5.1

$$\begin{aligned} a &= (-2,\, 0,\, 0,\, 1)\,, \\ b &= (5,\, 0,\, 3,\, 4)\,. \end{aligned} \tag{1}$$

a) To find out if a is timelike, spacelike, or null-like, we compute $a^2 = a \cdot a = a_\mu a^\mu$.

$$a^2 = \eta_{\mu\nu} a^\mu a^\nu = -(-2)^2 + 1^2 = -3 < 0\,, \tag{2}$$

thus a is timelike. For the case of b;

$$b^2 = -5^2 + 3^2 + 4^2 = -25 + 9 + 16 = 0\,, \tag{3}$$

so b is null-like (lightlike).

b)

$$a - 5b = (-2 - 25,\, 0,\, -5 \cdot 3,\, 1 - 5 \cdot 4) = (-27,\, 0,\, -15,\, -19) \tag{4}$$

c)

$$a \cdot b = \eta_{\mu\nu} a^\mu a^\nu = 10 + 4 = 14 \tag{5}$$

5.3

$$\frac{\mathrm{d}x}{\mathrm{d}t} = V\,, \quad x(0) = 0. \tag{6}$$

We wish to express x^μ as a function of the proper time τ. We use the relation

$$c^2(\mathrm{d}\tau)^2 = (c\mathrm{d}t)^2 - (\mathrm{d}x)^2\,, \tag{7}$$

and this gives us

$$\begin{aligned} c^2(\mathrm{d}\tau)^2 &= \left(c^2 - v^2\right)(\mathrm{d}t)^2 \\ &= \left(\frac{c^2}{v^2} - 1\right) v^2 (\mathrm{d}x)^2 \end{aligned} \tag{8}$$

so

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,, \tag{9}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\,. \tag{10}$$

$$\tag{11}$$

We integrate this up, and choose $t(\tau = 0) = 0$, in order to obtain:

$$t(\tau) = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\tau\,, \tag{12}$$

$$x(\tau) = \frac{v\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = v\gamma\tau\,. \tag{13}$$

5.4

The four-velocity is given by

$$u = (\gamma,\, \gamma \mathbf{V}) \; . \tag{14}$$

The four-acceleration is defined by

$$a = \frac{\partial u}{\partial \tau} \, . \tag{15}$$

We will now express this in terms of the three-velocity $\mathbf{V}$ and the three-acceleration $\mathbf{A} = \frac{\partial \mathbf{V}}{\partial t}$. The first step is to change derivation variable and then apply the product rule for differentiation:

$$a = \frac{\partial t}{\partial \tau} \frac{\partial u}{\partial t} = \gamma \left(\dot{\gamma},\, \dot{\gamma} \mathbf{V} + \gamma \mathbf{A} \right) \; . \tag{16}$$

We also need to calculate $\dot{\gamma}$:

$$\dot{\gamma} = \frac{\partial \gamma}{\partial t} = \frac{\partial}{\partial t} \left(1 - \mathbf{V}^2 \right)^{-1/2} = \gamma^3 \frac{-2 \mathbf{V} \cdot \mathbf{A}}{-2} = \gamma^3 \mathbf{V} \cdot \mathbf{A}, \tag{17}$$

and we are done.

Next, we will compute the inner product of a and u:

$$a \cdot u = -\gamma \dot{\gamma} + \dot{\gamma} \gamma \mathbf{V}^2 + \gamma^2 \mathbf{V} \cdot \mathbf{A} \, . \tag{18}$$

Insertation of the explicit expression for $\dot{\gamma}$, enables us to write

$$a \cdot u = \gamma^2 \mathbf{V} \cdot \mathbf{A} \left(-\gamma^2 + \gamma^2 \mathbf{V}^2 + 1 \right) = \gamma^2 \mathbf{V} \cdot \mathbf{A} \left[\gamma^2 \left(-1 + \mathbf{V}^2 \right) + 1 \right] = 0 \, . \tag{19}$$

The two four-vectors are orthogonal.

5.6

a) The velocity is given by

$$V_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{gt}{\sqrt{1 + g^2 t^2}} \tag{20}$$

As $|gt|$ is always less than $\sqrt{1 + g^2 t^2}$, the speed never exceeds 1; the speed of the light. The speed of the particle approaches this as $t \to \infty$.

b) According to Hartle, the four-velocity is given by

$$u = (\gamma,\, \gamma V_x \, \mathbf{i}) \, , \qquad \gamma = \frac{1}{\sqrt{1 - V_x^2}} \tag{21}$$

It would be nice to simplify this expression. To do this, we take a closer look at γ:

$$\gamma = \sqrt{\frac{1}{1 - V_x^2}} = \sqrt{\frac{1}{\frac{1+g^2t^2}{1+g^2t^2} - \frac{g^2t^2}{1+g^2t^2}}} = \sqrt{1 + g^2 t^2} \, . \tag{22}$$

The four-velocity can then be written

$$u = \left(\sqrt{1 + g^2 t^2},\, gt \, \mathbf{i} \right) \; . \tag{23}$$

c) Using the zero component of the four-velocity, we can derive an expression for the proper time τ:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{1+g^2t^2} \implies \int_{t_0}^{t} \frac{\mathrm{d}t'}{\sqrt{1+g^2t'^2}} = \int_{\tau_0}^{\tau} \mathrm{d}\tau' \,. \tag{24}$$

The above integral is a standard integral, which can be evaluated by substitution; set $t = \frac{1}{g}\sinh u$. (check this!) The result is

$$\tau = \frac{1}{g}\sinh^{-1}(gt)\,, \tag{25}$$

where we have defined a coordinates such that $t_0 = \tau_0 = 0$. The four-velocity can now be expressed in terms of the proper time:

$$u = (\cosh(g\tau),\, \sinh(g\tau)\mathbf{i}) \tag{26}$$

Next, we also wish to derive an expression for x as a function of the proper time. This is simple at this point, we apply the expression for the three-velocity:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \sinh(g\tau) \qquad \implies x = \frac{1}{g}\cosh(g\tau) + x_0\,. \tag{27}$$

Setting $x(0) = 0$, this becomes $x = \frac{1}{g}\left[\cosh(g\tau) - 1\right]$.

d) Using relation (5.41) in Hartle, we find the four-momentum :

$$p = mu = (m\cosh(g\tau),\, m\sinh(g\tau)\mathbf{i}) \tag{28}$$

The four-force is then

$$u = \frac{\mathrm{d}p}{\mathrm{d}\tau} = \left(\frac{m}{g}\sinh(g\tau),\, \frac{m}{g}\cosh(g\tau)\mathbf{i}\right)\,. \tag{29}$$

The three-force is given by

$$m\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = m\frac{g}{\sqrt{1+g^2t^2}}\,\mathbf{i} - \frac{g^3t^2}{\left(\sqrt{1+g^2t^2}\right)^3}\,\mathbf{i} = \frac{m}{g\gamma}\left(1 - \frac{g^2t^2}{\gamma^2}\right)\mathbf{i}\,. \tag{30}$$

5.7

We first look at the four-force exerted in the instantaneous rest frame (with propert time τ) of the particle. (we neglect the $\mathbf{i}$ in this excercise). It is given by

$$f' = (0, mg) \tag{31}$$

As this is a four-vector it can be Lorentz-transformed

$$f_\mu = L_{\mu\nu} f'_\nu\,, \tag{32}$$

and we get

$$\begin{aligned} f &= (-\sinh\theta,\, \cosh\theta mg) \\ &= (\gamma\frac{\mathrm{d}x}{\mathrm{d}t}mg,\, \gamma mg)\,. \end{aligned} \tag{33}$$

The generalization of Newton's law is

$$m\frac{\mathrm{d}u}{\mathrm{d}\tau} = f\,, \tag{34}$$

where $u = (\gamma, \gamma\frac{\mathrm{d}x}{\mathrm{d}t})$ is the four-velocity. For the spatial component this yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} = \gamma g\ . \tag{35}$$

We wish to solve this for x expressed in terms of τ, we should thus find an expression for γ, which does not depend on t:

$$\begin{aligned} \frac{1}{\gamma^2} &= 1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 1 - \frac{1}{\gamma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 \\ \Longrightarrow \gamma &= \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2} \end{aligned} \tag{36}$$

We now return to (35) and write $\frac{\mathrm{d}x}{\mathrm{d}\tau} = V$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V = g\sqrt{1+V^2} \implies \int \frac{\mathrm{d}V}{\sqrt{1+V^2}} = g\int \mathrm{d}\tau \tag{37}$$

This integral is straightforward to evaluate (using $v = \sinh z$), and the result is

$$v = \sinh g\tau\,, \tag{38}$$

where we have chosen τ such that $V(0) = 0$. Integration with respect to τ, gives us

$$x = \frac{1}{g}\left[\cosh(g\tau) - 1\right]\ . \tag{39}$$

Here we have chosen coordinates such that $x(0) = 0$. For γ we now have

$$\gamma = \sqrt{1 - V^2} = \cosh g\tau = \frac{\mathrm{d}t}{\mathrm{d}\tau}\,. \tag{40}$$

Integrating this, we get

$$t = \frac{1}{g}\sinh(g\tau) \tag{41}$$

The equation for the world-line $x(t)$ is found by plugging (41) into (39)

$$x = \frac{1}{g}\left[\sqrt{1+\sinh^2 g\tau} - 1\right] = x = \frac{1}{g}\left[\sqrt{g^2t^2+1} - 1\right]\ . \tag{42}$$

The world-line is plotted in Fig. 1.