Formalities.
Solutions should be handed in Monday 07.10., latest 14.00, by Inspera. If Inspera doesn't work, put your solutions in my mailbox (in D5-166), sent them by email or hand them in in the lectures.

1. Scaling relation for main-sequence (MS) stars.
a.) Write down the four equations of stellar structure and the three "material relations", assuming radiative energy transfer (i.e. no convection), an ideal gas law, a power law for the energy generation rate $\propto \rho T^n$, and a constant opacity.
b.) Introduce the mass fraction $x = M(r)/M$ and characteristic quantities (i.e. characteristic radius R_*, pressure P_*, ...). Show that by introducing $r = f_1(x)R_*$, $P = f_2(x)P_*$, ..., you can split these equations into non-linear differential equations for the dimensionless functions and algebraic equations for the characteristic quantities.
c.) Derive from the algebraic equations the following scaling relations:

$$L \propto M^3 \qquad \text{and} \qquad R \propto M^{\frac{n-1}{n+3}}.$$

d.) Derive the slope of the MS in the Hertzsprung-Russel diagram for stars generating energy mainly by *i*) the pp chain ($n = 4$) and *ii*) the CNO cycle ($n = 16$).
e.) The minimmal temperature for the ignition of the pp chain is $T_{\rm min} = 4 \times 10^6\,$K. Show that the central temperature scales as $T_c \propto M^{4/(n+3)}$. Use the Sun to fix the proportionalty constant and derive the lower end of the MS.
f.) How does the expected life-time of MS stars scale?
g.) How does your results in d.) and e) compare to observations?

a.) We write the equations differentiating w.r.t. the enclosed mass $M(r) \equiv M_r$. Then

$$\frac{\mathrm{d}r}{\mathrm{d}M_r} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

$$\frac{\mathrm{d}P}{\mathrm{d}M_r} = -\frac{GM_r}{4\pi r^4} \tag{2}$$

$$\frac{\mathrm{d}T}{\mathrm{d}M_r} = -\frac{3\kappa}{4ac}\frac{L}{(4\pi r^2)^2 T^3} \tag{3}$$

$$\frac{\mathrm{d}L}{\mathrm{d}M_r} = \varepsilon_0 \rho T^n \tag{4}$$

$$P = \mathcal{R}/\mu \rho T \tag{5}$$

b.) Introducing characteristic quantities, it is

$$r = f_1(x)R_*, \qquad P = f_2(x)P_*, \qquad \rho = f_3(x)\rho_*,, \qquad T = f_4(x)T_*, \qquad L = f_5(x)L_*. \tag{6}$$

The starred quantities, $R_*, P_*, \ldots$, carry the dimension of the original functions. We consider the hydrostatic equilibrium equation as an example for this procedure: Introducing x and P_* leads to

$$\frac{P_*}{M}\frac{\mathrm{d}f_2}{\mathrm{d}x} = -\frac{GMx}{4\pi f_1^4 R_*^4}. \tag{7}$$

The unknown P_* has to be proportional to GM^2/R_*^4. Choosing the proportionalty constant as one, we obtain

$$\frac{\mathrm{d}f_2}{\mathrm{d}x} = -\frac{x}{4\pi f_1^4}, \qquad P_* = GM^2/R_*^4 \tag{8}$$

and similarly

$$\frac{\mathrm{d}f_1}{\mathrm{d}x} = \frac{1}{4\pi f_1^2 f_3}, \qquad \rho_* = M/R_*^3 \tag{9}$$

$$f_2 = f_1^3 f_4, \qquad T_* = \mu P_*/(\mathcal{R}\rho_*) \tag{10}$$

$$\frac{\mathrm{d}f_4}{\mathrm{d}x} = -\frac{3f_5}{4f_4^3(4\pi f_1^2)^2} \qquad L_* = \frac{ac}{\kappa}\frac{T_*^4 R_*^4}{M} \tag{11}$$

$$\frac{\mathrm{d}f_5}{\mathrm{d}x} = f_3 f_4^n, \qquad L_* = \varepsilon_0 \rho_* T_*^n M \tag{12}$$

c.) Using $P_* = GM^2/R_*^4$ and $\rho_* = M/R_*^3$ in $T_* = \mu P_*/(\mathcal{R}\rho_*)$, it follows

$$T_* = \frac{\mu GM}{\mathcal{R}R_*}. \tag{13}$$

Inserting this into (11) gives

$$L_* = \frac{ac}{\kappa}\frac{T_*^4 R_*^4}{M} = \frac{ac}{\kappa}\left(\frac{G\mu}{\mathcal{R}}\right)^4 M^3 \propto \mu^4 M^3\,. \tag{14}$$

Thus the luminosity at a given x in stars with the different mass scales as the cube of their mass ratios. In particular, for $x = 1$, we obtain $L \propto M^3$. Additionally, we see the μ^4 dependence as in the Eddington model.
Combine next the two relations for L_*,

$$L_* = \frac{ac}{\kappa}\,\frac{T_*^4 R_*^4}{M} = \varepsilon_0 \rho_* T_*^n M. \tag{15}$$

Inserting then $\rho_* = M/R_*^3$ and $T_* = \mu P_*/(\mathcal{R}\rho_*)$ gives

$$R_*^4 \propto \rho_* T_*^{n-4} M^2 \propto \frac{M}{R_*^3}\left(\frac{M}{R_*}\right)^{n-4} M^2 \tag{16}$$

or

$$R_* \propto M^{\frac{n-1}{n+3}}.$$

The relation holds again in particular for $x = 1$, implying $R \propto M^{\frac{n-1}{n+3}}$.

d.) We use in $L = 4\pi R^2 \sigma T_{\rm eff}^4$ first $R \propto M^{\frac{n-1}{n+3}}$ and then $M \propto L^{1/3}$, obtaining

$$L^{1-\frac{2n-2}{3n+9}} \propto T_{\rm eff}^4.$$

Taking the log, we find

$$\log L \simeq 5.5 \log T_{\rm eff} + \text{const.}, \qquad n = 4,$$

$$\log L \simeq 8.4 \log T_{\rm eff} + \text{const.}, \qquad n = 16.$$

e.) We use $T_* = \frac{\mu G M}{\mathcal{R} R_*}$ and $R_* \propto M^{\frac{n-1}{n+3}}$ to get $T_c \propto M^{4/(n+3)}$ at $x = 0$. For $n = 4$, it is $T_c \propto M^{4/7}$. Using the Sun for calibration and requiring $T_c > T_{\rm min}$, we obtain

$$\frac{M}{M_\odot} > \left(\frac{T_{\rm min}}{T_{c,\odot}}\right)^{7/4} \sim 0.1.$$

This gives $L_{\rm min} \simeq 10^{-3} L_\odot$ as lower end of the MS.

f.) Use e.g. the values given in the appendix of the script to plot the HR diagramm for the MS and compare to the slopes endpoint obtained.
You should find that our result deviates from reality: the main reason is the temperature dependence of the opacity.

g.) Neglecting the fact that the fraction of hydrogen converted into helium depends on convection, it is

$$\tau \propto \frac{M}{L} \propto M^{-2}$$

2. Chandrasekhar theory for white dwarf stars.
Assume that the pressure of white dwarf (WD) stars is given by completely degenerate (non-interacting) electrons. In the general case, where the relativity parameter $x = p_F/(mc)$ is neither zero or one, the E.o.S. is not a polytrope and the Lane-Emden equation has to be generalised. Chandrasekhar showed that writing the pressure and the density of degenerate electrons as function of the relativity parameter x as $P(x) = Af(x)$ and $\rho(x) = Bx^3$, one can derive

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}z}{\mathrm{d}r}\right) = -\frac{\pi G B^2}{2A}(z^2 - 1)^{3/2}. \tag{17}$$

with $z^2 = x^2 + 1$.
a.) Make this equation dimensionless, introducing $x_c \equiv x(r = 0)$ and $z_c \equiv z(r = 0)$, and new variables

$$r = \alpha \eta \quad \text{and} \quad z = z_c \phi$$

satisfying

$$\alpha^2 = \frac{2A}{\pi G}\frac{1}{(B z_c)^2} \quad \text{and} \quad z_c^2 = x_c^2 + 1$$

and show that it can be written as

$$\frac{1}{\eta^2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right) = -\left(\phi^2 - \frac{1}{z_c^2}\right)^{3/2}. \tag{18}$$

b.) Specifiy the boundary conditions.
c.) Solve this equation numerically for ten values of $1/z_0^2$ btween zero and one, find the resulting M-R relation of WDs, and plot it. Give a short interpretation: existence of minimal/maximal masses, how reliable is the result?

b.) The relation $y = y_c\phi$ implies $\phi = 1$ at the center. Moreover, as for $x = 0$, the derivative of ϕ has to be zero at the center. Finally, the radius of the star is defined by $\rho(R) = 0$ and thus $x(R) = 0$. Hence $z(R) = 1$ and $\phi(\eta(R)) = z/z_c = 1/z_c$.
Thus the integration starts with $\phi(0) = 1$, and $\phi'(0) = 0$ at the center $\eta = 0$ of the star, choosing a value of z_c, until one reaches the stellar radius $R = r(\eta_1)$ when $\phi = 1/z_c$.

c.) The stellar radius is given by $R = \alpha\eta_0$ or

$$R = \sqrt{\frac{2A}{\pi G}}\frac{1}{Bz_c}\eta_1,$$

where η_1 is defined by $\phi = 1/z_c$. As in the case of the "ordinary" Lane-Emden equation, we express first ρ by ϕ,

$$\rho(x) = Bx^3 = B(z^2-1)^{3/2} = Bz_c^3\left(\phi^2 - \frac{1}{z_c^2}\right)^{3/2}.$$

Then we replace r and ρ in

$$M = 4\pi\int_0^R \mathrm{d}r r^2\rho = 4\pi\alpha^3 Bz_c^3\int_0^{\phi_1}\mathrm{d}\eta\eta^2\left(\phi^2 - \frac{1}{z_c^2}\right)^{3/2}. \tag{19}$$

Then we replace $(\cdots)^{3/2}$ using the generalised Lane-Emden equation,

$$M = 4\pi\alpha^3 Bz_c^3\left(-\eta^2\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right)\Bigg|_{\eta_1} \tag{20}$$

In the last step, we insert the definition of α and z_c, obtaining

$$M = \frac{4\pi}{B^2}\left(\frac{2A}{\pi G}\right)^{3/2}\left(-\eta^2\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right)\Bigg|_{\eta_1}. \tag{21}$$

Now we can evaluate numerically M and R for a given value of z_c, see the table for some results.
Interpretation: There exist solutions between $M = 0$ and $M_{\mathrm{Ch}} \simeq 5.84M_\odot$. The upper limit corresponds to $x = 1$ (ultra-relativistic limit), which is unstable according to the virial theorem.
Reliability: Main assumption is that interactions can be neglected. This is not true in particular for low masses: For $M \to 0$, we expect $\rho \simeq$ const. instead of $\rho \to 0$ allowing for the existence of planets—their stability is caused by electrostatic interactions between electrons and ions. In the other extreme, $M \to M_{\mathrm{Ch}}$ implies $\rho \to \infty$ and at some point weak interactions may play a role.

Table 1: Numerical constants from the integration of the generalised Lane-Emden equation.

$1/z_c^2$	z_c	ξ_1	$-\eta^2\,(\mathrm{d}\phi/\mathrm{d}\eta)_{\eta_1}$	ρ_c/μ_e	$\mu_e^2 M\mathrm{g/cm^3}$	$\mu_e R/\mathrm{km}$
0	∞	6.70	2.02	∞	5.84	0
0.01	9.95	5.36	1.93	9.48e8	5.60	4.17
0.02	7	4.99	1.86	3.31e8	5.41	5.50
0.05	4.36	4.46	1.71	7.98e7	4.95	7.76
0.1	3	4.07	1.52	2.59e5	4.40	10.0
0.2	2	3.73	1.24	7.70e6	3.60	13.0
0.3	1.53	3.58	1.03	3.43e6	2.99	16.0
0.5	1	3.53	0.707	9.63e5	2.04	19.5
0.8	0.5	4.04	0.309	1.21e5	0.89	28.2
1	0	∞	0	0	0	∞