrm{LT}} = \frac{2GJ}{c^2 z^3}.$$

in the same direction as the Earth rotates. Our calculation was done in the frame of the gyroscope. A Lorentz transformation to the frame of an observer on Earth does not affect the transverse spin components s^x and s^y, while the time-dilatation effect gives a $1/c^2$ correction, i.e. is of the size as the already neglected Φ corrections.
Interpretation: Einstein developed GR partly under the influence of Mach's ideas. For an discussion of the interpretation of the Lense-Thirring effect within Mach's principle(s) see e.g. Ref. [1].

[1] Hermann Bondi and Joseph Samuel. The Lense-Thirring effect and Mach's principle. *Physics Letters A*, 228(3):121–126, 1997.