8.2

a) A distance S on the sphere is given by

$$S = \int_1^2 a\sqrt{(\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\phi)^2} = \int_{\sigma_1}^{\sigma_2} a\sqrt{\left(\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2}\,\mathrm{d}\sigma\,. \tag{1}$$

We identify the Lagrangian:

$$L = a\sqrt{\left(\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2}\,. \tag{2}$$

The Euler-Lagrange equations (Eq. 8.10 in Hartle) give us

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\sin^2\theta\frac{1}{L}\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right] = 0\,, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\frac{1}{L}\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}\right] - \frac{1}{L}\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2 = 0\,. \tag{4}$$

As in the examples in Hartle, we use that

$$\frac{\mathrm{d}S(\sigma)}{\mathrm{d}\sigma} = L\,, \tag{5}$$

since this enables us to simplify our equations with a change of differentation variable

$$\frac{\mathrm{d}}{\mathrm{d}S}\left[\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}S}\right] = 0\,, \tag{6}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}S^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}S}\right)^2 = 0\,. \tag{7}$$

By performing the differentiation, these equations can be written in a form suited for identifying the Christoffel symbols. For the first one:

$$\sin^2\theta\frac{\mathrm{d}^2\phi}{\mathrm{d}S^2} + \sin\theta\cos\theta\frac{\mathrm{d}\phi}{\mathrm{d}S}\frac{\mathrm{d}\theta}{\mathrm{d}S} = 0 \tag{8}$$

$$\Longrightarrow \frac{\mathrm{d}^2\phi}{\mathrm{d}S^2} + \cot\theta\frac{\mathrm{d}\phi}{\mathrm{d}S}\frac{\mathrm{d}\theta}{\mathrm{d}S} = 0 \tag{9}$$

The second is already in the suited form. Using Eq. (8.14) in Hartle, we can read of the Christoffel symbols:

$$\Gamma^{\phi}_{\theta\phi} = \cot\theta\,, \quad \Gamma^{\phi}_{\theta\theta} = 0\,, \quad \Gamma^{\phi}_{\phi\phi} = 0\,, \tag{10}$$

and

$$\Gamma^{\theta}_{\theta\phi} = 0\,, \quad \Gamma^{\theta}_{\theta\theta} = 0\,, \quad \Gamma^{\phi}_{\phi\phi} = -\sin\theta\cos\theta\,. \tag{11}$$

b) To fully describe the movement of a particle we need to specify the initial velocity and position, we therefore choose a coordinate system (or starting position) such that $\phi(0) = 0$, $\theta(0) = \frac{\pi}{2}$, and $\dot{\theta}(0) = 0$, $\dot{\phi}(0) = \alpha$.

By inspection we, see that $\theta(S) = \frac{\pi}{2}$, is a solution to Eq. (7). Using this, Eq. (9) becomes

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}S^2} = 0\,, \quad \Longrightarrow \phi = \alpha S\,, \tag{12}$$

where we have imposed the initial conditions.

8.5

The metric is

$$g_{tt} = -1\,, \quad g_{rr} = 1\,, \quad g_{\theta\theta} = b^2 + r^2\,, \quad g_{\phi\phi} = (b^2 + r^2)\sin^2\theta\,, \tag{13}$$

and all other terms are zero. We see that all differentiation with respect to t and ϕ gives zero.

The general formula for the Christoffel symbols is

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)\,. \tag{14}$$

First, we set $\beta = t$, then the first and third term in Eq. (14) vanishes as g_{tt} is a constant, and all other contribution are zero due to the diagonal metric. The second term vanishes because no part of the metric depend on t. By the symmetry of the Christoffel symbols, it is the same for $\gamma = t$.

We also set $\delta = t$, and this means we must set $\alpha = t$ due to the diagonal metric. All terms vanish by the same argument as above. No nonzero Christoffel symbols refer to t.

Second, we set $\delta = r = \alpha$;

$$g_{rr}\Gamma^{r}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{r\beta}}{\partial x^{\gamma}} + \frac{\partial g_{r\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial r}\right)\,. \tag{15}$$

If $r = \beta$, this is zero, because g_{rr} is a constant, and this is the only part of the metric contributing. By symmetry, the same goes for $r = \gamma$.

Next we choose $\beta = \theta$. The two first terms in (15) vanish as the metric is diagonal, and the last term only contributes if $\gamma = \theta$. So,

$$g_{rr}\Gamma^{r}_{\theta\theta} = \Gamma^{r}_{\theta\theta} = -\frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial r} = -\frac{1}{2}(2r) = -r\,. \tag{16}$$

In a similar manner

$$\Gamma^{r}_{\phi\phi} = -\frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r} = -\frac{1}{2}(-2r\sin^2\theta) = -r\sin^2\theta\,. \tag{17}$$

Third, we set $\delta = \theta = \alpha$;

$$g_{\theta\theta}\Gamma^{\theta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\theta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\theta\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial \theta}\right)\,. \tag{18}$$

For $r = \beta$, the first term will not contribute as the metric is diagonal, and the last term will not contribute as g_{rr} is a constant. The only contribution comes when $\gamma = \theta$. Thus,

$$g_{\theta\theta}\Gamma^{\theta}_{r\theta} = \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial r} = r\,, \tag{19}$$

and

$$\Gamma^{\theta}_{r\theta} = \frac{r}{g_{\theta\theta}} = \frac{r}{b^2 + r^2} = \Gamma^{\theta}_{\theta r}\,. \tag{20}$$

The last equality comes from the symmetry of the lower indices. When setting $\beta = \theta$, the only unexplored option is $\gamma = \phi$. Differentiation with respect to ϕ vanish for the first term, the two last vanish as the metric is diagonal.

For $\beta = \phi$, we have

$$g_{\theta\theta}\Gamma^{\theta}_{\phi\gamma} = -\frac{1}{2}\frac{\partial g_{\phi\gamma}}{\partial\theta}\,. \tag{21}$$

Thus,

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{g_{\theta\theta}}\frac{1}{2}(b^2+r^2)2\sin\theta\cos\theta = -\sin\theta\cos\theta\,, \tag{22}$$

the other terms vanish.

Fourth, and finally, we set $\delta = \phi = \alpha$;

$$g_{\phi\phi}\Gamma^{\phi}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\phi\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\phi\gamma}}{\partial x^{\beta}}\right) \tag{23}$$

If $\beta \neq \phi$, then $\gamma = \phi$, for non vanishing contributions, also $\beta = \gamma = \phi$ vanish. If $\beta = r$;

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{g_{\phi\phi}}\frac{1}{2}\left(\frac{\partial g_{\phi\phi}}{\partial r}\right) = \frac{r}{b^2+r^2}\,. \tag{24}$$

If $\beta = \theta$;

$$\Gamma^{\phi}_{\theta\phi} = \frac{1}{g_{\phi\phi}}\frac{1}{2}\left(\frac{\partial g_{\phi\phi}}{\partial\theta}\right) = \frac{\sin\theta\cos\theta}{\sin^2\theta} = \cot\theta\,. \tag{25}$$

8.9

The Lagrangian for the geodesic is given by

$$L = \sqrt{-X^2\left(\frac{\mathrm{d}T}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}X}{\mathrm{d}\sigma}\right)^2}\,. \tag{26}$$

Using the Euler-Lagrange equations (Eq. 8.10 in Hartle), and the fact that $L = \frac{\mathrm{d}\tau}{\mathrm{d}\sigma}$, we get the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(X^2\frac{\mathrm{d}T}{\mathrm{d}\tau}\right) = 0 \tag{27}$$

$$\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} + X\left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)^2 = 0 \tag{28}$$

The first line gives us

$$\frac{\mathrm{d}T}{\mathrm{d}\tau} = \frac{A}{X^2}\,. \tag{29}$$

It is often useful to use the conservation law $u \cdot u = -1$;

$$-X^2\left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}X}{\mathrm{d}\tau}\right)^2 = -1\,. \tag{30}$$

We insert Eq (29) into this and obtain

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} = \pm\sqrt{\frac{A^2}{X^2} - 1}\,. \tag{31}$$

This can be integrated (absorbing the constant in τ):

$$\tau = \pm \int \frac{X\,\mathrm{d}X}{\sqrt{A^2 - X^2}} = \pm\sqrt{A^2 - X^2}\,, \tag{32}$$

then

$$X = \pm\sqrt{A^2 - \tau^2}\,. \tag{33}$$

To find an equation for T, we insert this into 29, and obtain

$$T = \int \mathrm{d}\tau\, \frac{A}{A^2 - \tau^2}\,. \tag{34}$$

We try the substitution $\tau = A\tanh u$:

$$\frac{\mathrm{d}\tau}{\mathrm{d}u} = A\left(1 - \tanh^2 u\right)\,. \tag{35}$$

and we find that $T + B = u$, so that $\tau = A\tanh(T)$. Thus,

$$X = \pm\sqrt{A^2 - \tau^2} = \frac{A'}{\cosh(T+B)} \tag{36}$$

To show that this is timelike, we compute

$$\mathrm{d}S^2 = -X^2\mathrm{d}T^2 + \mathrm{d}X^2\,. \tag{37}$$

We first look at

$$\mathrm{d}X = \mathrm{d}T\,\frac{\mathrm{d}}{\mathrm{d}T}\frac{A'}{\cosh(T+B)} = (A')\frac{\sinh(T+B)}{\cosh^2(T+B)}\mathrm{d}T\,, \tag{38}$$

and this gives us

$$\mathrm{d}S^2 = (A')^2\left(-\frac{\cosh^2(T+B)}{\cosh^4(T+B)} + \frac{\sinh^2(T+B)}{\cosh^4(T+B)}\right)\mathrm{d}T^2 = -A'^2\frac{1}{\cosh^4(T+B)} < 0\,, \tag{39}$$

which shows that it is timelike.

8.12

a) We investigate the metric

$$\mathrm{d}S^2 = \frac{1}{y^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)\,. \tag{40}$$

A given point on the x-axis is given by $a = (x_0, 0)$ and given point on the upper half-plane is given by $b = (x_1, y_1)$. The distance between these two point is given by

$$l = \int_a^b \mathrm{d}S = \int_a^b \frac{1}{y}\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_0^{y_1} \frac{1}{y}\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y\,. \tag{41}$$

It is obvious that

$$l > \int_0^{y_1} \mathrm{d}y\,\frac{1}{y}\,, \tag{42}$$

and this is logarithmic divergent. Therefore the distance between any point on the x-axis to any point on the positive half-plane is infinite.

b) Using Euler-Lagrange equations and $1 = u \cdot u = \frac{1}{y^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}S}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}S}\right)^2\right]$, we find

$$\frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{1}{y^2}\frac{\mathrm{d}x}{\mathrm{d}S}\right) = 0\,, \tag{43}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}S}\right) + \frac{1}{y} = 0\,. \tag{44}$$

c) First of all, it is cleat that $x = \text{const}$ is a solution to Eq. (43). From $u \cdot u = 1$;

$$\frac{\mathrm{d}y}{\mathrm{d}S} = \pm y\,. \tag{45}$$

The solution is $y = Ae^{\pm S}$, where A is some constant, and therefore y can take any value. We have found the vertical lines.

Eq. (43) can also be written

$$\frac{\mathrm{d}x}{\mathrm{d}S} = By^2\,, \tag{46}$$

this is found by integrating on both sides. B is some constant. Inserting this into $u \cdot u = 1$, we get

$$B^2y^4 + \left(\frac{\mathrm{d}y}{\mathrm{d}S}\right)^2 = y^2\,, \tag{47}$$

changing variables, this can also be written

$$\begin{aligned} B^2y^4 + (By^2)^2\left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2\left(\frac{\mathrm{d}y}{\mathrm{d}S}\right)^2 &= y^2 \\ B^2y^2 + B^2y^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 1\,, \end{aligned} \tag{48}$$

which gives us

$$\int \frac{By}{\sqrt{1 - B^2y^2}}\mathrm{d}y = \int \mathrm{d}x\,. \tag{49}$$

This integration is straightforward, and the result is

$$-\frac{1}{B}\sqrt{1 - B^2y^2} = x - x_0\,, \tag{50}$$

where x_0 is some integration constant. Squaring on both sides yields the equation for a circle cantered on the x-axis:

$$1 = B^2(x - x_0)^2 + B^2y^2\,. \tag{51}$$

d) For the vertical lines, we found the solutions in **c)** . For the circles it is natural to begin with (46):

$$\frac{\mathrm{d}x}{\mathrm{d}S} = By^2 = 1 - B^2(x - x_0)^2\,, \tag{52}$$

which gives us

$$\begin{aligned} S &= \int \mathrm{d}x\, \frac{1}{1 - B^2(x - x_0)^2} = \frac{1}{2} \int \mathrm{d}x \left(\frac{1}{1 - B(x - x_0)} + \frac{1}{1 + B(x - x_0)} \right) \\ &= \frac{1}{2}\frac{1}{B} \ln\left(\frac{1 + B(x - x_0)}{1 - B(x - x_0)} \right)\,, \end{aligned} \tag{53}$$

solving this for x yields

$$x = x_0 + \frac{1}{B}\frac{e^{(2BS)} - 1}{e^{(2BS)} + 1}\,. \tag{54}$$

In turn this gives us

$$y = \frac{1}{B}\sqrt{1 - \left(\frac{e^{(2BS)} - 1}{e^{(2BS)} + 1} \right)^2} = \frac{1}{B}\sqrt{\frac{4e^{2BS}}{(e^{2BS} + 1)^2}} = \frac{2}{B}\frac{e^{BS}}{e^{2BS} + 1}\,. \tag{55}$$

This result is in accordance with **a**). Only when $S \to \pm\infty$ is $y = 0$.