9.1

To determine the coordinate r, we use Eq. (9.5) in Hartle:

$$r = \sqrt{A/(4\pi)} \tag{1}$$

This tells us that $r_{max} = 10M$ and $r_{min} = 6M$. The spatial part of the metric is, after setting $\mathrm{d}\theta = \mathrm{d}\phi = 0$:

$$\mathrm{d}S^2 = \frac{1}{1 - \frac{2M}{r}} \mathrm{d}r^2 \,. \tag{2}$$

The physical thickness is then

$$a = \int_{6M}^{10M} \mathrm{d}r \sqrt{\frac{1}{1 - \frac{2M}{r}}} \,. \tag{3}$$

We have seen a similar integral previously in exercise 7.18. Following the same steps, we end up with

$$\begin{aligned} a &= M\left[\sinh(2y) + 2y\right] \Big|_{\cosh^{-1}(\sqrt{3})}^{\cosh^{-1}(\sqrt{5})} \\ &= M4\left(\sqrt{6} - \sqrt{5}\right) + 2M\left(\cosh^{-1}(\sqrt{5}) - \cosh^{-1}(\sqrt{3})\right) \,. \end{aligned} \tag{4}$$

9.6

We start with Eq. (9.29) in Hartle ($l = 0$):

$$\mathcal{E} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{M}{r} \tag{5}$$

Since the particle has zero momentum at infinity $\mathcal{E} = 0$, thus

$$\mathrm{d}r\sqrt{r} = \pm\sqrt{2M}\mathrm{d}\tau \,, \tag{6}$$

Therefore the time it takes for the observer to pass between $6M$ and $3M$ is

$$-\sqrt{2M}\Delta\tau = \int_{6M}^{2M} \mathrm{d}r\,\sqrt{r} \,, \tag{7}$$

where the negative sign is chosen to get a positive time. The result is

$$\Delta\tau = \sqrt{\frac{1}{2M}}\frac{2}{3}\left[(6M)^{3/2} - (2M)^{3/2}\right] = 4M\left(\sqrt{3} - \frac{1}{3}\right) \,. \tag{8}$$

9.7

In this exercise we will calculate the speed at which the particle pass by a stationary observer at $r = 6M$, we will do this by calculating the energy measured by the observer, as $E = -p \cdot u_{obs}$. First, to find the four-velocity of the observer, we note the observer is spatially at rest, and use the fact that

$$u_{obs} \cdot u_{obs} = g_{tt}\left(u^0_{obs} u^0_{obs}\right) = -1 \,, \tag{9}$$

and we find

$$u_{obs} = \left(\frac{1}{\sqrt{1-\frac{2M}{6M}}}, 0 \right) = \left(\sqrt{3/2}, 0\right) . \tag{10}$$

Second, to find the zero component of the four-momentum of the paricle we use Eq. (9.21) in Hartle:

$$e = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}, \qquad \Longrightarrow \qquad m\frac{3e}{2} = mu_0 = p_0 . \tag{11}$$

Then the observed energy is

$$E = -p \cdot u_{obs} = me\sqrt{3/2} \tag{12}$$

Finally, we use the relation $E = \gamma m$ to find the speed of the particle

$$\frac{1}{1-v^2} = e^2\frac{3}{2}, \qquad \Longrightarrow \qquad v = \sqrt{1 - \frac{2}{3e^2}} \tag{13}$$

The first particle, $e = 1$, has the speed $v_1 = \frac{1}{\sqrt{3}}$, while the second $e = 2$ has $v_2 = \sqrt{\frac{5}{6}}$, and therefore $\frac{v_2}{v_1} = \sqrt{\frac{5}{2}}$.

9.8

a) The angular velocity is given by Eq. (9.46) in Hartle

$$\Omega = \sqrt{\frac{M}{r^3}} = \frac{1}{7^{3/2}}\frac{1}{M}, \tag{14}$$

this gives us the period ($2\pi = \Delta T\Omega$):

$$\delta T = 14\pi\sqrt{7}M \tag{15}$$

If we know l, we can calculate the angular speed by use of Eq. (9.22) in Hartle. To find the l wich corresponds to a circular orbit, we differentiate the effective potential:

$$\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}r} = \frac{M}{r^2} - l^2\frac{1}{r^3} - \frac{3M}{r^4} = 0 \tag{16}$$

This can be rearranged into

$$l^2 = \frac{Mr^2}{r-3M} = \frac{77^2}{4}M^2 \qquad \Longrightarrow \qquad l = \sqrt{7}\frac{7}{2}M . \tag{17}$$

We can calculate the angular velocity. We assume that $\theta = \pi/2$, so that $\dot{\theta} = 0$. (A choice of coordinate system). Then,

$$l = r^2\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 7^2M^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}, \tag{18}$$

so

$$\frac{\sqrt{7}}{14M} = \frac{\mathrm{d}\phi}{\mathrm{d}\tau}, \tag{19}$$

and we find the period as measured by the observer

$$\delta\tau = \frac{2\pi}{\frac{\mathrm{d}\phi}{\mathrm{d}\tau}} = 4\pi\sqrt{7}M . \tag{20}$$

9.9

Actually, most of the steps in this excercise has been performed in the previous excercise. We use that (assuming $\theta = \frac{\pi}{2}$)

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{l}{r^2}\,, \tag{21}$$

and the condition that we are in the minimum of the effective potential (hence circular orbits);

$$l^2 = \frac{Mr^2}{r-3M}\,, \tag{22}$$

and we obtain

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{1}{r}\sqrt{\frac{M}{r-3M}} = \left(\frac{M}{r^3}\right)^{1/2}\frac{1}{\sqrt{1-3M/r}}\,. \tag{23}$$

9.10

We solve this exercise with the same approach as in exercise 9.7. We first find the four-velocity of the stationary observer

$$u_{obs}\cdot u_{obs} = g_t t\left(u^0_{obs}u^0_{obs}\right) = -1\,, \quad \Longrightarrow \quad u^0_{obs} = \frac{1}{\sqrt{1-2M/r}}\,. \tag{24}$$

The four-velocity of the particle is

$$p = m\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, 0, 0\,, \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)\,. \tag{25}$$

Then $E = p \cdot u_{obs}$ yields

$$E = m\sqrt{1-2M/r}\frac{\mathrm{d}t}{\mathrm{d}\tau}\,. \tag{26}$$

To find $\frac{\mathrm{d}t}{\mathrm{d}\tau}$, we use that $\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\Omega$. Using the result of the previous exercises we find that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-3M/r}}\,, \tag{27}$$

so

$$E = \gamma m = m\sqrt{\frac{r-2M}{r-3M}}\,, \tag{28}$$

and this gives us

$$\frac{1}{\gamma^2} = 1 - v^2 = \frac{r-3M}{r-2M}\,. \tag{29}$$

Solving this for the linear velocity we obtain

$$v^2 = 1 - \frac{r-3M}{r-2M} = \frac{M}{r-2M}\,. \tag{30}$$

At ISCO, $r = 6M$:

$$v_{\mathrm{ISCO}} = \frac{1}{2}\,. \tag{31}$$