12.3

The Schwarzshild metric is given by

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\,. \tag{1}$$

We change variables as suggested by Hartle

$$t = v - r - 2M\log\left|\frac{r}{2M} - 1\right|\,, \tag{2}$$

the differential of this is

$$\mathrm{d}t = \mathrm{d}v + \left(\frac{2M}{2M-r} - 1\right)\mathrm{d}r = \mathrm{d}v + \frac{r}{2M-r}\mathrm{d}r = \mathrm{d}v - \frac{1}{1-2M/r}\mathrm{d}r\,. \tag{3}$$

Then we have

$$\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 = \left(1-\frac{2M}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r - \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2\,, \tag{4}$$

and this gives us the Schwarschild geometry in Eddington-Finkelstein coordinates:

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,. \tag{5}$$

12.4

We consider

$$\mathrm{d}s^2 = -\left(1-\frac{M}{r}\right)^2\mathrm{d}t^2 + \left(1-\frac{M}{r}\right)^{-2}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\,. \tag{6}$$

a) Motivated by the previous excercise we try the following change of variables

$$t = v - r - f(r), \tag{7}$$

and insert $[\mathrm{d}t = \mathrm{d}v - 1 - f'(r)\mathrm{d}r]$ into

$$\begin{aligned}
&-\left(1-\frac{M}{r}\right)^2\mathrm{d}t^2 + \left(1-\frac{M}{r}\right)^{-2}\mathrm{d}r^2 \\
&= -\left(1-\frac{M}{r}\right)^2\mathrm{d}v^2 + 2(1+f'(r))\left(1-\frac{M}{r}\right)^2\mathrm{d}v\mathrm{d}r \\
&\quad + \left[(1+f'(r))^2\left(1-\frac{M}{r}\right)^2 - \left(1-\frac{M}{r}\right)^{-2}\right]\mathrm{d}r^2
\end{aligned} \tag{8}$$

We want the last line above to vanish and therefore set

$$(1+f'(r))^2\left(1-\frac{M}{r}\right)^2 - \left(1-\frac{M}{r}\right)^{-2} = 0\,, \tag{9}$$

which leads to

$$f'(r) = \frac{r^2}{(r-M)^2} - 1 = \frac{r^2}{(r-M)^2} - \frac{r^2 - 2Mr + 2M^2 - M^2}{(r-M)^2} \tag{10}$$

$$= 4M\frac{r-M}{(r-M)^2} + \frac{M^2}{(r-M)^2} = 2\left(\frac{r}{M} - 1\right)^{-1} + \left(\frac{r}{M} - 1\right)^{-2} . \tag{11}$$

Integrating up, we find (we are free to choose the integration constant):

$$f(r) = 2M \log\left|\frac{r}{M} - 1\right| - M\left(\frac{r}{M} - 1\right)^{-1} . \tag{12}$$

Using this change of variables, the second term in the line-element becomes

$$2(1 + f'(r))\left(1 - \frac{M}{r}\right)^2 \mathrm{d}v\mathrm{d}r = 2\mathrm{d}v\mathrm{d}r\,. \tag{13}$$

At this point, it is thus clear that the geometry is not singular at $r = M$. The geometry is now specified by

$$\mathrm{d}s^2 = -\left(1 - \frac{M}{r}\right)^2 \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) . \tag{14}$$

12.5

For purely radial movement, we have according to Eq. (9.29) in Hartle:

$$\mathcal{E} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{M}{r} \tag{15}$$

At $r = R = 10M$, the observer is at rest, thus

$$\mathcal{E} = -\frac{1}{10}\,, \tag{16}$$

and we must therefore solve the equation

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2M}{r} - \frac{2M}{R}} \tag{17}$$

where the negative sign is chosen as we are dealing a fall towards the singularity. We write τ as the integral

$$\Delta\tau = -\int_R^0 \mathrm{d}r\,\frac{1}{\sqrt{\frac{2M}{r} - \frac{2M}{R}}} = \sqrt{\frac{R}{2M}}\int_0^R \mathrm{d}r\,\sqrt{\frac{r}{R-r}}\,. \tag{18}$$

We now make the substitution $r = R\sin^2(\theta)$ and get

$$\Delta\tau = \sqrt{\frac{R}{2M}}R\int_0^{\pi/2} \mathrm{d}r\,2\sin^2(\theta) = \frac{\pi}{2}\sqrt{\frac{R}{2M}}R = 5\sqrt{5}\pi M = 35.1M\,. \tag{19}$$

12.14

It is clear that the trajectory must be some form of geodesic, as this gives the extremal values of the proper time. Our starting point for this problem is Eq. (9.29) in Hartle

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \mathcal{E} + \frac{M}{r} - l^2\left(\frac{1}{2r^2} - \frac{M}{r^3}\right) . \tag{20}$$

As we are inside the Schwarschild radrius ($r < 2M$), the part proportional to l^2 is negative. Thus, with larger l the change in radial position increases, and we infer that the choice of $l = 0$ gives the longest time iside the black hole.

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \mathcal{E} + \frac{M}{r} . \tag{21}$$

Next, we wish to minimize $\mathcal{E} + \frac{M}{r}$, but our choice of $\mathcal{E}$ is restricted, as $\mathcal{E} = (e^2 - 1)/2$, and e is a real number. It is clear that $\mathcal{E} = -\frac{1}{2}$ is the best choice, as this makes the change in radial position as small as possible. This means that the observer begins at rest rest at $R = 2M$.

The situation is nearly identical with the case in problem 12.5, and we can follow the same steps in our calculation, and this leads to

$$\Delta\tau = \frac{\pi}{2}\sqrt{\frac{R}{2M}}R = \pi M \tag{22}$$

12.15

We will solve this problem by using energy-momentum conservation. The initial four momentum is given by p, the remaining four momentum is given by p_f, and the ejected four momentum is given by p_e.

$$p = p_e + p_f . \tag{23}$$

To get the most out of the ejected four-momentum, it should be lightlike. Thus

$$0 = p_e^2 = p \cdot p + p_f \cdot p_f - 2p_f \cdot p = -m^2 - m_f^2 - -2p_f \cdot p . \tag{24}$$

We now need to calculate the above product.

The four velocity of the hovering spaceship is stationary with respect to the spatial Schwarschild coordinates

$$u = \left(u^t, 0\right) . \tag{25}$$

Normalization of the four velocity, $u \cdot u = g_{tt}u^t u^t = -1$, gives us

$$\left(1 - \frac{2M}{R}\right)u_t^2 = -1 , \implies u_t = \frac{1}{\sqrt{1 - \frac{2M}{R}}} . \tag{26}$$

The four momentum is given by $p = mu$.

At this point, we see that we only need to the zero component of the four-momementum p_f. We utilize Eq. (9.21) in Hartle:

$$e = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} . \tag{27}$$

As we are interested in minimum energy, the particle should be at rest when $r \to \infty$, thus $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = 1$ and $e = 1$. This is a relativistic invariant, and we find that

$$\frac{1}{\left(1 - \frac{2M}{R}\right)} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = u_f^t(R)\,. \tag{28}$$

Then, the product $p_f \cdot p$ is given by

$$\begin{aligned} p_f \cdot p &= (mm_f)\, u_f \cdot u = g_{tt} u^t u_f^t \\ &= -(mm_f)\left(1 - \frac{2M}{r}\right) \frac{1}{\sqrt{1-\frac{2M}{R}}} \frac{1}{\left(1-\frac{2M}{R}\right)} = -(mm_f) \frac{1}{\sqrt{1-\frac{2M}{R}}} \end{aligned} \tag{29}$$

The fraction f is given by $m_f = fm$. Finally, Eq. (24) can be solved

$$f^2 - 2f \frac{1}{\sqrt{1-\frac{2M}{R}}} + 1 = 0\,, \tag{30}$$

and the solution is,

$$f = \frac{1 - \sqrt{2M/R}}{\sqrt{1 - 2M/R}}\,, \tag{31}$$

where the $-$ sign, comming from the $\pm$ arising when solving this second order equation, is chosen. This is to ensure that the fraction f does not exceed 1. This expression gives us the maximum fraction to escape to infinity.

To find the limit as $R \to 2M$, we can parameterize it by $1 - x = \frac{2M}{R}$, and let $x \to 0$. Then,

$$f = \lim_{x \to 0} \frac{1 - \sqrt{1-x}}{\sqrt{x}} = \lim_{x \to 0} \frac{1 - (1 - \frac{1}{2}x)}{\sqrt{x}} = \lim_{x \to 0} \sqrt{x} = 0 \tag{32}$$

So, nothing can escape to infinity if the starpoint is at the horizon.

12.22

b) The singularity is parameterized by

$$V = \sqrt{U^2 + 1}. \tag{33}$$

In this problem the particle is destroyed in the singularity when $U = 0$, thus $V = 1$. The starting position is described by the equation for the howering observer

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = \frac{1}{4} = U^2 - V^2\,. \tag{34}$$

Accoring to Eq. (12.17) in Hartle, the time t is given by

$$\tanh\left(\frac{t}{4M}\right) = \frac{U}{V}\,. \tag{35}$$

At $t = 0$, $V = 0$, therefore $U = \frac{1}{2}$, at the event when the observer leaves the spaceship. So the straight line in the Kruskal-Szerkes diagram is given by

$$V = 1 - 2U\,. \tag{36}$$

This means that $\left|\frac{\mathrm{d}V}{\mathrm{d}U}\right| = 2 > 1$, and therefore the observer is following a timelike trajectory.