Chapter 1

Stars

1.1 Generalities

This book was not meant to be about stars. But stars are the most familiar, best studied, and arguably most important objects in the astrophysicist's universe. They are therefore the building blocks of many theories of more exotic objects. More fundamentally, the study of stars is the study of the competition between gravity and pressure. Astrophysics is distinguished from nearly all of the rest of physics by the importance of gravity, so that an understanding of the principles of stellar structure is necessary in order to understand most other astronomical objects.

The study of stellar structure and evolution is an elaborate and mature subject. The underlying physical principles are mostly well-known, and have been developed in great detail. Powerful numerical methods produce quantitative results for the properties and evolution of stars. Numerous texts and a very extensive research literature document this field. I refer the reader to three standard texts; although not new they have aged very well, and it would be both pointless and presumptuous to attempt to improve on them. Chandrasekhar (1939) reviews the classical mathematical theory of stellar structure, whose beginnings are now more than a century old. Schwarzschild

(1958) presents a less mathematical description of the physical principles of stellar structure and evolution, with more attention to the observed phenomenology. This is probably the best book for a general introduction to the properties of stars and their governing physics. I recommend it (supplemented by any of the numerous recent descriptive astronomy books) as a reference for the physicist without astronomical background. Clayton (1968) is particularly concerned with processes of nucleosynthesis and thermonuclear energy generation.

There are still a number of outstanding problems in the theory of ordinary stars. Many of these arise from a single area of theoretical difficulty: the problem of quantitatively describing turbulent flows. This problem arises in the formation of stars from diffuse gas clouds, in stellar atmospheres, for rotating stars and accretion discs (which may be thought of as the limiting case of rapidly rotating stars), in interacting binary stars, in stars with surface abundance anomalies, and in stellar collapse and explosion. If turbulent flows have a material effect on the properties of a star, quantitative theory must usually be supplemented by rough approximations, and confident calculation becomes uncertain and approximate phenomenology. This is even more true of the more exotic objects which are the subject of this book.

The problems of turbulent flow appear in two distinct forms. In the first form, a turbulent flow arises in an otherwise well-understood configuration, and may even resemble the turbulent flows known to hydrodynamicists; the problem is the calculation of some property, usually an effective transport coefficient, of the flow. The most familiar example of this is turbulent convection in the solar surface layers. In the second form, the initial or boundary conditions of a flow are not known; it may not be turbulent in the hydrodynamicist's sense of eddies or nonlinear wave motion on a broad range of length scales, but quantitative calculation is still impossible. The formation of stars is an example of this kind of flow. A variety of assumptions, approximations, and models, generally of uncertain validity and unknown accuracy, are used to study turbulent flows in astrophysics.

This chapter on stars has two purposes. One is to illustrate some of those physical principles of stellar structure which are useful in understanding stars and other astrophysical objects. The other is to develop the kind of rough (often order-of-magnitude) estimates and dimensional analysis which are widely used in modelling novel astrophysical phenomena. Some of this material follows Schwarzschild (1958).

1.2 Phenomenology

Hundreds of years of observations of stars have produced an enormous body of data and revealed a wide variety of phenomena which are discussed in numerous texts and monographs and a voluminous research literature. Here we will summarize only the tiny fraction of those data essential to the astrophysicist who wishes to use stars in models of high energy astrophysical phenomena.

The luminosities and surface temperatures of stars are often described by their place on a Hertzsprung-Russell diagram, such as that shown in Figure 1.1. In this theoretician's version the abscissa is the stellar effective surface temperature T_e, defined as the temperature of a black body which radiates the same power per unit area as the actual stellar surface; the ordinate is the stellar photon luminosity in units of the Solar luminosity $L_\odot = 3.9 \times 10^{33}$ erg/sec. There are also observers' versions in which the abscissa is a "color index," a directly observable measure of the spectrum of the emitted radiation, and the ordinate may be the absolute or apparent stellar magnitude in some observable part of the spectrum. Accurate conversion between these

Figure 1.1. Hertzsprung-Russell diagram.

two versions requires a quantitative knowledge of the spectrum of emitted radiation, which is approximately (but not exactly) that of a black body.

Most stars are found to lie on a narrow strip called the main sequence. These stars (occasionally referred to as dwarves) produce energy by the thermonuclear transmutation of hydrogen into helium near their centers. Their positions along the main sequence are determined by their masses, which vary monotonically from about $30M_\odot$ (where the solar mass $M_\odot = 2 \times 10^{33}$ gm) at the upper left to $0.1M_\odot$ in the lower right. The Sun lies on the main sequence near its middle.

Stars found above and to the right of the main sequence are called giants and supergiants; their higher luminosities (and their names) are accounted for by large radii, ranging in extreme cases up to 10^{14} cm, about 1000 times that of the Sun. These stars have exhausted the hydrogen at their centers and produce energy by thermonuclear reactions in shells close to, but outside, their centers. Stars of nearly equal ages (such as the members of a single cluster of stars, formed nearly simultaneously) will be distributed along a narrow track in the giant and supergiant region, a track whose form reflects their complex evolutionary path. Stars of a broad range of ages, such as the totality of stars in the solar neighborhood, will mostly be found on the main sequence; those in the giant and supergiant regions will be broadly distributed rather than lying on a narrow track. There are no sharp distinctions among main sequence (dwarf) stars, giants, and supergiants, and intermediate cases are found.

Degenerate (traditionally called white) dwarves are faint, dense stars in whose interiors the electrons are Fermi-degenerate, resembling the state of an ideal metal or metallic liquid. They generally produce negligible thermonuclear energy, having converted essentially all their hydrogen (and probably also their helium) to heavier elements. Their meager luminosity is supplied by their thermal energy content, possibly augmented by the latent heat of crystallization, the gravitational energy released by the sedimentation of their heavier elements, and other minor sources. They cool steadily as these energy sources are exhausted. Degenerate dwarves move to the lower right along a track parallel to lines of constant radius as they cool. Their radii depend on their masses (roughly as their reciprocals), but because their masses are believed to span a moderate range (perhaps $0.4M_\odot$ to $1.2M_\odot$) they all lie in a strip of moderate width. These masses are less than those which these stars had when young, but the amount of mass lost is controversial and may range from a few percent of to nearly all the initial mass. It is not known whether the mass in the degenerate dwarf stage is a monotonic function of or even determined by the mass at birth; it may be random and unpredictable. Very few stars other than degenerate dwarves are found much below and to the left of the main sequence; most of these few are probably evolving rapidly into degenerate dwarves.

An extrapolation of the main sequence to the lower right leads to stars of mass too low to produce thermonuclear energy, generally called brown dwarves. These objects slowly evolve into degenerate dwarves of very low mass and lie near (but above, because of their low masses) an extrapolation of the degenerate dwarf strip. Jupiter may be regarded as an extreme case. These objects are nearly unobservable because of their low luminosities, and only a few, if any, can be identified with confidence. Their properties are uncertain because the properties of matter under brown dwarf conditions are not well known; few data are available to test the uncertain calculations.

Objects at the upper left end of the main sequence are very rare, with their rarity increasing with increasing mass and luminosity. As a consequence, extrapolation beyond masses of $50M_\odot$ is largely limited to theory.

1.3 Equations

A star may be defined as a luminous self-gravitating gas cloud. If it is also spherical, in hydrostatic equilibrium, and in thermal steady state it is described by the classical equations of stellar structure:

$$\frac{dP(r)}{dr} = -\frac{\rho(r)GM(r)}{r^2} \tag{1.3.1}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \tag{1.3.2}$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r) \tag{1.3.3}$$

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)L(r)}{16\pi acT^3(r)r^2}. \tag{1.3.4}$$

Here $P(r)$ is the pressure, $M(r)$ is the mass enclosed by a sphere of radius r, $\rho(r)$ is the density, $L(r)$ is the luminosity produced within a sphere of radius r, $\epsilon(r)$ is the rate of nuclear energy release per gram, $T(r)$ is the temperature, $\kappa(r)$ is the Rosseland mean opacity (defined in **1.7.2**) in $\mathrm{cm}^2/\mathrm{gm}$, and a is the radiation constant. The first three of these equations are elementary; (1.3.4) is derived in **1.7**.

Numerous assumptions and approximations have been made: spherical symmetry, Newtonian gravity, a star in a stationary (unchanging) state, and a flow of energy by the diffusion of radiation only. Various of these assumptions may be relaxed if the equations are appropriately modified. It is frequently necessary to allow for the transport of energy by turbulent convection (most familiarly, in the outer layers of the Sun) or by conduction (in electron-degenerate matter).

These equations must be supplemented by three constitutive relations, derived from the microscopic physics of the stellar material. For any given chemical composition they take the form:

$$P = P(\rho, T) \tag{1.3.5}$$

$$\epsilon = \epsilon(\rho, T) \tag{1.3.6}$$

$$\kappa = \kappa(\rho, T). \tag{1.3.7}$$

These equations of stellar structure may be solved numerically, which is necessary to obtain quantitative results. It is illuminating, however, to make order-of-magnitude estimates. If we did not have computers available (and were unwilling to integrate these equations numerically by hand), or did not know the quantitative form of the constitutive relations, these rough estimates would be the best that we could do. Until the development of quantitative theories of thermonuclear reactions and opacity, no detailed calculation was possible. Even today, rough estimates are the basis of most qualitative understanding. In novel circumstances they are the first step toward building a quantitative model.

1.4 Estimates

1.4.1 Order of Magnitude Equations In order to make rough approximations to the differential equations (1.3.1–4) we replace them by algebraic equations in which the variables P, M, L, and T represent their mean or characteristic values in the star, the continuous variable r is replaced by the stellar radius R, and the derivative d/dr is replaced by the multiplicative factor $1/R$. In most cases this level of approximation produces useful rough results, although it is occasionally disastrous; with intelligent choice of the numerical constants it can be remarkably accurate, though usually only when a quantitative solution is available as a guide.

The equations become:

$$P = \rho \frac{GM}{R} \tag{1.4.1}$$

$$M = \frac{4}{3}\pi R^3 \rho \tag{1.4.2}$$

$$L = \frac{4}{3}\pi R^3 \rho \epsilon \tag{1.4.3}$$

$$T^4 = \frac{3\kappa \rho L}{16\pi acR}. \tag{1.4.4}$$

We now assume the perfect nondegenerate gas constitutive relation for pressure

$$\begin{aligned} P &= P_g + P_r \\ &= \frac{\rho N_A k_B T}{\mu} + \frac{aT^4}{3}, \end{aligned} \tag{1.4.5}$$

where P_g and P_r are the gas and radiation pressures respectively, μ is the mean molecular weight (the number of atomic mass units per free particle), N_A is Avogadro's number per gram, k_B is Boltzmann's constant, and a is the radiation constant. Combination of (1.4.1), (1.4.2), and (1.4.5) (ignoring the radiation pressure term in 1.4.5, an excellent approximation for stars like the Sun) yields results for the characteristic values of ρ, P, and T:

$$\rho = \frac{3M}{4\pi R^3} \tag{1.4.6}$$

$$P = \frac{3GM^2}{4\pi R^4} \tag{1.4.7}$$

$$T = \frac{GM}{R}\frac{\mu}{N_A k_B}. \tag{1.4.8}$$

1.4.2 Application to the Sun In Table 1.1 we compare the numerical estimates for ρ, P, and T obtained by substituting the solar mass, radius, and molecular weight, to the quantitative values found for the center of the Sun in a numerical integration (Schwarzschild 1958) of the equations (1.3.1)–(1.3.7). More recent calculations (Bahcall, *et al.* 1982) produce slightly different numbers, but the difference is of no importance when we are examining the validity of order-of-magnitude estimates. We use $R = 6.95 \times 10^{10}$ cm, $M = 2 \times 10^{33}$ gm, and $\mu = 0.6$.

The estimated value of T is remarkably accurate (probably fortuitously so), while the estimates of ρ and P are low by two orders of

Table 1.1

	Estimate	Solar Center
ρ (gm/cm^3)	1.42	134
P (dyne/cm^2)	2.73×10^{15}	2.24×10^{17}
T (°K)	1.39×10^7	1.46×10^7
κ (cm^2/gm)	2.18×10^3	1.07
ϵ (erg/gm/sec)	1.95	14

magnitude (note that the estimated ρ is nothing more than the mean stellar density). This large discrepancy reflects the concentration of mass towards the center of a star, and is a consequence of the compressibility of gases and the inverse-square law of Newtonian gravity. The discrepancy also reflects a deliberate obtuseness on our part in comparing the estimated values of ρ and P to the calculated central values. Had we been more cunning we could have chosen to compare to a suitable chosen "mean" point in the numerical integration, and would have obtained truly impressive (but deceptive) agreement.

In all stars the central density greatly exceeds the mean density. In stars of similar structure this ratio is nearly constant, and the greatest use of eqs. (1.4.6–8) is as scaling relations among stars of differing mass and radius. Rough estimates and qualitative understanding may be obtained readily; numerical integrations are always possible when quantitative results are needed.

For giant and supergiant stars the ratio of central to mean density may be as much as 10^{16}. Such enormous ratios indicate a complete breakdown of the approximations (1.4.1–4); the interior structure of such stars is very different from that of stars like the Sun; it can be roughly described by simple relations, but requires an understanding of their peculiar structure. In fact, their condensed central cores and very dilute outer layers may each be separately described by equations (1.4.6–8) with reasonable accuracy; disaster strikes only when one attempts to describe both these regions together.

Equations (1.4.3) and (1.4.4) may also be used to estimate κ and ϵ given the estimates for ρ, P, and T. For the Sun we use $L = 3.9 \times 10^{33}$ erg/sec. These numerical values are also compared in Table 1.1 to quantitative values at the Solar center (Schwarzschild, 1958). The estimated value of ϵ is just the Solar (mass-weighted) mean; the actual central value is several times higher because thermonuclear reaction rates are steeply increasing functions of temperature, which peaks at the center. The estimated value of κ is far wrong; this is in part because of the hundredfold concentration of density at the center, and in part because of the concentration into a small central core of thermonuclear energy generation. Equation (1.3.4) shows that using an erroneously low estimated ρ and high R produces an erroneously large estimate for κ.

Except for temperature, our rough estimates have been very inaccurate. Approximations like those of equations (1.3.6) and (1.3.7) are still useful, particularly when only scaling laws are needed for a qualitative understanding. They can also produce semiquantitative results when some additional understanding is inserted into the equations in the form of intelligently chosen numerical coefficients. We have deliberately refrained from doing so in order to show the pitfalls as well as the utility of rough estimates; when aided by intuition and guided by experience they can do much better.

1.4.3 Minimum and Maximum Stellar Surface Temperatures The observed range of stellar surface temperatures is approximately 2500°K to 50,000°K. These limits each have simple explanations.

The continuum opacity of stellar atmospheres is largely attributable to bound-free (photoionization) and free-free (inverse bremsstrahlung) processes. For the visible and near-infrared photons carrying most of the black-body flux at low stellar temperatures the most important bound-free transition is that of the H^- ion, which has a threshold of 0.75 eV. At temperatures of a few thousand degrees matter consists largely of neutral atoms and molecules, and the small equilibrium (Saha equation) free-electron density is very sensitive to temperature, dropping precipitously with further decreases in temperature. The H^- abundance, in equilibrium with the free electrons, drops nearly as steeply. The atmosphere approaches the very transparent molecular gas familiar from the Earth's atmosphere. As a consequence of this steep drop in opacity, the photosphere (the layer in which the emitted radiation is produced) of a very cool star forms at a temperature around 2500°K, below which there is hardly enough opacity and emissivity to absorb or emit radiation. This temperature bound is insensitive to other stellar parameters, and amounts to an outer boundary condition on integrations of the stellar structure equations for cool stars.

The maximum stellar surface temperature has a different explanation. In luminous stars the radiation pressure far exceeds the gas pressure, and the luminosity is nearly the Eddington limiting luminosity L_E (**1.11**), at which the outward force of radiation pressure equals the attraction of gravity:

$$L \approx L_E \equiv \frac{4\pi c G M}{\kappa}, \tag{1.4.9}$$

where κ is the opacity. Under these conditions the opacity is predominantly electron scattering, and $\kappa = 0.34 \text{ cm}^2/\text{gm}$, essentially independent of other parameters. The effective (surface) temperature T_e is then approximately given by

$$T_e^4 = \frac{cGM}{\kappa \sigma_{SB} R^2}, \tag{1.4.10}$$

where σ_{SB} is the Stefan-Boltzmann constant. In order to estimate R we approximate the pressure by the radiation pressure

$$P \approx \frac{a}{3} T^4, \tag{1.4.11}$$

where T is an estimate of the central temperature. Note that here we neglect the gas pressure; in obtaining equation (1.4.8) we neglected the radiation pressure. Eliminating P and ρ from (1.4.1), (1.4.6), and (1.4.11) produces an estimate for R:

$$R^4 = \frac{9GM^2}{4\pi a T^4}. \tag{1.4.12}$$

Substituting this result in (1.4.10) gives

$$T_e^4 = \frac{T^2 c}{\kappa \sigma_{SB}} \sqrt{\frac{4}{9} \pi a c G}. \tag{1.4.13}$$

Because thermonuclear reaction rates are usually very steeply increasing functions of temperature, the condition that thermonuclear energy production balances radiative losses acts as a thermostat; detailed calculation shows that $T \approx 4 \times 10^{7}$°K, nearly independent of other parameters for these very massive and luminous stars. Numerical evaluation of (1.4.13) then gives

$$T_e \approx 90{,}000^\circ K. \tag{1.4.14}$$

This numerical value is about twice as large as the results of detailed calculations, but they confirm the qualitative result of a mass-independent upper bound to T_e for hydrogen burning stars.

1.5 Virial Theorem

For stars (defined as self-gravitating spheres in hydrostatic equilibrium) it is easy to prove a virial theorem, so named because it is closely related to the virial theorem of point-mass mechanics. Begin with the equation (1.3.1) of hydrostatic equilibrium and assume it is always valid:

$$-\rho(r)\frac{GM(r)}{r^2} = \frac{dP(r)}{dr}. \tag{1.5.1}$$

Multiply each side by $4\pi r^3$, and integrate over r, integrating by parts:

$$\begin{aligned} -\int_0^R \rho(r)\frac{GM(r)}{r}4\pi r^2 dr &= \int_0^R \frac{dP(r)}{dr}4\pi r^3 dr \\ &= -\int_0^R 12\pi r^2 P(r)dr + 4\pi r^3 P(r)\Big|_0^R. \end{aligned} \tag{1.5.2}$$

The definition of the stellar radius R is that $P(R) = 0$. Hence

$$-\int_0^R \rho(r)\frac{GM(r)}{r}4\pi r^2 dr = -3\int_0^R P(r)4\pi r^2 dr. \tag{1.5.3}$$

The left hand side is the integrated gravitational binding energy of the star E_{grav}. For a gas which satisfies a relation $P \propto \rho^\gamma$ for adiabatic processes we can use the thermodynamic relation (see **1.9.1**)

$$P = (\gamma - 1)\mathcal{E}, \tag{1.5.4}$$

where $\mathcal{E}$ is the internal energy per unit volume. If we denote the integrated internal energy content of the star by E_{in} we obtain

$$E_{grav} = -3(\gamma - 1)E_{in}. \tag{1.5.5}$$

Denoting the total energy $E = E_{in} + E_{grav}$ we have

$$E = E_{in}(4 - 3\gamma) = E_{grav}\left(\frac{3\gamma - 4}{3\gamma - 3}\right) \leq 0. \tag{1.5.6}$$

The inequality comes from the requirement that a star be energetically bound. This simple relation is very useful in qualitatively understanding stellar stability and energetics.

For perfect monotonic nonrelativistic gases (including the fully ionized material which constitutes most stellar interiors) $\gamma = 5/3$; this applies even if the electrons are Fermi-degenerate. For a perfect gas of relativistic particles or photons $\gamma = 4/3$; this is a good description of gases whose pressure is largely that of radiation. Gases in which new degrees of freedom appear as the temperature is raised (for example, those undergoing dissociation, ionization, or pair production) may have still lower values of γ, approaching 1. Interatomic forces reduce γ if attractive, or increase it if repulsive (as for the nucleon-nucleon repulsion of neutron star matter).

If $\gamma = 5/3$, as is accurately the case for stars like the Sun, and more roughly so for most degenerate (white) dwarves and for neutron stars, then $E = \frac{1}{2}E_{grav} = -E_{in} < 0$. Such a star is gravitationally bound with a large net binding energy, and resists disruption. It is also stable and resists dynamical collapse, because in a smaller and denser state $|E_{grav}|$ and $|E|$ would be larger. In order to reach such a state it would have to reduce its total energy E, but on dynamical time scales energy is conserved. Energy can only be lost by slow radiative processes (including emission of neutrinos); in most cases it is stably replenished from thermonuclear sources.

A star with $\gamma > 4/3$ may be thought of as having negative specific heat, because an injection of energy increases E, which reduces $|E|$, $|E_{grav}|$ and E_{in} (see 1.5.6). Because temperature is a monotonically increasing function of E_{in} (and depends only on E_{in} for perfect nondegenerate matter) this injection of energy leads to a reduction in temperature; similarly, the radiative loss of energy from the stellar surface, if not replenished internally, leads to increasing internal temperature. The reason for this somewhat surprising behavior, described as a negative effective specific heat, is the fixed relation (1.5.5) between E_{in} and E_{grav}, which holds so long as the assumption of hydrostatic equilibrium is strictly maintained. The negative effective specific heat is also the reason thermonuclear energy release, which increases rapidly with temperature, is usually stably self-regulating.

In a degenerate star the relation between E_{in} and temperature is complicated by the presence of a Fermi energy and the effective specific heat is positive when thermonuclear or radiative processes are considered; thermonuclear energy release is either insignificant or unstable, and radiation produces steady cooling. On dynamical time scales processes are adiabatic and the star is stable, just as is a nondegenerate star. E_{in} is related to the Fermi energy which varies in proportion to the temperature for adiabatic processes, and the effective specific heat is again negative.

A star with $\gamma = 4/3$ has $E = 0$; the addition of 1 erg is sufficient to disrupt it entirely, and the removal of 1 erg to produce collapse. Of course, stars with γ exactly equal to 4/3 do not exist (and cannot exist, for this reason), but as γ approaches 4/3 a star becomes more and more prone to various kinds of instability. Stars with γ very close to 4/3 include very massive stars whose pressure is almost entirely derived from radiation, and degenerate dwarves near their upper mass (Chandrasekhar) limit.

A star with $\gamma < 4/3$ would have positive energy and would be exploding or collapsing. Such stars do not exist, but localized regions with $\gamma < 4/3$ do. They are found in cool stellar atmospheres (especially those of giants and supergiants) in which matter is partly ionized, and possibly in the cores of evolved stars which are hot enough for thermal pair production or dense enough for nuclei to undergo inverse β-decay. Such regions tend to destabilize a star, though the response of the entire star must be calculated to determine if it is unstable; instability is a property of an entire star in hydrostatic equilibrium, not of a subregion of it.

1.6 Time Scales

A star is characterized by a number of time scales. The shortest is the hydrodynamic time scale t_h, which is defined

$$t_h \equiv \sqrt{\frac{R^3}{GM}}. \tag{1.6.1}$$

This is approximately equal to the time required for the star to collapse if its internal pressure were suddenly set to zero. The fundamental mode of vibration has a period comparable to t_h, as does a circular Keplerian orbit skimming the stellar surface. For phenomena with time scale much longer than t_h the star may be considered to be in hydrostatic equilibrium, and eq. (1.3.1) applies. On shorter time scales the application of (1.3.1) is in general not justified. For the Sun $t_h \approx 26$ minutes.

The thermal time scale t_{th} is defined

$$t_{th} \equiv \frac{E}{L}, \tag{1.6.2}$$

where E is the total energy (gravitational plus internal) of the star, as defined in **1.5**, and L is its luminosity. This is the time which would be required for a star to substantially change its internal structure if its thermonuclear energy supply were suddenly set to zero. For phenomena with time scales longer than t_{th} the star may be considered to be in thermal equilibrium, and eq. (1.3.3) applies. The application of (1.3.3) on shorter time scales is in general not justified. For the Sun $t_{th} \approx 2 \times 10^7$ years.

The longest time scale is the thermonuclear time t_n, defined by

$$t_n \equiv \frac{M\varepsilon c^2}{L}, \tag{1.6.3}$$

where εc^2 is the energy per gram available from thermonuclear reactions of stellar material. This measures the life expectancy of a star in a state of thermal equilibrium. After a time of order t_n its fuel will be exhausted and its production of radiant energy will end; a wide variety of ultimate fates are conceivable, including cooling to invisibility, explosion, and gravitational collapse. For ordinary stellar composition $\varepsilon \approx 0.007$; about 3/4 of this is accounted for by the conversion of hydrogen to helium and about 1/4 by the conversion of helium to heavier elements. For the Sun $t_n \approx 10^{11}$ years; its actual life will be about ten times shorter because after the exhaustion of the hydrogen in a small region at the center, L will begin to increase rapidly and its remaining life will be brief. The Sun is presently near the midpoint of its life.

There is an additional time scale t_E which characterizes stars in general. In **1.4.3** we saw that there is a characteristic luminosity L_E (Eq. 1.4.9) which serves as an upper bound on stellar luminosities. Define the Eddington time t_E as the thermonuclear time t_n for a hypothetical star of luminosity L_E. Then

$$t_E \equiv \frac{\varepsilon c \kappa}{4\pi G} = \frac{2\varepsilon e^4}{3Gc^3 m_e^2 m_p \mu_e}, \tag{1.6.4}$$

where we have written the electron scattering opacity κ in terms of fundamental constants and μ_e is the mean number of nucleons per electron. For ordinary stellar composition $t_E \approx 3 \times 10^6$ years. This is an approximate lower bound on the lifespan of a star. Because it nearly four orders of magnitude shorter than the age of the universe, luminous stars have passed through many generations, manufacturing nearly all the elements heavier than helium. The luminosities of stars range over at least nine orders of magnitude, so lower luminosity stars have lifetimes very much longer than t_E, and even much longer than the present age of the universe.

A quantity analogous to the Eddington time is also an important parameter in the study of rapidly accreting masses (for example, in models of X-ray sources and quasars; Salpeter 1964). The luminosity is given by $L = \dot{M}c^2\varepsilon$. The Salpeter time is defined as the e-folding time of the mass M, if $L = L_E$:

$$t_S \equiv \frac{M}{\dot{M}} = \frac{\varepsilon c \kappa}{4\pi G}. \tag{1.6.5}$$

It is usually estimated that $\varepsilon \sim .1$, so that $t_S \sim 4 \times 10^7$ years. This is the characteristic lifetime of such a luminous accreting object.

Finally, there is a simple "light travel" time scale t_{lt} which may be defined for any object of size R:

$$t_{lt} \equiv \frac{R}{c}. \tag{1.6.6}$$

It is generally not possible for an object of size R to change substantially (by a factor of ~ 2) its emission on a time scale shorter than t_{lt}, because that is the shortest time in which signals from a single triggering event can propagate throughout the object, and hence the shortest time on which its emission can vary coherently. A small change, by a factor $1 + \delta$ with $\delta \ll 1$, can occur in a time $\sim \delta t_{lt}$. If the velocity of propagation were the sound speed (or, equivalently, a free-fall speed) rather than c, then t_{lt} would be the hydrodynamic time t_h given by (1.6.1).

The time scale t_{lt} is chiefly used in models of transient or rapidly variable objects in high energy astrophysics, such as variable quasars and active galactic nuclei, γ-ray bursts, and rapidly fluctuating X-ray sources. The observation of a substantial variation in the radiation of an object in a time t_{var} is evidence that its size R satisfies

$$R \lesssim ct_{var}. \tag{1.6.7}$$

Such an upper bound on R may then be combined with the luminosity to place a lower bound on the radiation flux and energy density within the object, and therefore to constrain models of it.

These arguments contain loopholes. It is possible to synchronize clocks connected to energy release mechanisms and distributed over a large volume so that they all simultaneously trigger a sudden release of energy (because the clocks are at rest with respect to each other there is no difficulty in defining simultaneity). A distant observer would not see the energy release to be simultaneous, but rather spread over a time t_{lt}, where R is the difference in the path lengths between him and the various clocks. However, if the clocks have appropriately chosen delays which cancel the differences in path lengths, he will see the signals of all the clocks simultaneously, violating (1.6.7). This would require a conspiracy among the clocks which is unlikely to occur except by intelligent design, and would produce a signal violating (1.6.7) only for observers in a narrow cone.

Other loopholes are more likely to occur in nature. A strong brief pulse of laser light propagating through a medium with a population inversion depopulates the excited state at the moment of its passage. Nearly all of the medium's stored energy may appear in a thin sheet of electromagnetic energy, whose thickness may be much less than R, and whose duration measured by an observer at rest may violate (1.6.7). This is a familiar phenomenon in the laser laboratory, in which nanosecond (or shorter) pulses of light may be produced by arrays of lasing medium more than a meter long.

Analogous to a thin sheet of laser light is a spherical shell of relativistic particles streaming outward from a central source (Rees 1966). If they produce radiation collimated outward (radiation produced by relativistic particles is usually directed nearly parallel to the particle velocity) the shell of particles will be accompanied by a shell of radiation. This radiation shell will propagate freely, and will eventually sweep over a distant observer, who may see a rapidly varying source of radiation whose duration violates (1.6.7). The factor by which it is violated depends on the detailed kinematics of the radiating particles. In general, (1.6.7) is inapplicable when there is bulk relativistic motion, even if only of energetic particles; conversely, its violation implies bulk relativistic motion.

1.7 Radiative Transport

1.7.1 Fundamental Equations The most important means by which energy is transported in astrophysics is by the flow of radiation from regions of high radiant energy density to those of lesser; radiation carries energy from stellar interiors to their surfaces, and from their surfaces to dark space. The complete theory of this process is unmanageably and incalculably complex and cumbersome, but a variety of approximations make it tractable and useful. Fortunately, these approximations are well justified in most (but not all) circumstances of interest, so that the theory is not only tractable but also powerful and successful. Here we will be concerned principally with the simplest limit, applicable to stellar interiors, in which matter is dense and opaque, and radiation diffuses slowly. There is another, even simpler limit, that of vacuum, through which radiation streams freely at the speed c. Between these limits there are the more complex problems of radiative transport in stellar atmospheres (by definition, the regions in which the observed photons are produced). This is a large field of research blessed with an abundance of observational data; several texts exist (for example, Mihalas 1978).

Consider in spherical coordinates the propagation of a beam of radiation, so that r measures the distance from the center of the coordinate system and ϑ is the angle between the beam and the local radius vector. In general, the radiation intensity I will depend on the point of measurement (r, θ, ϕ) (note that ϑ must be distinguished from the polar angle θ), on the polarization, and the the photon frequency ν. In most cases it is possible either to assume spherical symmetry (so that there is no dependence on θ and ϕ), or to treat the problem at different θ and ϕ locally, so that these angles enter only as parameters of the solution, like the chemical composition of the star being studied. In either case it is not necessary to consider θ and ϕ explicitly, and they will be ignored, along with any dependence of the intensity on the azimuthal angle φ of its propagation direction. Problems in which these approximations are not permissible are difficult, and generally their solution requires Monte Carlo methods (in which the paths of large numbers of test photons are followed on a computer in order to determine the mean flow of radiation). I also neglect polarization because it does not significantly affect the flow of radiative energy; it is worth calculating in some stellar atmospheres because it is sometimes observable for nonspherical stars or during eclipses (symmetry implies that the radiative flux integrated over the surface of a spherical star is unpolarized). The frequency dependence of the radiation field is important, although it will not always be written explicitly.

In travelling a small distance dl a beam loses a fraction $\kappa\rho dl$ of its intensity, where κ is the mass extinction coefficient (with dimensions of cm^2/gm), and ρ is the matter density. We consider a beam with intensity $I(r,\vartheta)$ (with dimensions erg/cm^2/sec/steradian, where the element of solid angle refers to the direction of propagation, not to the geometry of the spherical star); the power crossing an element of area ds normal to the direction of propagation, and propagating in an element $d\Omega$ of solid angle, is $I(r,\vartheta)dsd\Omega$. In the short path dl a power $I(r,\vartheta)\kappa\rho dldsd\Omega$ is removed from the beam by matter in the right cylinder defined by ds and dl, where we have taken $d\Omega \ll ds/dl^2$. Matter also emits radiation, and the volume emissivity j is defined so that the power emitted by the volume $dlds$ into the beam solid angle $d\Omega$ is $j\rho dlds\frac{d\Omega}{4\pi}$. The units of j are erg/gm/sec and the emission is assumed isotropic, as is the case unless there is a very large magnetic field.

After travelling the distance dl the radiation field transports energy out of the cylinder with a power $I(r+dr,\vartheta+d\vartheta)dsd\Omega$, where it has been essential to note that a straight ray (we neglect refraction) changes its angle to the local radius vector as it propagates. In a steady state the energy contained in the cylinder does not change with time, so that the sum of sources and sinks is zero:

$$I(r,\vartheta)dsd\Omega - I(r,\vartheta)\kappa\rho dldsd\Omega + j\rho dlds\frac{d\Omega}{4\pi} - I(r+dr,\vartheta+d\vartheta)dsd\Omega = 0. \tag{1.7.1}$$

¿From elementary geometry

$$dr = dl\cos\vartheta \tag{1.7.2a}$$

$$d\vartheta = -dl\sin\vartheta/r. \tag{1.7.2b}$$

These equations are a complete description of the trivial problem of the propagation of a ray in vacuum, and may be combined and integrated to yield the solution

$$r = r_\circ \csc\vartheta, \tag{1.7.3}$$

where $r_\circ$ is the distance of closest approach of the ray to the center of the sphere. If the polar axis of the spherical coordinates is chosen to pass through the point at which the ray is tangent to the sphere of radius $r_\circ$ then the path of the ray in spherical coordinates is given by

$$\theta = \pi/2 - \vartheta = \pi/2 - \sin^{-1}(r_\circ/r). \tag{1.7.4}$$

If we expand $I(r,\vartheta)$ in a Taylor series:

$$I(r+dr,\vartheta+d\vartheta) = I(r,\vartheta) + \frac{\partial I(r,\vartheta)}{\partial r}dr + \frac{\partial I(r,\vartheta)}{\partial\vartheta}d\vartheta + \cdots, \tag{1.7.5}$$

keep only first order terms in small quantities, and substitute this and the expressions 1.7.2 into 1.7.1, we obtain the basic equation of radiative transport:

$$\frac{\partial I_\nu(r,\vartheta)}{\partial r}\cos\vartheta - \frac{\partial I_\nu(r,\vartheta)}{\partial\vartheta}\frac{\sin\vartheta}{r} + \kappa_\nu\rho I_\nu(r,\vartheta) - \frac{j_\nu\rho}{4\pi} = 0. \tag{1.7.6}$$

The subscript ν denotes the dependence of I, κ, and j on photon frequency; properly I_ν and j_ν are defined per unit frequency interval. Henceforth we do not make this subscript or the arguments (r, ϑ) explicit unless they are being discussed.

We are usually more interested in quantities like the energy density of the radiation field and the rate at which it transports energy than in the full dependence of I on angle. Fortunately, these quantities may be represented as angular integrals over I, and are intrinsically much simpler quantities which satisfy much simpler equations than (1.7.6). Only in the very detailed study of stellar atmospheres is the full angular dependence of I significant. The following quantities are important:

$$\frac{4\pi}{c} J \equiv \mathcal{E}_{rad} \equiv \frac{1}{c} \int I \; d\Omega \tag{1.7.7a}$$

$$H \equiv \int I \cos\vartheta \; d\Omega \tag{1.7.7b}$$

$$\frac{4\pi}{c} K \equiv P_{rad} \equiv \frac{1}{c} \int I \cos^2\vartheta \; d\Omega. \tag{1.7.7c}$$

In (1.7.7a) and (1.7.7c) two symbols have been defined because both are in common use. Sometimes H is defined as $\frac{1}{4\pi}$ times the definition in (1.7.7b). The integrals in (1.7.7) are called the angular moments of I; clearly an infinite number of such moments may be defined, but these three are usually the only important ones. It is evident that $\mathcal{E}_{rad}$ is the energy density of the radiation field, H is the radiation flux (the rate at which radiation carries energy across a unit surface normal to the $\vartheta = 0$ direction), and P_{rad} is the radiation pressure. As defined these quantities are functions of frequency, but formally identical relations apply to their integrals over frequency.

In general the n-th moment (where n is the power of $\cos\vartheta$ appearing in the integrand) is a tensor of rank n; the scalar expressions of (1.7.7b) and (1.7.7c) refer to the z component of the flux vector and the zz component of the radiation stress tensor, where $\hat{z}$ is the unit vector along the $\vartheta = 0$ axis. In practice, the z component of H is usually the only nonzero one and the stress tensor is usually nearly isotropic so that it may be described by a scalar P_{rad}.

It is now easy to obtain differential equations for the simpler quantities $\mathcal{E}_{rad}$, H, P_{rad} by taking angular moments of equation (1.7.6); that is, by applying $\int \cos^n \vartheta \; d\Omega$ to the entire equation and carrying out the integrals. The zeroth and first moments are

$$\frac{dH}{dr} + \frac{2}{r}H + c\kappa\rho\mathcal{E}_{rad} - j\rho = 0 \tag{1.7.8a}$$

$$\frac{dP_{rad}}{dr} + \frac{1}{r}(3P_{rad} - \mathcal{E}_{rad}) + \frac{\kappa\rho}{c}H = 0. \tag{1.7.8b}$$

There is an evident problem with this procedure: we have two equations for the three quantities $\mathcal{E}_{rad}$, H, and P_{rad}. If we obtain a third equation by taking the second moment of (1.7.6) we must evaluate integrals like $\int I \cos^3 \vartheta \; d\Omega$, which introduce a fourth quantity, the third moment of I. It is evident that this problem will not be solved exactly by taking any finite number of moments; it arises very generally in moment expansions in physics.

In practice moment expansions are truncated; only a small finite number of moments are taken, and some other information, usually approximate, is used to supply the missing equation. In order to do this expand I in a power series in $\cos\vartheta$:

$$I = I_0 + I_1 \cos\vartheta + I_2 \cos^2\vartheta + \cdots. \tag{1.7.9}$$

We could also expand in Legendre polynomials, which would have the advantage of being orthogonal functions, but for the argument to be made here this is unnecessary. Substitute this power series into (1.7.6), and equate the coefficients of each power of ϑ in the resulting expression to zero. There results an infinite series of algebraic equations whose first three members are:

$$\frac{I_1}{r} + \kappa\rho I_0 = \frac{j\rho}{4\pi} \tag{1.7.10a}$$

$$\frac{\partial I_0}{\partial r} + \frac{2I_2}{r} + \kappa\rho I_1 = 0 \tag{1.7.10b}$$

$$\frac{\partial I_1}{\partial r} - \frac{I_1}{r} + \frac{3I_3}{r} + \kappa\rho I_2 = 0. \tag{1.7.10c}$$

We now need only to estimate the order of magnitude of the I_n, so we may replace $\frac{\partial}{\partial r}$ by $1/l$ and r by l where l is a characteristic length (noting that $\frac{\partial}{\partial r}$ and $-1/r$ do not cancel because this is only an order-of-magnitude replacement—instead, their sum is still of order $1/l$). Again, we have one more variable than equations. However, these equations have an approximate solution for which terms involving the extra variable become insignificant. This solution is

$$I_0 \approx \frac{j}{4\pi\kappa} \tag{1.7.11a}$$

$$I_n \sim I_0(\kappa\rho l)^{-n} \qquad n \geq 1. \tag{1.7.11b}$$

The factor $(\kappa\rho l)$ is generally very large ($\sim 10^{10}$ in the Solar interior) so the higher terms in (1.7.9) become small exceedingly rapidly. As a result (1.7.11a) holds very accurately, while (1.7.11b) is only an order of magnitude expression. It is evident that the terms in (1.7.10) which bring in more variables than equations (those of the form nI_n/r) are smaller than the other terms by a factor of order $(\kappa\rho l)^{-2}$ and are completely insignificant. (1.7.11b) is a rough approximation only because of the replacement of $\frac{\partial}{\partial r}$ by $1/l$, not because of the neglect of the terms of the form nI_n/r.

Because of (1.7.11b), (1.7.9) may be truncated after the $n = 1$ term, and $\mathcal{E}_{rad}$, H, and P_{rad} expressed to high accuracy in terms of I_0 and I_1 alone, reducing the three variables to two. The important result is that

$$P_{rad} = \frac{4\pi}{3c} I_0 = \frac{1}{3}\mathcal{E}_{rad}. \tag{1.7.12}$$

This relation between P_{rad} and $\mathcal{E}_{rad}$ is known as the Eddington approximation. By relating two of the moments of the radiation field it "closes" the moment expansion (1.7.8). It holds to high accuracy everywhere except in stellar atmospheres (in which $\kappa\rho l \sim 1$).

It might be thought that more accurate results could be obtained by taking more terms in the moment expansions. In stellar interiors this is unnecessary. Where (1.7.12) is not accurate, taking higher terms does not lead to rapid improvement. Expansions which do not converge rapidly often do not converge at all. A numerical description of the full ϑ dependence of I is a better approach.

The form of (1.7.12) is no surprise; it expresses the relation between radiation pressure and energy density in thermodynamic equilibrium, which should hold deep in a stellar interior. Similarly, if the matter at any point is locally in thermal equilibrium and there are no photon scattering processes the right hand side of (1.7.11a) equals (by the condition of detailed-balance) the black-body radiation spectrum (also called the Planck function) B_ν:

$$\frac{j_\nu}{4\pi\kappa_\nu} = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_BT) - 1}. \tag{1.7.13}$$

The condition that the matter is in local thermal equilibrium (abbreviated LTE) holds to high accuracy in stellar interiors. It may fail in stellar atmospheres where the radiation field is strongly anisotropic, being mostly directed upward; such a radiation field is not in equilibrium (the Planck function is isotropic), and may drive populations of atomic levels away from equilibrium. This often produces observable effects in stellar spectra, but does not have significant effects on the gross energetics of radiative energy flow.

Scattering presents a different problem. It is simple enough to include scattering out of the beam in the opacity κ, but the source term j is more difficult, because radiation is scattered *into* the beam from all other directions (and, in some cases, from other frequencies).

In general, a term of the form

$$\int d\Omega' d\nu' \frac{d\sigma(\Omega, \Omega', \nu, \nu')}{d\Omega'} I(\Omega', \nu') \tag{1.7.14}$$

must be added to j_ν in (1.7.6), where σ is the scattering cross-section, and the solid angles Ω and Ω' describe the pairs of angles (ϑ, φ) and (ϑ', φ'). The azimuthal angles must be included to completely describe the geometry of scattering. This term is complicated; worse, it turns the relatively simple differential equation (1.7.6) into an integral equation which is much harder to solve. If the radiation field equals the Planck function, as is accurately the case in stellar interiors, then the relation (1.7.13) holds even in the presence of scattering, and it is not necessary to consider the messy integral (1.7.14).

In stellar interiors we may use the Eddington approximation (1.7.12) to reduce equations (1.7.8) to the form

$$\frac{d(Hr^2)}{dr} + c\kappa\rho\mathcal{E}_{rad} - j\rho = 0 \tag{1.7.15a}$$

$$H + \frac{c}{3\kappa\rho}\frac{d\mathcal{E}_{rad}}{dr} = 0. \tag{1.7.15b}$$

1.7.2 Spectral Averaging and Energy Flow In stellar interiors we are concerned with the flow of energy, and not with its detailed frequency dependence. We therefore wish to consider frequency integrals of our previous results. Define the luminosity $L \equiv \int 4\pi r^2 H_\nu \, d\nu$, and note that in steady state there is no net exchange of energy between the radiation and the matter, so that $\int j_\nu d\nu = \int c\kappa_\nu \mathcal{E}_{rad\nu} d\nu$. Then (1.7.15a) states that L is independent of r. For a star in steady state (as we have assumed) this is just the conservation of energy. In discussing radiative transport we have neglected nuclear energy generation; if it were included we would obtain (1.3.3).

It is more interesting to integrate (1.7.15b) over frequency. Define $H_{av} \equiv \int H_\nu d\nu$ and $\mathcal{E}_{av} \equiv \int \mathcal{E}_{rad\nu} d\nu$ so that

$$\begin{aligned} H_{av} &= -\frac{c}{3\rho} \int \frac{1}{\kappa_\nu} \frac{d\mathcal{E}_{rad\nu}}{dr} d\nu \\ &= -\frac{c}{3\rho} \frac{d\mathcal{E}_{av}}{dr} \frac{\displaystyle\int \frac{1}{\kappa_\nu} \frac{d\mathcal{E}_{rad\nu}}{dr} d\nu}{\displaystyle\int \frac{d\mathcal{E}_{rad\nu}}{dr} d\nu}. \end{aligned} \tag{1.7.16}$$

Because the radiation field I_ν is very close to that of a black body B_ν we may write $\mathcal{E}_{rad\nu} = \frac{4\pi}{c} B_\nu$. Then (1.7.16) may be written in the simple form

$$H_{av} = -\frac{c}{3\kappa_R \rho} \frac{d\mathcal{E}_{av}}{dr}, \tag{1.7.17}$$

where we have defined the Rosseland mean opacity

$$\kappa_R \equiv \frac{\displaystyle\int \frac{dB_\nu}{dr} d\nu}{\displaystyle\int \frac{1}{\kappa_\nu} \frac{dB_\nu}{dr} d\nu} = \frac{\displaystyle\int \frac{dB_\nu}{dT} d\nu}{\displaystyle\int \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu}. \tag{1.7.18}$$

These integrals may be computed from the atomic properties of the matter and the Planck function.

The Rosseland mean κ_R is a harmonic mean, and therefore is sensitive to any "windows" (frequencies at which κ_ν is small), but is insensitive to spectral lines at which κ_ν is large. This behavior is very different from that of the frequency-integrated microscopic emissivity of matter (which gives the power radiated by low density matter for which absorption in unimportant); this emissivity is proportional to the arithmetic mean of κ_ν so that lines are important but windows are not. The spectrum of matter usually contains many absorption lines, but not windows, because there generally are processes which provide some absorption across very broad ranges of frequency. The Rosseland mean is therefore not very sensitive to uncertainties in

κ_ν, which is fortunate, because κ_ν is hard to calculate accurately. Because of the frequency dependences of $\frac{dB_\nu}{dT}$ and of typical κ_ν, κ_R is most sensitive to the values of κ_ν at frequencies for which $\frac{h\nu}{k_B T} \sim$ 3–10.

¿From (1.7.17) we obtain

$$H_{av} = -\frac{c}{\kappa_R \rho} \frac{dP_r}{dr}, \tag{1.7.19}$$

where P_r is the frequency-integrated radiation pressure. This relates the rate at which radiation carries energy to the gradient of radiation pressure. If the black body relation $P_r = \frac{a}{3} T^4$ is substituted in (1.7.19) and the definition of L is used then (1.3.4) is obtained.

In general $0 > \frac{dP_r}{dr} \geq \frac{dP}{dr}$ (unless the gas pressure were to *increase* outward, an unlikely event which would require that the density also increase outward, an unstable situation; see **1.8.1**). The equation of hydrostatic equilibrium (1.3.1) gives $\frac{dP}{dr}$, so that (1.7.19) implies an upper bound on H_{av} and on L for a star in hydrostatic equilibrium. This is the origin of the Eddington limit on stellar luminosities L_E used in **1.4.3**.

1.7.3 Scattering Atmospheres An interesting application of these equations is to the problem of an atmosphere in which the opacity is predominantly frequency-conserving scattering, rather than absorption. This is a good approximation for hot luminous stars, X-ray sources, and the hotter parts of accretion discs, but also for visible radiation in very cool stellar and planetary atmospheres. Define the single-scattering albedo ϖ of the material as the fraction of the opacity attributable to scattering; then $1 - \varpi \ll 1$ is the fraction attributable to absorption.

Begin with equations (1.7.8), assume a nearly isotropic radiation field and the Eddington approximation (1.7.12), and consider the case of a plane-parallel atmosphere of uniform temperature, so that $\frac{1}{r} \ll \frac{d}{dr}$ and B is independent of space. Equations (1.7.8) become

$$\frac{dH}{dr} + c\kappa\rho\mathcal{E}_{rad} - j\rho = 0 \tag{1.7.20a}$$

$$\frac{dP_{rad}}{dr} + \frac{\kappa\rho}{c}H = 0. \tag{1.7.20b}$$

The source term j is now given by

$$j = 4\pi\kappa B(1-\varpi) + \kappa\mathcal{E}_{rad}c\varpi; \tag{1.7.21}$$

substitution leads to

$$\frac{1}{\kappa\rho}\frac{dH}{dr} + \mathcal{E}_{rad}c(1-\varpi) - 4\pi B(1-\varpi) = 0. \tag{1.7.22}$$

Define the optical depth τ by

$$d\tau \equiv -\kappa\rho dr, \tag{1.7.23}$$

with $\tau = 0$ outside the atmosphere (above essentially all its material); this definition is used in all radiative transfer problems. Equations (1.7.22) and (1.7.20b) become

$$\frac{dH}{d\tau} = (\mathcal{E}_{rad}c - 4\pi B)(1-\varpi) \tag{1.7.24a}$$

$$\frac{1}{3}\frac{d\mathcal{E}}{d\tau} = \frac{H}{c}. \tag{1.7.24b}$$

Differentiation of (1.7.24b) and substitution into (1.7.24a) leads to

$$\frac{d^2(\mathcal{E}_{rad} - 4\pi B/c)}{d\tau^2} = 3(1-\varpi)(\mathcal{E}_{rad} - 4\pi B/c). \tag{1.7.25}$$

Applying the boundary condition that $\mathcal{E}_{rad} \to 4\pi B/c$ as $\tau \to \infty$ leads to the solution

$$\mathcal{E}_{rad} = \frac{4\pi B}{c}\left[1 - \exp\left(-\sqrt{3(1-\varpi)}\tau\right)\right]. \tag{1.7.26}$$

One consequence of this result is that the radiation field does not approach the black body radiation field until $\tau \gtrsim [3(1-\varpi)]^{-1/2} \gg 1$; in an atmosphere with largely absorptive opacity the corresponding condition is $\tau \gtrsim 1$.

Another consequence is found when we compute the emergent radiant power $H(\tau = 0)$ from (1.7.24b):

$$H = \frac{4\pi B}{3}\sqrt{3(1-\varpi)}. \qquad (1.7.27)$$

This should be compared to the result for a black body radiator $H = \pi B$, which is obtained from (1.7.7b) if $I = B$ for $\vartheta \leq \pi/2$, and $I = 0$ for $\vartheta > \pi/2$. The scattering atmosphere radiates a factor of $\frac{4}{3}\sqrt{3(1-\varpi)} \ll 1$ as much power as a black body at the same temperature. This may be described as an emissivity $\varsigma = \frac{4}{3}\sqrt{3(1-\varpi)} \ll 1$ of the scattering atmosphere; by the condition of detailed balance such an atmosphere has an angle-averaged albedo (the fraction of incident flux returned to space after one or more scatterings) of $1-\varsigma$. If it has an effective temperature T_e, its actual temperature $T \approx \varsigma^{-1/4} T_e \approx 0.81(1-\varpi)^{-1/8} T_e$, where we have assumed that ϖ and ς are not strongly frequency dependent.

The high albedo of a medium whose opacity is mostly scattering is observed in everyday life when one adds cream to coffee. The extract of coffee we drink is a nearly homogeneous substance whose opacity is almost entirely absorptive; its albedo is very low. The mixture of coffee and cream is visibly lighter in appearance because of the high scattering cross-sections of globules of milk fat. The reduced emissivity of the mixture is unobservable, because the Planck function is infinitesimal at visible wavelengths and room temperature.

Equation (1.7.27) appears to imply $\varsigma > 1$ if $\varpi \to 0$, but this thermodynamically impossible result is incorrect because the assumption of the Eddington approximation is invalid for $\tau \lesssim 1$, which is the important region in determining the emergent flux from an absorbing atmosphere. In a scattering atmosphere, optical depths up to $[3(1-\varpi)]^{-1/2} \gg 1$ are important; the Eddington approximation is valid over most of this range.

1.8 Turbulent Convection

If we heat the bottom and cool the top of a reservoir of fluid at rest, heat will flow upward. The central regions of stars are heated by thermonuclear reactions and their surfaces are cooled by radiation. If the rate of heat flow is low, it will flow by a combination of radiation and conduction. Conduction is usually dominant in everyday liquids and in degenerate stellar material, and radiation is usually dominant in gases, at high temperatures, and in nondegenerate stellar interiors. At high heat fluxes a new process appears, in which macroscopic fluid motions transport warmer material upward and cooler material downward. This process is called convection. For limited parameter ranges convection may take the form of a laminar flow, but in astronomy it is almost always turbulent, if it occurs at all. We must ask when it occurs and what are its consequences.

1.8.1 Criteria Two criteria must be satisfied in order to have convection. The first is that viscosity not be large enough to prevent it. This is an important effect in small laboratory systems, and successful quantitative theories exist, but in stellar heat transport the influence of viscosity is negligible; if convection takes place at all Reynolds numbers usually exceed 10^{10}.

The more important criterion is that the thermodynamic state of the stellar interior be such that convective motions release energy, rather than requiring energy to drive them. In other words, convection will occur if it carries heat from hotter regions to cooler ones (given the well-justified assumption that viscosity is a negligible retarding force), but not if it were to carry heat from cooler regions to hotter ones.

To make this criterion more quantitative we compare the thermodyamic state of the star at two radii separated by a small radius increment dr; at r_l the pressure is P_l and the density is ρ_l, while at r_u the pressure is P_u and the density is ρ_u. We assume that the chemical composition is uniform and that densities and opacities are high enough that radiative transport of energy is negligible on the timescales of convective motions; these assumptions are usually (but not always) justified in stellar interiors, but fail in stellar atmospheres. We also relate adiabatic variations in the pressure and density of the fluid by an equation of state of the form

$$P \propto \rho^\gamma. \tag{1.8.1}$$

Such a fluid is known as a "γ-law" gas; γ is discussed in **1.9.1** and is usually between 4/3 and 5/3. It is here only necessary to assume that the form (1.8.1) holds for adiabatic processes over small ranges of P and ρ; this will be the case for any fluid except near a phase transition.

Now consider raising an element of fluid from the lower level to the upper one, with all fluid velocities slow (much slower than the sound speed) so that the fluid element remains in hydrostatic equilibrium with its mean surroundings. When it reaches the upper level it has a density ρ'_u given by

$$\rho'_u = \rho_l \left(\frac{P_u}{P_l}\right)^{\frac{1}{\gamma}} \approx \rho_l \left(1 + \frac{1}{\gamma P_l}\frac{dP}{dr}dr\right). \tag{1.8.2}$$

If $\rho'_u > \rho_u$ then the raised fluid element is denser than its surroundings and will tend to fall back to its initial position. In this case the fluid is stable against convective displacement. A more quantitative analysis would calculate the frequency of sinusoidal perturbations of the horizontal fluid layers (analogous to water surface waves, but allowing for the continuous variation of P and ρ), and would find their frequency to be real.

If $\rho'_u < \rho_u$ the raised fluid is less dense than its surroundings, and experiences a further buoyancy force which accelerates its rise. A similar calculation of the density of a fluid element descending from the upper layer shows that for it $\rho'_l > \rho_l$, so negative buoyancy accelerates its descent. In this case the fluid is unstable, and convective motions begin. In the more quantitative analysis the perturbations of the horizontally layered structure have imaginary frequencies of both signs, and grow exponentially.

For small dr we may write $\rho_u \approx \rho_l + \frac{d\rho}{dr} dr$ so that the stability condition becomes

$$-\frac{1}{\gamma P}\frac{dP}{dr} < -\frac{1}{\rho}\frac{d\rho}{dr}. \qquad (1.8.3)$$

This awkward-appearing form with minus signs on each side has been chosen because the derivatives are both negative.

The definition of an incompressible fluid is that $\gamma \rightarrow \infty$; then the stability criterion (1.8.3) becomes $\frac{d\rho}{dr} < 0$, a familiar result. It is apparent that for compressible fluids as well $\frac{d\rho}{dr} > 0$ would make stability impossible (because the equation 1.3.1 of hydrostatic equilibrium requires $\frac{dP}{dr} < 0$). For an adiabatic equation of state of the form (1.8.1) the entropy $S \propto \ln(P/\rho^\gamma)$, and the stability condition takes the form

$$0 < \frac{dS}{dr}. \qquad (1.8.4)$$

These stability conditions are local; it is clear that if an unstable interchange is possible between two widely separated layers (1.8.3) and (1.8.4) will be violated for at least a portion of the region between the layers.

The bound (1.8.3) may be transformed into a bound on $\frac{dT}{dr}$ by use of (1.4.5); the result is messy unless one of the terms in (1.4.5) is negligible. More generally, if $P \propto \rho^\alpha T^\beta$ (in contrast to 1.8.1, this refers to the functional form of $P(\rho, T)$, and *not* to its variation under adiabatic processes) we can readily obtain

$$-\left(1 - \frac{\alpha}{\gamma}\right) \frac{1}{P} \frac{dP}{dr} > -\frac{\beta}{T} \frac{dT}{dr}. \tag{1.8.5}$$

This is known as the Schwarzschild criterion for stability.

In this derivation we have assumed uniform chemical composition and have ignored angular momentum. Either of these may make the problem much more difficult. For example, if the matter in layer l has higher molecular weight than that in layer u this will tend to stabilize the fluid against convection. A more subtle process called semi-convection may still occur even when ordinary convection does not; it depends on the ability of energy to flow radiatively out of the denser fluid, and thus to separate itself from the stabilizing influence of the higher molecular weight. Semi-convection is one of a large class of "double-diffusive" and "multi-diffusive" processes known to astrophysicists and geophysicists.

The criterion (1.8.5) shows that there is instability when $\left|\frac{dT}{dr}\right|$ is large, and (1.3.4) shows that this tends to occur when κ or L/r^2 are large. Detailed calculations show that (1.8.3–5) are violated in the outer layers of stars with cool surfaces (including the Sun) because at low temperatures κ is large, and near the energy-producing regions of luminous stars, where L/r^2 is large.

1.8.2 Consequences Suppose (1.8.3–5) are violated; what then? It is clear that the interchange of elements of matter which are unstable against interchange will tend to reduce ρ_u and to increase ρ_l, and to increase S_u and to decrease S_l. The limiting state of this process is to turn the violated inequalities (1.8.3–5) into equalities whch then describe the variation of P, ρ, T, and S in the star. Any one of these equalities (they are all equivalent) then replaces (1.3.4) in describing the thermal structure of the star. In other words, the effect of convective instability is to eliminate the conditions which gave rise to it. This is a natural and plausible hypothesis which is widely assumed in turbulent flow problems. It cannot be exactly true; some small excess $\left|\frac{dT}{dr}\right|$ must remain to drive the convective flow.

A crude argument exists to estimate the accuracy of this approximation; the estimate is based on an adaptation of Prandtl's mixing length theory of turbulent flows. Although reality is surely more complex, imagine that the turbulent flow is composed of discrete fluid elements which rise or fall without drag forces (but remain in pressure equilibrium with their surroundings) for a distance ℓ from their origins. After travelling this distance they mix with their new surroundings and lose their identity. Denote the excess of the temperature gradient over the value given by (1.8.5) (taken as an equality) by $\Delta\nabla T$; it is this quantity (called the superadiabatic temperature gradient) we must estimate. After a rising fluid element has travelled a distance dr its temperature exceeds that of its mean surroundings by an amount $\Delta\nabla T dr$; its own thermodynamic state has varied exactly adiabatically and it remains in pressure equilibrium with its mean surroundings (both by assumption). A falling fluid element is similarly cooler than its mean surroundings by $\Delta\nabla T dr$. The combination of rising warmer fluid and falling cooler fluid produces a mean convective heat flux

$$H_{conv} \sim \Delta\nabla T dr c_P \rho v, \tag{1.8.6}$$

where v is a typical flow velocity and c_P is the specific heat at constant P.

In order to estimate v we use the assumption that the only forces acting on fluid elements are those of buoyancy. We have

$$\frac{\Delta\nabla\rho}{\rho} = \left(\frac{\beta}{\gamma - \alpha}\right)\frac{\Delta\nabla T}{T} \sim \frac{\rho}{T}\Delta\nabla T, \tag{1.8.7}$$

and the buoyancy force (which is proportional to dr) leads to a velocity

$$v^2 = \frac{GM(r)}{r^2}\frac{\Delta\nabla\rho}{\rho}(dr)^2 \sim \frac{GM(r)}{r^2}\frac{\Delta\nabla T}{T}(dr)^2. \tag{1.8.8}$$

Now evaluate these expressions after fluid elements have travelled half of the mixing length, so that $dr = \ell/2$:

$$H_{conv} \sim \frac{c_P\rho\ell^2}{4}\sqrt{\frac{GM(r)}{r^2 T}}(\Delta\nabla T)^{3/2}. \tag{1.8.9}$$

A sensible choice of ℓ is a matter of guesswork; it is usually taken to be comparable to the pressure scale height $\left|\frac{d\ln P}{dr}\right|^{-1}$. Observations of the Solar surface show that the convective motions are very complex. The visible surface is divided into a network of small polygonal cells, called granules, which are columns of rising fluid bounded by regions of descending fluid. There is also a larger scale pattern of supergranulation. These observations do not provide direct evidence concerning the vertical mixing length, and flows in the observable Solar atmosphere (where the scale height is small) may not resemble those in deeper layers.

If ℓ is the pressure scale height and $H_{conv} = L/(4\pi r^2) - H_{av}$ (where H_{av} is the radiative flux calculated in **1.7**) then we can evaluate $\Delta\nabla T$ and v at various places in a star. Our results may be manipulated to yield

$$\Delta\nabla T \sim \left|\frac{dT}{dr}\right|\left(\frac{\ell}{r}\right)^{-4/3}\left(\frac{t_h}{t_{th}}\right)^{2/3} \tag{1.8.10a}$$

$$v^2 \sim c_s^2 \left(\frac{T_c}{T}\right) \left(\frac{\ell}{r}\right)^{2/3} \left(\frac{t_h}{t_{th}}\right)^{2/3} \tag{1.8.10b}$$

where the thermal time t_{th} has been redefined (from 1.6.2) to include only the thermal energy content of the convective region, T_c is the central temperature, and c_s is the sound speed. For the convective regions of the Sun (but not its surface layers) $\Delta \nabla T \sim 10^{-6} \left| \frac{dT}{dr} \right|$ and $v \sim 10^{-4} c_s \sim 30$ m/sec. Thus the adiabatic approximation to the structure of a convective zone—the adoption of (1.8.3–5) as equalities—is usually justified to high accuracy, even though the estimates (1.8.6–9) are very crude. Similarly, characteristic hydrodynamic stresses are $\sim \rho v^2 \sim 10^{-8} P$, which establishes that the assumption that fluid elements remain in hydrostatic equilibrium also holds to high accuracy. The time for fluid to circulate through the Solar convective region is $\sim \ell/v \sim 1$ month, which is short enough to guarantee complete mixing.

These approximations break down in the surface layers of stars, as shown by equations (1.8.10). In these layers the scale height and ℓ become small, as do ρ, T, and t_{th} ($t_{th} \approx c_P \rho T \ell / H$). It is not possible to calculate quantitatively the structure of these layers. This problem is most severe for cool giants and supergiants, where T and especially ρ become very small. Their surfaces may not be spherical or in hydrostatic equilibrium, but may rather consist of geysers or fountains of gas which erupts, radiatively cools, and then falls back.

It is important to realize that H_{conv} (1.8.9) is not directly related to or limited by the pressure gradient, unlike the radiative H_{av} (1.7.17). This means that in stellar interiors convection may carry a nearly arbitrarily large luminosity, and the Eddington limit L_E does not apply.

Near stellar surfaces this problem is more complicated because there $\Delta \nabla T$ becomes large for large H_{conv}. In the low densities of stellar atmospheres convection is incapable of carrying a large heat flux because the thermal energy content of the matter is low, and energy must flow by radiation. For hot stars the opacity is essentially constant and radiative transport in the upper atmosphere imposes the upper bound L_E on the stellar luminosity. For cool giants and supergiants the opacity in the upper atmosphere may be extremely small, and no simple bound on the luminosity exists. The actual luminosity of fully convective stars is determined by these surface layers in which the approximation of nearly adiabatic convection breaks down, and no satisfactory theory exists.

1.9 Constitutive Relations

Each of the constitutive relations (1.3.5–7) is an extensive field of research which extends far beyond the scope of this book. This section presents only the sketchiest overview of a few qualitative conclusions which should be familiar to every astrophysicist.

1.9.1 Adiabatic Exponent Here we derive a few useful results. Because stars are large and opaque, and t_{th} is usually long, we are often concerned with the properties of matter undergoing adiabatic processes.

Consider a perfect gas which satisfies the equation of state (1.4.5)

$$P = \frac{\rho N_A k_B T}{\mu} \tag{1.9.1}$$

where we now neglect radiation pressure. For a gram of gas undergoing a reversible process

$$dQ = d\mathcal{U} + P dV \tag{1.9.2}$$

where dQ is an infinitesimal increment of heat, $\mathcal{U}(V, T)$ is the internal energy per gram, and $V \equiv 1/\rho$ is the volume per gram. We define a perfect gas by the condition that $\mathcal{U}$ depend only on T: $\mathcal{U}(V,T) = \mathcal{U}(T)$.

The specific heats at constant pressure and at constant volume, c_P and c_V respectively, are defined:

$$c_P \equiv \left.\frac{dQ}{dT}\right|_P \tag{1.9.3a}$$

$$c_V \equiv \left.\frac{dQ}{dT}\right|_V, \tag{1.9.3b}$$

where the subscript denotes the thermodynamic variable to be held constant. From (1.9.2), using (1.9.1) to eliminate P

$$c_V = \frac{d\mathcal{U}}{dT} \tag{1.9.4a}$$

$$c_P = \frac{d\mathcal{U}}{dT} + \frac{N_A k_B}{\mu}. \tag{1.9.4b}$$

The definition of an adiabatic process is that $dQ = 0$. From the preceding equations and definitions we find for such a process

$$0 = c_V dT + (c_P - c_V)\frac{T}{V}dV. \tag{1.9.5}$$

Defining $\gamma \equiv c_P/c_V$ yields

$$0 = d\ln T + (\gamma - 1)d\ln V. \tag{1.9.6}$$

Integrating this equation, using the definition of V and (1.9.1), yields

$$P \propto \rho^\gamma. \tag{1.9.7}$$

The ratio of specific heats depends on the atoms or molecules making up the gas. By explicit calculation of $\mathcal{U}$ for a perfect gas it is easy to see that

$$\gamma = \frac{q+2}{q} \tag{1.9.8}$$

where q is the number of degrees of freedom excited per atom or molecule. For a monatomic gas $q = 3$, for a diatomic gas in which the vibrational degrees of freedom are not excited (such as air under ordinary conditions) $q = 5$, while for a gas of large molecules or one undergoing temperature-sensitive dissociation or ionization $q \rightarrow \infty$. In stellar interiors we may usually take $q = 3$ and $\gamma = 5/3$, except in regions of partial ionization or where radiation pressure or relativistic degeneracy are important.

In this simple derivation it was necessary to assume a perfect gas and to exclude radiation pressure. These may be included, but lead to much more complex results. For a gas consisting only of radiation this derivation is invalid because $c_P \rightarrow \infty$; T is a unique function of P so that at fixed P no amount of added energy can raise the temperature.

¿From the relation (1.9.7) describing adiabatic processes we can derive a relation between P and the internal energy per volume $\mathcal{E}$. Taking logarithmic derivatives of (1.9.7) and using the definition of V we obtain

$$VdP = -\gamma P dV. \tag{1.9.9}$$

Adding PdV to each side gives

$$VdP + PdV = -(\gamma - 1)PdV \tag{1.9.10a}$$

$$d\left(\frac{PV}{\gamma - 1}\right) = -PdV. \tag{1.9.10b}$$

In an adiabatic process the work done by the fluid on the outside world is $-PdV$, so that (1.9.10b) has the form of a condition of conservation of energy for the fluid, with the left hand side being the increment in internal energy. Then the internal energy per unit volume $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{P}{\gamma - 1}. \tag{1.9.11}$$

The order of the manipulations between (1.9.7) and (1.9.11) may be reversed, so that these two relations are equivalent.

It is important to note that the equivalence between (1.9.7) and (1.9.11) does not require the assumption of a perfect gas or the definition of the specific heats, so that it applies even where it is not possible to derive γ as a ratio of specific heats. The most important application of this is to radiation. From (1.7.12) (or 1.7.7), for a black body radiation field $\mathcal{E}_{rad} = 3P_{rad}$, so that $\gamma = 4/3$ and (1.9.7) describes adiabatic processes in a gas of equilibrium radiation.

1.9.2 Degeneracy The matter in degenerate dwarves, the cores of some giant and supergiant stars, and in neutron stars is Fermi-degenerate. By this we mean that the thermal energy $k_B T$ is much less than the Fermi energy ϵ_F (or, more properly, the chemical potential of the degenerate species), so that states with energies up to ϵ_F are nearly all occupied, and those with higher energies are nearly all empty. This resembles the familiar metallic state of matter. The degenerate species is usually the electron; in neutron stars free neutrons are also degenerate, hence their name.

The density n_d of the degenerate fermion species is given by

$$n_d = 2\left(\frac{4}{3}\pi p_F^3\right)\frac{1}{h^3}, \tag{1.9.12}$$

where p_F is the momentum corresponding to the Fermi energy ϵ_F. This is a standard result of elementary statistical mechanics, obtained by counting volumes in phase space, or by calculating the eigenstates of free particles in a box. The factor of 2 comes from the statistical weight of spin 1/2 particles.

For noninteracting nonrelativistic particles of mass m_d we have

$$\epsilon_F = \frac{p_F^2}{2m_d} \propto n_d^{2/3}, \tag{1.9.13}$$

while characteristic Coulomb energies vary with density as $\epsilon_C \propto e^2 n_d^{1/3}$. Thus at high densities $\epsilon_F \gg \epsilon_C$ and degenerate electrons may be accurately treated as non-interacting particles. This makes the calculation of their equation of state easy and accurate, because the complex band structure of ordinary metals (for which $\epsilon_F \sim \epsilon_C$) may be neglected. The cohesion of ordinary metals (the fact that they have $P = 0$ at finite n_d) requires that ϵ_C be comparable to ϵ_F.

The pressure and internal energy of noninteracting degenerate nonrelativistic particles are found by integrating over their distribution function:

$$\begin{aligned} P &= \int_0^{p_F} p_x v_x \frac{2}{h^3} d^3p \\ &= \frac{1}{3}\int_0^{p_F} m_d v^2 \frac{2}{h^3} d^3p \\ &= \frac{8\pi p_F^5}{15 m_d h^3} \\ &\propto \rho^{5/3} \end{aligned} \tag{1.9.14a}$$

$$\begin{aligned} \mathcal{E} &= \int_0^{p_F} \frac{m_d v^2}{2} \frac{2}{h^3} d^3p \\ &= \frac{3}{2}P, \end{aligned} \tag{1.9.14b}$$

where we have used the fact that $\langle p_x v_x \rangle = \frac{1}{3}\langle p_x v_x + p_y v_y + p_z v_z \rangle = \frac{1}{3}\langle pv \rangle$ for a distribution function which is isotropic in 3-dimensional momentum space; here unsubscripted p and v denote their magnitudes. The relation between $\mathcal{E}$ and P, which corresponds to $\gamma = 5/3$, depends only on the fact that the particle energy $\epsilon_p = \frac{1}{2}pv$, and not on the form of the distribution function; hence it applies to all noninteracting gases of nonrelativistic particles, whether degenerate, nondegenerate, or partially degenerate ($\epsilon_F \approx k_B T$).

If the density is very high most of the particles are relativistic, $\epsilon_p \approx pc$ and $v_x \approx c p_x / p$. If we assume this relation holds exactly over the entire distribution function then

$$\begin{aligned} P &= \int_0^{p_F} \frac{p_x^2 c}{p} \frac{2}{h^3} d^3p \\ &= \frac{1}{3} \int_0^{p_F} pc \frac{2}{h^3} d^3p \\ &= \frac{2\pi c p_F^4}{3h^3} \\ &\propto \rho^{4/3} \end{aligned} \tag{1.9.15a}$$

$$\begin{aligned} \mathcal{E} &= \int_0^{p_F} pc \frac{2}{h^3} d^3p \\ &= 3P. \end{aligned} \tag{1.9.15b}$$

The relation between $\mathcal{E}$ and P, which corresponds to $\gamma = 4/3$, depends only on the relativistic relation $\epsilon_p = pc$, and not on the form of the distribution function; hence it applies to all noninteracting relativistic gases whether degenerate or not; it even applies to bosons, which is why we recover the relation (1.7.12) for photons.

Between the nonrelativistic and relativistic limits is a regime in which neither (1.9.14) nor (1.9.15) is accurate, and $4/3 < \gamma < 5/3$. This transition occurs for $p_F \approx m_d c$, which by (1.9.12) occurs at a density

$$n_d \approx \frac{8\pi m_d^3 c^3}{3h^3}. \tag{1.9.16}$$

For degenerate electrons this corresponds to $\rho \approx 2 \times 10^6$ gm/cm^3, while for neutrons $\rho \approx 10^{16}$ gm/cm^3. These are, to order of magnitude, the characteristic densities of degenerate dwarves and neutron stars respectively.

The regions in the ρ - T plane in which various approximations to the equation of state hold are shown in Figure 1.2. Quantitative calculations exist for the intermediate cases. The regions occupied by the centers and deep interiors of ordinary stars and of degenerate dwarves are shown.

Figure 1.2. Equation of State Regimes.

The results (1.9.14) and (1.9.15) are only rough approximations for degenerate neutrons, because neutrons interact by strong nuclear forces, which are attractive at relatively large distances (several $\times 10^{-13}$ cm) but which are strongly repulsive at shorter distances.

1.9.3 Opacity A quantitative calculation of the opacity of stellar material requires elaborate calculations involving the absorption cross-sections of the ground and many excited states of many ionic species. Such calculations have been performed, and their results are available for quantitative work. It is still important to be aware of a few qualitative principles.

In all ionized matter free electrons scatter radiation, a process called Thomson or Compton scattering. For nondegenerate electrons, in the limits $h\nu \ll m_e c^2$ and $k_B T \ll m_e c^2$ the scattered radiation has the same frequency as the incident radiation, and carries no net momentum. The scattering is not isotropic, but for all $0 \leq \psi \leq \pi/2$ scattering by angles ψ and by $\pi - \psi$ is equally likely; for most purposes it may be treated as if it were isotropic. The total scattering cross-section (**2.6.3**) is $\frac{8\pi e^4}{3m_e^2 c^4} = 6.65 \times 10^{-25}$ cm^2. For matter of the usual stellar composition (70% hydrogen by mass) this produces an electron scattering opacity

$$\kappa_{es} = 0.34 \text{ cm}^2/\text{gm}. \tag{1.9.17}$$

Because this opacity is essentially independent of frequency and temperature in fully ionized matter, (1.9.17) is usually a lower bound on the Rosseland mean opacity. The only circumstances in which the opacity of stellar matter may be significantly less than this value are when it is degenerate (electron scattering is suppressed because most outgoing electron states are occupied), or when it is cool enough that most of the electrons are bound to atoms. The total opacity drops below the value given by (1.9.17) for $T \lesssim 6000°$K.

A free electron moving in the Coulomb field of an ion may absorb radiation; this process is called free-free absorption or inverse bremsstrahlung. Its quantitative calculation is rather lengthy, but a simple semiclassical result is informative. This may be obtained by using the classical expression (2.6.12) or (2.6.15) for the power radiated by an accelerated charge (an electron in the Coulomb field of the ion) to calculate the emissivity, and using the condition of detailed-balance (1.7.13) to obtain from this the opacity. The resulting cross-section per electron is proportional to $n_i v^{-1} \nu^{-3}$, where n_i is the ion density, v is the electron velocity, and ν is the photon frequency. For a typical electron v will be comparable to the thermal velocity, so $v \propto T^{1/2}$, and for a representative photon $h\nu \propto T$.

Rough numerical evaluation of the Rosseland mean leads to

$$\kappa_R \sim 10^{23} \frac{\rho}{T^{7/2}} \text{ cm}^2/\text{gm}; \tag{1.9.18}$$

this expression is only approximate. The functional form of (1.9.18) is known as Kramers' law.

The photoionization of bound electrons (from both ground and excited states) produces bound-free absorption. Its frequency dependence above its energy threshold is usually similar to the ν^{-3} of free-free absorption, but the abundances of the various ions, ionization states, and excitation levels must be considered too. The resulting mean opacity roughly follows Kramers' law, and is of the same order of magnitude as that attributable to free-free absorption.

Any Kramers' law opacity is large at low temperature and high density. At high temperature or low density electron scattering is the principal opacity. The dividing line is approximately given by $T \sim 5 \times 10^6 \rho^{2/7}$ °K. At low temperatures ($T \lesssim 10000$°K) the number of free electrons becomes small and most photons have insufficient energy to ionize atoms; consequently, the opacity drops precipitously and falls below κ_{es}.

The serious user of quantitative opacity information will use the tables which have been computed, but a few further qualitative points should be made:

Because the Rosseland mean is a harmonic mean, the various contributions to the mean opacity are not additive unless they have the same frequency dependence.

Absorption opacities contain a factor $\left[1 - \exp(-h\nu/k_B T)\right]$ whose physical origin is the effect of stimulated emission. This must be included when the Rosseland mean is computed; it is implied by the factor of this form contained in B_ν in (1.7.13); LTE of the atomic and ionic levels has been assumed.

Scattering opacities do not contain a stimulated emission factor if the scattering conserves frequency. The total rate of scattering from state i to state f is proportional to $n_i(1+n_f)$, where n_i and n_f are the occupation numbers of the corresponding photon states; $n_i n_f$ is the rate of stimulated scattering. ¿From this must be subtracted the rate $n_f(1+n_i)$ of scatterings from f to i. The net rate is proportional to $n_i - n_f$, where n_i gives the scattering rate implied by the scattering cross-section without any stimulated scattering term, and n_f gives the the scattering contribution to the source term j. The absence of an explicit stimulated scattering factor is of little importance in stellar interiors, but may be significant in laser experiments in which n_i and n_f may be very large.

Degenerate matter, like ordinary metals, is a good conductor of heat, and in it the radiative transport of energy is usually insignificant. Because the conductive heat flux is proportional to the temperature gradient, a relation like (1.3.4) may be defined in which κ includes also the effects of conduction.

1.9.4 Thermonuclear Energy Generation Many nuclear reactions are involved in the thermonuclear production of energy and the transmutation of lighter elements into heavier ones. Each presents special problems. Here I briefly discuss a few general principles. Quantitative calculation of reaction rates in stellar interiors requires more careful attention to many details; see, for example, Clayton (1968) and Harris *et al.* (1983).

The radius of a nucleus containing A nucleons is approximately given by

$$R \approx 1.4 \times 10^{-13} A^{1/3} \text{cm}. \tag{1.9.19}$$

The electrostatic energy required to bring two rigid and unpolarizable spherical nuclei of radii R_1 and R_2 and atomic numbers Z_1 and Z_2

into contact, if their charges are concentrated at their centers, is

$$E_C = \frac{Z_1 Z_2 e^2}{R_1 + R_2} \approx \frac{Z_1 Z_2}{A_1^{1/3} + A_2^{1/3}} \text{ MeV}. \qquad (1.9.20)$$

Once the nuclei touch strong attractive nuclear forces take over. In the centers of main sequence stars $k_B T$ is in the range $\frac{1}{2} - 4$ KeV so that it is evident that conquering the Coulomb barrier is the chief obstacle to thermonuclear reactions.

The Coulomb barrier is overcome by tunnelling, in a manner first calculated by Gamow; nuclei with energies much less than E_C may (infrequently) react. We work in the center-of-mass frame of the two nuclei, so that $m = \frac{M_1 M_2}{M_1 + M_2}$ is their reduced mass, r their separation, and $k = \sqrt{2mE_\circ}/\hbar$ and $E_\circ$ are the wave-vector and kinetic energy at infinite separation. The barrier tunnelling probability P_0 is calculated in the W. K. B. approximation as

$$P_0 \sim \exp\left(-2\int_R^{r_\circ} \sqrt{\frac{2me^2 Z_1 Z_2}{\hbar^2 r} - k^2}\, dr\right) \equiv \exp(-\mathcal{I}), \qquad (1.9.21)$$

where we write only the very sensitive exponential term, neglecting more slowly varying factors. Here $R = R_1 + R_2$ is the separation at contact (within which the nuclear interactions make the potential attractive), $r_\circ = \frac{2me^2 Z_1 Z_2}{\hbar^2 k^2}$ is the classical turning point (at which the integrand is zero), and the subscript 0 indicates that we consider only the $l = 0$ partial wave. Higher angular momentum states produce much smaller P_l.

The exponent in (1.9.21) may be calculated:

$$\begin{aligned} \mathcal{I} &= 2k \int_R^{r_\circ} \sqrt{\frac{r_\circ}{r} - 1}\, dr \\ &= 4kr_\circ \int_{\sqrt{R/r_\circ}}^1 \sqrt{1 - \zeta^2}\, d\zeta, \end{aligned} \qquad (1.9.22)$$

where $\zeta \equiv \sqrt{r/r_\circ}$. Now $\sqrt{R/r_\circ} \ll 1$ so that we may expand the integral in a power series in $\sqrt{R/r_\circ}$ with the result:

$$\begin{aligned}\mathcal{I} &= 4kr_\circ \left(\int_0^1 \sqrt{1-\zeta^2}\, d\zeta - \int_0^{\sqrt{R/r_\circ}} 1\, d\zeta + \cdots \right) \\ &= 4kr_\circ \left(\frac{\pi}{4} - \sqrt{\frac{R}{r_\circ}} + \cdots \right).\end{aligned} \tag{1.9.23}$$

The leading term in (1.9.23) does not depend on R at all; this is fortunate because it implies that to a good approximation the result is independent of the nuclear sizes or to the form of the potential near nuclear contact, where it is poorly known. We now have

$$\mathcal{I} = \frac{\pi Z_1 Z_2 e^2}{\hbar}\sqrt{\frac{2m}{E_\circ}} - 4\frac{e}{\hbar}\sqrt{2mZ_1Z_2R} + \cdots. \tag{1.9.24}$$

The second term is independent of energy; it affects the reaction rate but we do not consider it further. The third and higher terms are small. The first term is large and after exponentiation makes the reaction rate a sensitive function of $E_\circ$.

We now must average the reaction rate over the thermal equilibrium distribution of nuclear kinetic energies. When we transform variables from the velocities of the reacting nuclei to the center-of-mass and relative velocities v_{cm} and v_{rel}, we find that the kinetic energy $\frac{1}{2}M_1v_1^2 + \frac{1}{2}M_2v_2^2 = \frac{1}{2}(M_1+M_2)v_{cm}^2 + \frac{1}{2}mv_{rel}^2$, so that the distribution function of the relative motion of the reduced mass m is Maxwellian at the particle temperature T. Then the total reaction rate is given by the average over the distribution function $\langle \sigma v_{rel} \rangle$, where σ is the reaction cross-section and contains the critical factor $\exp(-\mathcal{I})$. Aside from slowly varying factors this leads to

$$\langle \sigma v_{rel} \rangle \sim \int_0^\infty \exp\left(-\frac{E}{k_BT} - \frac{B}{\sqrt{E}} \right) dE, \tag{1.9.25}$$

where $B \equiv \pi Z_1 Z_2 e^2 \sqrt{2m}/\hbar$.

The first term in the exponent in (1.9.25) declines rapidly with increasing E, while the second increases rapidly. For $B^2 \gg k_B T$ (almost always the case) their sum has a fairly narrow maximum, and when exponentiated the peak is very narrow. We therefore find the maximum and expand around it. By elementary calculus

$$-\frac{E}{k_B T} - \frac{B}{\sqrt{E}} = -\frac{3E_G}{k_B T} - \frac{3}{8}\frac{B}{E_G^{5/2}}(E - E_G)^2 + \cdots, \tag{1.9.26}$$

where the Gamow energy E_G has been defined

$$E_G \equiv \left(\frac{B k_B T}{2}\right)^{2/3}. \tag{1.9.27}$$

Now the integral in (1.9.25) may be carried out by taking only the first two terms of (1.9.26) and extending the lower limit of integration to $-\infty$, with the result

$$\begin{aligned} \langle \sigma v_{rel} \rangle &\sim \sqrt{\frac{8\pi E_G^{5/2}}{3B}} \exp\left(-\frac{3E_G}{k_B T}\right) \\ &\sim \exp\left[-3\left(\frac{\pi^2 Z_1^2 Z_2^2 e^4 m}{2\hbar^2 k_B T}\right)^{1/3}\right], \end{aligned} \tag{1.9.28}$$

where in the last expression the slowly varying factor has been dropped, as similar factors were before, leaving only the dominant exponential dependence. This result gives the dominant temperature dependence of nonresonant thermonuclear reactions.

Under typical conditions of interest the argument of the cube root in (1.9.28) is $\sim 10^4$. It is therefore apparent that P_0 and $\langle \sigma v_{rel} \rangle$ are very small, as must be the case, in order that the nuclei in a dense stellar interior survive for 10^6–10^{10} years before reacting. It is then evident that the reaction rate is a steeply increasing function of T, and a steeply decreasing function of $Z_1 Z_2$. The sensitivity to T

implies that thermonuclear energy generation acts nearly as a thermostat when in a star whose effective specific heat is negative (see **1.5**), and tends to produce rapid instability when the effective specific heat is positive (as is the case in degenerate matter or for thin shells). It also means that when energy is produced by a given nuclear reaction T is a weak function of the other parameters. The sensitivity to Z_1Z_2 implies that in most circumstances the reactions which proceed most rapidly are those with the smallest product Z_1Z_2.

Real nuclear physics makes the problem more complex. If the reaction of interest is resonant at near-thermal energies (as some important ones are) this may increase the reaction rates by a large factor. The peculiar properties of nuclei with $A = 2$, 5, and 8 are also worthy of note:

The only stable nucleus with $A = 2$ is the deuteron. To produce it from protons requires the reaction

$$p + p \rightarrow D + e^+ + \nu_e. \tag{1.9.29}$$

Because this reaction depends on the weak interaction (it amounts to a β-decay from an unbound diproton state), its rate is many orders of magnitude lower than would otherwise be the case. Yet there is no other direct way of combining two protons; the diproton is not a bound nucleus at all, but is better described as a pole of the p-p scattering matrix. Were the diproton bound, stars (and the universe) would be very different. Because (1.9.29) is so slow, a catalytic process known as the CNO cycle proceeds more rapidly in stars more massive than the Sun, even though it requires reactions with $Z_1Z_2 = 7$.

There are no stable nuclei with $A = 5$ or 8, so that helium nuclei cannot react with each other or with protons. More exotic reactions (such as ^{3}He + ^{4}He, or He + Li) also do not cross the $A = 8$ barrier. The only way to build nuclei heavier than $A = 8$ is by the process

$$\alpha + \alpha + \alpha \rightleftharpoons {}^{12}\mathrm{C}^* \rightarrow {}^{12}\mathrm{C} + \gamma + \gamma', \tag{1.9.30}$$

where the asterisk denotes the 7.654 MeV excited state and the right hand side indicates two successive radiative decays. This process is resonant because the energy of $^{12}C^*$ is only $E_* = 379$ KeV above that of three α-particles. In (1.9.30) the decay rate Γ_α of $^{12}C^*$ to the left is much faster than that Γ_γ to the right; the excited state is in thermal equilibrium with the α-particles, and its density n_* may be calculated from the Saha equation, with the result:

$$n_* = n_\alpha^3 \left(\frac{h^2}{2\pi k_B T}\right)^3 \left(\frac{3m_\alpha}{m_\alpha^3}\right)^{3/2} \exp(-E_*/k_B T), \qquad (1.9.31)$$

where n_α and m_α are the α-particle density mass.

The exponential in (1.9.31) contains the critical temperature dependence, which is characteristic of resonant reaction rates and is even steeper than that of (1.9.28). The factor P_0 need not be calculated explicitly because it enters in both directions on the left hand side of (1.9.30). A steady state abundance of $^{12}C^*$ is achieved in a time $\sim \Gamma_\alpha^{-1} \sim 10^{-15}$ sec. In practice, (1.9.30) proceeds through the unbound ^{8}Be nucleus (a scattering resonance only 92 KeV above the energy of 2 α-particles), rather than through a triple collision, but this does not affect the thermodynamic argument or the result. The reaction rate is $n_* \Gamma_\gamma$. The presence of an excited state of ^{12}C at the right energy to facilitate (1.9.30) is the reason carbon is a relatively abundant element in the universe; this is apparently fortuitous unless one attributes it to divine intervention, or argues that if it were not there we would not be present to observe its absence.

1.10 Polytropes

The solution of the equations (1.3.1–4) of stellar structure is complicated, because the equation of hydrostatic equilibrium (1.3.1) is coupled to the equation of energy flow (1.3.4) through (1.3.3) and the constitutive relation among P, ρ, and T. This problem is now readily handled numerically, even if some of the assumptions (most importantly, that of a thermal steady state) made in deriving (1.3.1–4) are relaxed. In the early (pre-computer) decades of stellar structure research this was not possible, and calculations of models simplified still further were performed. These methods are of more than historical interest, because the very simplified models which they produced are still powerful qualitative tools in understanding stars. They cannot replace modern computational methods of obtaining quantitative results, but they are much more transparent than a table of numbers, and therefore are very helpful to the astrophysicist who needs a qualitative understanding of the properties of self-gravitating configurations of matter.

A *polytrope* is a solution of the equation of hydrostatic equilibrium (1.3.1) under the assumption that the pressure P and the density ρ are everywhere related by the condition

$$P = K\rho^{\frac{n+1}{n}}. \qquad (1.10.1)$$

The quantity n is called the polytropic index.

This relation is formally identical to the adiabatic relation (1.9.7) if $\gamma = \frac{n+1}{n}$, but their meanings are quite different. Equation (1.9.7) describes the variation of the properties of a fluid element undergoing an adiabatic process. Equation (1.10.1) constrains the variations of P and ρ with radius in a star, because if r is introduced as a parameter it relates $P(r)$ and $\rho(r)$. A star may be described by (1.10.1) even if the thermodynamic properties of its constituent matter are described by an adiabatic exponent γ different from $\frac{n+1}{n}$.

Equations (1.10.1) and (1.9.7) are equivalent if a star is neutrally stable (equivalently, marginally unstable) against convection, so that the actual dependence of P on ρ in the star is the same as the adiabatic one. This will be the case in a star which is completely convectively mixed, as is believed to be the case for very low mass main-sequence stars ($M \lesssim 0.2M_{\odot}$). The envelopes of red giants and supergiants are mixed, and also resemble polytropes if the gravitational influence of their dense cores may be neglected (a fair approximation if the envelope is very massive). In each of these cases $n \approx 3/2$; the deep convective envelope is a consequence of the high radiative opacity in the surface layers. Very luminous and massive stars also possess extensive mixed inner regions, and their envelopes are not far from convective instability. For these stars $n \approx 3$; convection is a consequence of their large luminosity.

The assumption of (1.10.1) in place of (1.3.4) permits the stellar structure equations to be reduced to a single nonlinear ordinary differential equation characterized by the parameter n. This equation is readily integrated numerically (even without computers!). Eliminating M from (1.3.1) and (1.3.2), we obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{1.10.2}$$

Dimensionless variables are defined: $\phi^n \equiv \rho/\rho_c$ and $\xi \equiv r/\alpha$, where ρ_c is the central density, and the characteristic length (not the radius) $\alpha \equiv \left[\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2}$. Substitution of these variables and (1.10.1) into (1.10.2) yields the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n. \tag{1.10.3}$$

The boundary conditions at $\xi = 0$ are $\phi = 1$ and $\frac{d\phi}{d\xi} = 0$. The surface is defined as the smallest value of ξ for which $\phi = 0$ (the solution for larger ξ is of no physical significance). Once a numerical integration in the dimensionless variables has been tabulated, it is readily applied to a star of specified ρ_c and K by using the definitions of ϕ and ξ.

Polytropes with certain values of n are of special interest. The ratios of the central density ρ_c to the mean density $\langle\rho\rangle$ indicate the degree to which mass is concentrated in their centers, and are a convenient one-parameter description of their structure.

If $n = 0$ then (1.10.1) corresponds to an incompressible fluid (only one value of ρ is permitted) and $\rho_c/\langle\rho\rangle = 1$. The definitions of ϕ, α, and K become indeterminate; with a little care they could be redefined, but there are easier ways of calculating the radius and pressure distribution of a sphere of incompressible fluid.

If $n = 1$ (1.10.3) is linear and may be integrated analytically, with the result $\phi = \sin\xi/\xi$. Here $\rho_c/\langle\rho\rangle = 3.29$.

If $n = 3/2$ (1.10.1) corresponds to an adiabatic star with $\gamma = 5/3$, and is therefore a good description of fully convective stars with this equation of state. The calculated $\rho_c/\langle\rho\rangle = 5.99$ is the lowest such value which may be obtained for stars composed of perfect gases.

If $n = 3$ (1.10.1) corresponds to an adiabatic star with $\gamma = 4/3$, and is therefore a good description of fully convective (or nearly convective) stars with this equation of state. It also turns out that an $n = 3$ polytrope is a fair description of the density structure $\rho(r)$ of stars in the middle and upper main sequence. Their deep interiors have steeper density gradients than they would if they were convective, but the adiabatic γ is larger than that of a fully convective $n = 3$ polytrope (for which γ must be 4/3); these two effects roughly cancel. For an $n = 3$ polytrope $\rho_c/\langle\rho\rangle = 54.2$. In the present-day Sun this ratio is calculated to be close to 100, while when the Sun was young it was about 60 (the difference results from the depletion of hydrogen and the increase in the molecular weight in the core). The structure and properties of an $n = 3$ polytrope are widely used when a rough but convenient model of a star is needed for more complex calculations.

If $n = 5$ (1.10.3) may also be solved analytically, with the result $\phi = (1 + \xi^2/3)^{-1/2}$. For $n \geq 5$ the radius is infinite because ξ never drops to zero.

If $n \to \infty$ (1.10.1) approaches an isothermal equation of state. The definition of ϕ becomes improper, but (1.10.2) is readily integrated without using (1.10.3). At large r, $\rho \propto r^{-2}$ and $M(r) \propto r$, so that both the radius and the total mass diverge. Such configurations do not describe stars. The upper atmospheres of stars may be isothermal but their structure does not approach an $n = \infty$ polytrope except at very large radii and extremely small density. Long before this the assumption of hydrostatic equilibrium will have failed because of the forces applied by the interstellar medium. These $n = \infty$ polytropes may describe the structure of gravitating clusters of collisionless objects (clusters of stars or of galaxies, for example).

1.11 Mass-Luminosity Relations

In **1.4** we derived scaling relations and made order-of-magnitude estimates for the characteristic ρ, P, and T of a star of given mass M and radius R. We now make similar approximations to estimate the relation between L and M of a main sequence star. As in **1.4**, our results are not meant to be numerically accurate, but rather to be an illuminating guide to the governing physics of stars of various masses.

We begin by defining β, the ratio of the gas pressure to the total pressure:

$$P_g = \beta P \qquad (1.11.1a)$$

$$P_r = (1 - \beta)P. \qquad (1.11.1b)$$

The parameter β is a function of T and ρ and, in general, varies from place to place within a star. Here we assume that it is a constant throughout a given star. This is true for an $n = 3$ perfect gas polytrope, because in such a polytrope the variations in ρ and T are related by $\rho \propto T^n$, so that the two terms in (1.4.5) vary in proportion. Stars on the middle and upper main sequence are approximately described as $n = 3$ polytropes, so that for them our results, derived assuming a constant β, are fair approximations to reality.

Now rewrite (1.3.4) or (1.7.19) in the form

$$\frac{dP_r(r)}{dr} = \frac{\kappa(r)\rho(r)L(r)}{4\pi c r^2}, \tag{1.11.2}$$

and divide this equation by (1.3.1). The result is

$$\frac{dP_r}{dP} = \frac{\kappa(r)L(r)}{4\pi cGM(r)}. \tag{1.11.3}$$

Drop the explicit dependence on r, and use (1.11.1) to rewrite this in terms of a constant β:

$$L = \frac{4\pi cGM}{\kappa}(1 - \beta). \tag{1.11.4}$$

This equation is a fundamental relation among L, M, κ, and β. Because $\beta > 0$ it implies an upper limit on the radiative luminosity of a star.

In hot, luminous stars $\kappa \approx \kappa_{es}$ (1.9.17), so that

$$L = L_E(1 - \beta), \tag{1.11.5}$$

where the Eddington limiting luminosity L_E is defined

$$\begin{aligned} L_E \equiv \frac{4\pi cGM}{\kappa_{es}} &= 1.47 \times 10^{38} \frac{\text{erg}}{\text{sec } M_\odot} \\ &= 3.77 \times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot. \end{aligned} \tag{1.11.6}$$

Therefore, L_E is the upper limit to the radiative luminosity of hot stars. As discussed in **1.8**, it does not properly apply to the convective luminosity; it probably does still limit the luminosity of hot convective stars because their luminosity must flow radiatively through their atmospheres, where convection is ineffective. Cool supergiants may perhaps evade the limit (1.11.6) because κ may be very small in their cool atmospheres, but there is no evidence that they actually do so.

We can also express β in terms of ρ and T, and by so doing obtain a unique (though very approximate) relation between L and M. ¿From the definitions of P_g, P_r, and P (1.4.5) we obtain, after eliminating T,

$$P = \left[\frac{3}{a}\left(\frac{N_A k_B}{\mu}\right)^4 \frac{1-\beta}{\beta^4}\right]^{1/3} \rho^{4/3}. \tag{1.11.7}$$

Now use the relations (1.4.6,7) to express the dependence of P and ρ on M and R. In order to obtain a more useful numerical result we take the actual values of the coefficients which have been calculated for an $n = 3$ polytrope. The result is

$$\frac{1-\beta}{\beta^4} = 2.979 \times 10^{-3} \mu^4 \left(\frac{M}{M_\odot}\right)^2. \tag{1.11.8}$$

This is known as Eddington's quartic equation. From it we may obtain $\beta(M)$ and $L(M)$. Note that β and L do not depend explicitly on R.

At low masses ($M\mu^2 \ll 20 M_\odot$, which includes nearly all stars) $\beta \rightarrow 1$ and $1 - \beta \propto \mu^4 M^2$. ¿From (1.11.4), dropping the μ dependence, we obtain the mass-luminosity relation for constant κ:

$$L \propto M^3; \tag{1.11.9}$$

this describes main sequence stars with $\kappa \approx \kappa_{es}$ and holds for $M_\odot \ll M \ll 50 M_\odot$.

For stars of yet lower mass, κ is roughly described by Kramers' law (1.9.18). If we use (1.4.6,8) to determine T and ρ in Kramers' law, then

$$L \propto M^{11/2} R^{-1/2} \propto M^5, \tag{1.11.10}$$

where the last relation assumed $M \propto R$, which is implied by the approximation **(1.9.4)** that thermonuclear energy generation makes the central temperature nearly independent of M.

The Sun is very near the transition between (1.11.9) and (1.11.10), and has $\beta \approx 0.9996$. Very low mass stars ($M \lesssim 0.2 M_\odot$) are fully convective and their luminosity is determined by their surface boundary condition; the relations of this section do not apply.

Although these results are only approximate, it is evident that L is a steeply increasing function of M; massive stars are disproportionately luminous and short-lived, and low mass stars are disproportionately faint. Very massive stars are also much rarer in the Galaxy than low mass stars, so that they do not overwhelmingly dominate the total luminosity produced by stars; stars of moderate (Solar) mass are not insignificant. If one picks a photon of visible starlight in the Galaxy (or, similarly, chooses a star randomly on the sky), there is a significant chance that it will have come from a star of moderate mass. Very low mass stars, however, are so faint (1.11.10) that they contribute little to the starlight of the night sky.

For very large masses ($M \gtrsim 50 M_\odot$) $\beta \propto M^{-1/2} \rightarrow 0$ and $L \rightarrow L_E$, so that

$$L \propto M. \tag{1.11.11}$$

Stars this massive are very rare or nonexistent, but (1.11.11) represents a limiting relation which is approached by the most massive and luminous stars.

The relations in this section are inapplicable to stars far from the main sequence. In degenerate dwarfs the pressure is almost entirely that of electron degeneracy, which was not included in (1.4.5). As a result T is much lower than (1.4.8) would suggest for these dense stars, and L is lower by several orders of magnitude. This was a puzzle until electron degeneracy pressure was understood. White dwarfs slowly cool to a state in which $T = 0$, $\beta = 1$, and $L = 0$, in complete contradiction to (1.11.8).

The internal structures of giants and supergiants differ drastically from those of $n = 3$ polytropes, with $\rho_c / \langle \rho \rangle$ larger by many orders of magnitude. As a result, the approximate relations (1.4.6,7) fail completely. The structures of these stars are discussed in **1.13**. An analogue of (1.11.8) may be obtained if, instead of (1.4.7), we write

$$P \sim \frac{GM}{R_c} \frac{M}{R^3}, \tag{1.11.12}$$

where $R_c = \zeta R$ is the core radius. Then we obtain

$$\frac{1 - \beta}{\beta^4} \sim \frac{\mu^4 M^2}{\zeta^3}. \tag{1.11.13}$$

Because $\zeta \ll 1$, the limit $\beta \to 0$ is approached for much smaller M than would otherwise be the case; this crudely describes the high luminosity of giant and supergiant stars.

1.12 Degenerate Stars

The basic theory of cold degenerate stars was developed by Chandrasekhar, shortly after the development of quantum mechanics and the Pauli exclusion principle made possible the calculation of degenerate equations of state. His work was concerned with stars in which the electrons are degenerate, known to astronomers as white dwarves, and the discussion of this section generally refers to them. The results and conclusions are also qualitatively (but not quantitatively) applicable to neutron stars, in which degenerate neutrons contribute most of the pressure.

The theory of degenerate stars quantitatively predicts a relation between their masses and radii. It is possible to consider also a number of small effects not included in the basic theory, such as the effect of nonzero temperature, the structure of the nondegenerate atmosphere, the thermodynamics of the ion liquid and its crystallization, gravitational sedimentation in the atmosphere and in the deep interior, ... , and to make detailed predictions about luminosities, spectra, cooling histories, and other properties. Unfortunately, the quality of the extant data is inadequate to test either the basic mass-radius relation or these more sophisticated theories. Reliable masses are known for only a very few degenerate dwarves, and accurate radii for fewer (if any). Therefore, we are here concerned chiefly with their most basic properties, for which the theory, based only on quantum mechanics and Newtonian gravity, may be assumed with confidence.

In order to calculate the relation between the masses and radii of degenerate stars, we should calculate the zero-temperature equation of state $P(\rho)$ for arbitrary density, including the important regime of $\rho \sim 10^6$ gm/cm^3 lying between the relativistic (1.9.15) and nonrelativistic (1.9.14) limits. These calculations exist (see Chandrasekhar 1939), but a qualitative approach using the virial theorem may be more illuminating.

The total energy E of a star is

$$E = E_{grav} + E_{in}. \tag{1.12.1}$$

The quantitative value of each of these terms depends on the detailed forms of $\rho(r)$, $M(r)$, and $\mathcal{E}(r)$. Their scaling with M and R may be simply written, using relations like (1.4.6,7)

$$E_{grav} = -\int_0^R \rho(r)\frac{GM}{r}4\pi r^2\, dr \equiv -\mathcal{A}\frac{GM^2}{R} \tag{1.12.2a}$$

$$E_{in} = \int_0^R \mathcal{E}4\pi r^2\, dr \equiv \mathcal{B}K\left(\frac{M}{R^3}\right)^{\gamma} R^3, \tag{1.12.2b}$$

where $\mathcal{A}$ and $\mathcal{B}$ are dimensionless numbers of order unity, and we have written $P = K\rho^\gamma \propto (M/R^3)^\gamma$, as is appropriate for adiabatic changes. For our qualitative considerations, we will assume that $\mathcal{A}$ and $\mathcal{B}$ are independent of changes in R, although this is not accurate except in the extreme nonrelativistic and extreme relativistic limits.

To compute the dynamical equilibrium radius of the star we find the minimum of the function $E(R)$. If $\gamma = 5/3$ there is a stable minimum E at

$$R = \frac{2\mathcal{B}K}{\mathcal{A}GM^{1/3}}. \tag{1.12.3}$$

This result is strictly applicable only in the limit $\rho \rightarrow 0$ (in order that $\gamma = 5/3$ hold exactly), $R \rightarrow \infty$, and $M \rightarrow 0$.

(1.12.3) describes the mass-radius relation of low mass degenerate dwarves, for which $\gamma = 5/3$ is a good approximation. (1.12.3) applies also to any series of $n = 3/2$ polytropes with a given value of K (equivalently, with a given specific entropy); if one adds to the outside of such a star matter with the same K as that inside, it will shrink. If mass is removed it expands. This is true both of degenerate dwarves (for which $S = 0$) and of low mass nondegenerate stars. The appearance of M in the denominator of (1.12.3) may be surprising; it is a consequence of the compressibility of matter and the increase of the gravitational force with increasing mass.

For small bodies, like those of everyday life, the density is set by their atomic properties, (1.9.14) is inapplicable, and $R \propto M^{1/3}$ (this may be taken as the definition of a planet). Jupiter is near the dividing line between these two regimes, and thus has approximately the largest radius possible for *any* cold body.

If $\gamma = 4/3$ the condition of minimum E is an equation for M, in which R does not appear:

$$M = \left(\frac{\mathcal{B}K}{\mathcal{A}G}\right)^{3/2}. \tag{1.12.4}$$

Such a configuration is an $n = 3$ polytrope, and $\mathcal{A}$ and $\mathcal{B}$ may be calculated from the known properties of polytropes. We know (see **1.5**) that if $\gamma = 4/3$ then $E = 0$, independently of R, so the absence of R from (1.12.4) is no surprise. Because the binding energy is zero and independent of R the radius is indeterminate.

More remarkable is the fact that a solution exists for only one allowable mass! This mass is called the Chandrasekhar mass M_{Ch}. Numerical evaluation for the relativistic degenerate equation of state (1.9.15) gives

$$\begin{aligned} M_{Ch} &= 5.75 M_\odot / \mu_e^2 \\ &\sim \left(\frac{\hbar c}{G m_P^2} \right)^{3/2} m_P . \end{aligned} \tag{1.12.5}$$

Calculations of stellar evolution and nucleosynthesis indicate that real degenerate dwarves will be composed principally of carbon and oxygen; in the special case in which they are built up by the gradual accretion of matter supplied from the outside they may be principally helium. For all of these elements the molecular weight per electron $\mu_e = 2$. M_{Ch} is reduced slightly below the value given in (1.12.5) by some small effects; the final numerical result is $M_{Ch} = 1.40 M_\odot$ (Hamada and Salpeter 1961).

The unique mass (1.12.4,5) and indeterminate radius apply only in the limit $R \rightarrow 0$ and $\rho \rightarrow \infty$, because only in this limit is $\gamma = 4/3$ exactly. Between this singular solution and the low density limit (1.12.3) there are solutions in which $4/3 < \gamma < 5/3$, and the equation of state is only partly relativistic. These solutions are not polytropes (because γ is not constant within them), but are readily calculated. Observed degenerate dwarves are believed to lie in the range $0.4 M_\odot \lesssim M \lesssim 1.2 M_\odot$, and to be in this semirelativistic regime. Calculations show that for these masses $R \approx 6000 (M_\odot / M)$ km is a fair approximation; their characteristic density is $\rho \sim 2 \times 10^6$ gm/cm^3 (1.9.16). By using the virial theorem (**1.5**) we can also estimate the surface gravitational potential $GM/R \sim m_e c^2$ (actual calculated values are ~ 100 KeV/amu).

If $M > M_{Ch}$ no zero-temperature hydrostatic solutions exist. This is probably the most important result in astrophysics, because it means that stars more massive than M_{Ch} must either reduce their masses below M_{Ch}, end their lives in an explosion, or ultimately collapse.

Equations (1.12.3,4) apply to nondegenerate stars as well. For example, (1.12.4) describes the dependence of K on M for very massive stars, which approximate $n = 3$ polytropes because of the importance of radiation pressure. The factor K has larger values for nondegenerate matter than for degenerate matter, which has the lowest possible P at a given ρ.

The discussion of this section also applies qualitatively to neutron stars. Their characteristic density is determined by (1.9.16), and is $\sim (m_n/m_e)^3$ times larger than that of degenerate dwarves, and their radii are $\sim m_e/m_n$ times as large. Because K is independent of m_d in the relativistic regime (1.9.15), (1.12.4) predicts essentially the same limiting mass for neutron stars as for degenerate dwarves. Their surface gravitational potential $GM/R \sim m_n c^2$ (actual numerical values are believed to be ~ 100 MeV/amu). The strong interactions between neutrons make (1.9.14,15) and (1.12.4) rough approximations at best; the equation of state of neutron matter is controversial. However, the conclusion that as $\rho \rightarrow \infty$ the Fermi momentum $p_F \rightarrow \infty$ and $\gamma \rightarrow 4/3$, which implies an upper mass limit M_{Ch}^{ns}, is inescapable. The effects of general relativity are also significant, and tend to increase the strength of gravity and to reduce M_{Ch}^{ns}, though they are not as large as the uncertainties in the equation of state.

Most calculations agree that for neutron stars $R \approx 10$ km, approximately independent of mass for $0.5M_\odot \lesssim M \lesssim M_{Ch}^{ns}$. The value of M_{Ch}^{ns} is also controversial, but it is probably in the range

$1.40M_\odot < M_{Ch}^{ns} \lesssim 2.5M_\odot$. The lower bound on M_{Ch}^{ns} is firm, and is obtained from the observation of neutron stars of this mass in the binary pulsar PSR 1913+16, for which relativistic orbital effects permit accurate determination of the the pulsar mass (this is the only accurately determined neutron star mass). Because it is hard to imagine the production of neutron stars except as a consequence of the collapse of degenerate dwarves (or the degenerate dwarf cores of larger stars), it is likely that most neutron stars have $M \geq M_{Ch}$, which also implies $M_{Ch}^{ns} \geq M_{Ch}$. The upper bound on M_{Ch}^{ns} is less certain, but uncontroversial properties of the equation of state imply that it cannot much exceed $2.5M_\odot$.

It is frequently pointed out in nontechnical astronomy books that a teaspoon (5 cm^3) of typical white dwarf matter has a mass of about 10 tons. It is not usually added that the internal energy of this teaspoonful is equivalent to that released by about 20 megatons of high explosive.

1.13 Giants and Supergiants

Main sequence and degenerate stars may be approximately described as polytropes. For giants and supergiants polytropic models and the rough approximations of **1.4** fail completely. These stars contain dense cores, resembling degenerate dwarves, and very dilute extended envelopes. The ratio $\rho_c/\langle\rho\rangle$, which is 54.2 for an $n = 3$ polytrope, may be $\sim 10^{12}$ (or more, in extreme cases).

The development of giant structure in a star is the outcome of complex couplings among the equations (1.3.1–7). Their solutions, obtained numerically, are the only proper explanation of giant structure, but it is useful to consider rough arguments. If the core and envelope are considered separately, the approximations of **1.4**, and simple models, may still be qualitatively informative.

A main sequence star will eventually exhaust the hydrogen at its center, leaving a core of nearly pure helium. For stars of masses approximately equal to or exceeding that of the Sun, this happens in less than the age of the Galaxy. Stars have presumably been born throughout that time (there are few quantitative data), so that there now exist stars of a variety of masses which have helium cores. Because the star continues to radiate energy, in a thermal steady state hydrogen must continue to be transformed to helium. This will happen in the hottest part of the star which contains hydrogen, a thin shell just outside the helium core.

The helium core will be essentially inert. In steady state it is isothermal at the temperature of the hydrogen burning shell at its outer surface. Because of the thermostatic properties (**1.9.4**) of thermonuclear energy release, we may roughly regard this shell as having a fixed $T_\circ \approx 4 \times 10^{7\circ}$K.

Once the core has accumulated a significant fraction (typically 8%) of the stellar mass, its temperature $T_\circ$ is insufficient to satisfy the equation (1.3.1) of hydrostatic equilibrium. Equation (1.4.8) explains why; T is set by the shell temperature, and hence by the structure of the outer star, but the core has a larger value of μ (4/3 for helium) and its higher density leads to a large M/R. It then contracts, producing a higher T (this process is stable, by the arguments of **1.5**). Now heat flows outward, which leads to yet higher T (the negative effective specific heat discussed in **1.5**). The heat flow reduces the entropy of the core, until its equation of state approaches that of a degenerate electron gas; the core comes to resemble a degenerate dwarf inside the larger star.

Core contraction will be interrupted when the temperature becomes high enough ($T \gtrsim 10^{8\circ}$K) for reaction (1.9.30) to take place, and exothermically to convert helium to carbon (auxiliary reactions also produce oxygen and rarer elements). This leaves an inert carbon-oxygen core surrounded by a double shell, the outer shell burning hydrogen and the inner shell burning helium. Such double shells have a complex and unstable evolution, but this is irrelevant to our rough description of the structure of a giant star.

The combination of a degenerate dwarf core with a thermostatic boundary condition produces the extended low density envelope of a giant star. A simple argument uses the scale height of the matter overlying the core. If L is not close to L_E radiation pressure is unimportant (see 1.11.5). An isothermal gas, supported in hydrostatic equilibrium by gas pressure against a uniform acceleration of gravity $g = GM_c/R_c^2$, has a density which varies as

$$\rho \propto \exp(-r/h), \tag{1.13.1}$$

where the scale height h is

$$\begin{aligned} h &= \frac{R_c^2 N_A k_B T}{G M_c \mu} \\ &= \frac{R_c k_B T}{E_b \mu}, \end{aligned} \tag{1.13.2}$$

and E_b is the gravitational binding energy per nucleon. The matter is not accurately isothermal and g is not strictly constant, but for $h \ll R_c$ these are good approximations. The approximations made in **1.4** were equivalent to assuming $h \sim R$ everywhere in the stellar interior, and fail at the core-envelope boundary where $h \ll R_c \ll R$.

For a degenerate core with $M_c = 0.7 M_\odot$, at $r = R_c$ we find $h/R_c \approx 0.055 \ll 1$. As a result, the density drops by a large factor in the region just outside the core boundary, where g is large. If the envelope contains a significant amount of mass, as it will in most giants, then this low density requires it to have a large volume and a large radius. Very crudely, we might expect the radius to be larger than that of a main sequence star (which the envelope would otherwise resemble) by a factor $\sim \exp\bigl(R_c/(3h)\bigr) \sim 10^2 - 10^3$, which is consistent with the radii of large red giants. If the core is more massive the density will be yet lower and the radius yet larger. The actual radius and T_e of a red giant are determined by the surface boundary conditions on its outer convective zone.

This argument is not applicable when $L \approx L_E$, because then the scale height is larger by a factor $\beta^{-1} \gg 1$. Instead, we equate the pressure of radiation to the pressure produced by the weight of the overlying matter, so that

$$\frac{a}{3}T_\circ^4 \sim \frac{GM_c\rho}{R_c}. \tag{1.13.3}$$

For $M_c = 1.2M_\odot$ ($\beta \ll 1$ only as $M_c \to M_{Ch}$) and $T_\circ = 4 \times 10^{7\circ}$K we estimate $\rho \sim 0.02$ gm/cm^3. If the envelope roughly resembles an $n = 3$ polytrope, as is likely, then its radius will be $\sim 20R_\odot$. Such a star is not nearly as large as a red giant or supergiant, but possesses a less extreme form of their structure of dense core and extended envelope. Because of its high luminosity and moderate radius its surface temperature is high. These stars are found in a region of the Hertzsprung-Russell diagram between the red supergiants and the upper main sequence, called the horizontal branch (most horizontal branch stars are produced differently, when rapid helium burning increases R_c and h).

1.14 Spectra

The study of astronomical spectra is a large field of research. Here we only draw a few qualitative conclusions useful in modelling novel objects and phenomena.

The radiation we observe from stars is produced in their atmospheres, and its spectrum reflects the physical conditions there. These atmospheres may usually be approximated as plane-parallel layers, so that in the equation (1.7.6) of radiative transfer we may neglect the term containing $1/r$. Then

$$\frac{\partial I(\tau,\vartheta)}{\partial\tau}\cos\vartheta - I(\tau,\vartheta) + S(\tau) = 0, \tag{1.14.1}$$

where the source function $S(\tau) \equiv j(\tau)/4\pi\kappa(\tau)$, and the optical depth τ is defined by $d\tau \equiv \kappa\rho dr$, and $\tau \to 0$ as $r \to \infty$. I, j, κ, and τ all implicitly depend on ν. For $\cos\vartheta > 0$ this equation has the formal solution

$$I(\tau,\vartheta) = \int_\tau^\infty S(\tau')\exp\bigl[-(\tau'-\tau)\sec\vartheta\bigr]\sec\vartheta\; d\tau'. \tag{1.14.2}$$

The emergent flux is that at $\tau = 0$:

$$I(0,\vartheta) = \int_0^\infty S(\tau')\exp(-\tau'\sec\vartheta)\sec\vartheta\; d\tau'. \tag{1.14.3}$$

The emergent flux is a weighted average of S over the atmosphere, with most of the contribution coming from the range $0 \le \tau' \lesssim \cos\vartheta$.

The opacity κ_ν of matter typically has the form shown in Figure 1.3, with sharp atomic lines superposed on a slowly varying continuum. The lines are those of the species abundant in the atmosphere, which depend on its chemical composition, density, and (most sensitively) temperature. In hot stars the strong lines are those of species like He II and C III, in somewhat cooler stars those of He I or H I, in yet cooler stars Ca I and Fe I, and in the coolest stars those of molecules like TiO.

In the simplest stellar atmospheres matter is in thermodynamic equilibrium, there is no scattering, $S = B$ (the Planck function), and the temperature increases monotonically inward. Then I reflects the value of B in the region $\tau \sim 1$, and we may approximate $I_\nu(\tau = 0) \approx B_\nu\bigl(T(\tau_\nu = 2/3)\bigr)$. At a line frequency ν_l the opacity κ_{ν_l} is large and $\tau_{\nu_l} = 2/3$ high in the atmosphere, where T and B are low,

Figure 1.3. Varieties of Spectra.

while outside the line κ_ν is small and $\tau_\nu = 2/3$ much deeper in the atmosphere. The result is an absorption line spectrum, as shown in the figure.

In many stars the upper atmosphere is much hotter than the rest of the atmosphere. In the Sun the upper atmosphere and corona are heated by acoustic (or magneto-acoustic) waves generated within the convective zone. In a few stars a strong radiation flux from a luminous binary companion heats the upper atmosphere; this is found in some companions to strong X-ray sources. When the temperature profile is inverted in this manner there results an emission line spectrum, as shown in the figure. Often a weak emission line spectrum from the highest levels of the atmosphere is superposed on a stronger absorption line spectrum.

If line scattering opacity is important it may also produce an absorption line, regardless of the temperature gradient in the atmosphere. The mechanism is outlined in **1.7.3**; the presence of scattering reduces the emissivity. At such frequencies the diffuse reflectivity of the atmosphere is significant, so that a fraction of the flux is the (zero) reflected flux of the dark sky. If there is significant scattering opacity in the continuum, but the line opacity is absorptive, then the sky is reflected in the continuum and the line will appear in emission. These processes are known as the Schuster mechanism.

In a dilute gas cloud the upper limit in the integral (1.14.3) is τ_{max}, the total optical depth integrated through the cloud. Often the cloud is so rarefied and transparent that $\tau_{max} \ll 1$ at all frequencies. Then (1.14.3) may be approximated

$$I(0,\vartheta) \approx \frac{j_\nu}{4\pi} \sec\vartheta \int \rho \, dr. \qquad (1.14.4)$$

The frequency dependence of the emergent spectrum is that of the emissivity j_ν. Under these conditions LTE is usually inaccurate; the emergent spectrum qualitatively resembles that of the opacity κ_ν, although quantitative results require a calculation of the various atomic and ionic processes. There is an emission line spectrum in which the lines are extremely strong, carrying a significant fraction of the total flux. Such spectra are observed from interstellar clouds, winds flowing outward from stars, the debris of stellar explosions, stellar coronae, laboratory gas discharge lamps, and in other circumstances in which $\int \rho \, dr$ is very small. Because the emitting volume may be large, the total mass and radiated power need not be small, despite the low density.

These classes of spectra are very different, and may often be identified at a glance, even though they are not usually found in their pure states. This is useful in attempting to construct a rough model of a novel astronomical object, because the densities, dimensions, and directions of energy flow are readily constrained. Images are not available for many interesting astronomical objects, because of their small angular sizes, so that the first step in understanding them is the identification of their components and the construction of a rough model of their geometry, their physical parameters, and of the important physical processes.

1.15 Mass Loss

Spectroscopic observations show that many stars lose mass. Typically, the observations show emission lines whose Doppler widths indicate the flow velocity. In most cases the line shape does not directly establish that the mass is flowing outward, only that the star is surrounded by a dilute cloud of gas with the appropriate distribution of velocities; it is usually not possible to determine from the data which velocities are found at which points in space, but outflow is often the only plausible interpretation. In some cases the outflowing gas absorbs an observable amount of the stellar line radiation, and the resulting complex (P Cygni) line profiles may be interpreted unambiguously as mass outflow.

Some stars are observed in ordinary photographs (or infrared images) to be surrounded by luminous gas clouds they have expelled; in some cases these clouds have visibly expanded since the first photographs were taken. Many different kinds of stars lose mass by a variety of mechanisms and at widely varying rates. Even the Sun loses mass at the very small rate of $\sim 10^{-15} M_{\odot}$/year in the Solar wind, produced by the thermal expansion of its hot corona. All stars with convective surface layers are expected to have coronae, whose mass loss rates should be much greater in larger stars with lower surface gravity.

It is known that some stars born with M substantially larger than M_{Ch} have evolved into degenerate dwarves; this establishes that, in some cases, a star may lose the greater part of its mass. In this section I briefly and qualitatively discuss mass loss mechanisms which may occur in luminous stars, where the mass loss rate is often high. Most of these processes are not understood quantitatively.

In a very luminous star the radiation pressure approaches the total pressure, and $\beta \rightarrow 0$ (1.11.1). How closely a star approaches the neutrally stable limit $\beta = 0$ depends on the detailed calculation of its structure; we know (see 1.11.8) that very massive stars and giant stars with dense degenerate cores have small β. ¿From the equation of hydrostatic equilibrium we have

$$-\frac{\beta G M \rho}{r^2} = \frac{dP_g}{dr}, \qquad (1.15.1)$$

so that in this limit the gradient of the gas pressure becomes zero. Essentially the entire weight of the matter is supported by the gradient of radiation pressure; in other words, the force of gravity and the force of radiation pressure cancel. If $\beta = 0$ exactly, nothing is left to resist the gradient of P_g, and the stellar material will float off into space. This argument suggests that very luminous stars are likely to lose mass.

This conclusion is at least qualitatively correct, and may be reached on simple energetic grounds by noting that as $\beta \rightarrow 0$ we have $\gamma \rightarrow 4/3$, and that if $\gamma = 4/3$ the binding energy $E = 0$ (see **1.5**). It is possible to show, by manipulation of the stellar structure equations, that $n = 3$ polytropes (which stars approach as $\beta \rightarrow 0$)

with a constant β are neutrally stable against convection if $\gamma = 4/3$ (also approached as $\beta \rightarrow 0$); it is unsurprising that a star with zero binding energy should be neutrally stable against the interchange of its parts.

Should L exceed $4\pi cGM/\kappa$, the star becomes unstable against convection, and if convection is efficient it carries the excess flux. The radiative luminosity does not exceed $4\pi cGM/\kappa$ and the gradient of radiation pressure does not exceed the force of gravity. In fact, $L > 4\pi cGM/\kappa$ in the envelopes of many cool giants and supergiants, where κ is large; these stars generally do not lose mass rapidly. Only if convection is incapable of carrying the heat flux does excess radiation pressure drive a mass efflux.

It is comparatively easy to disrupt a star with $\beta \ll 1$ if it can be disturbed, but reliable calculation is difficult. Possible disturbances include fluctuations and instability in the nuclear energy generation rate (known to occur in supergiants with degenerate cores and double burning shells), and the inefficient convection present in the outer layers of cool giants and supergiants. Such stars may lose their entire envelopes in response to modest disturbances (most notably in the formation of planetary nebulae by supergiant stars), but it is also necessary to consider less dramatic mass loss processes. These are easier to observe (because they last longer) and to calculate.

The most important factor leading to steady mass loss is probably an increase in κ in optically thin regions above the photosphere. Because the density and optical depth are low, convection cannot transport heat effectively, and probably does not take place. Instead, matter can actually be subject to a force of radiation pressure exceeding that of gravity (a situation which would not occur in a stellar interior in hydrostatic equilibrium). At least two kinds of physical processes, changing ionization balance and grain formation, may produce such an abrupt jump in κ.

The temperature of a grey body (one whose opacity is independent of frequency) just outside a photosphere will be lower than that of one just inside by a factor of about $2^{-1/4} = 0.84$; outside, the black body radiation field only fills the 2π steradians of outward-directed rays, while the 2π steradians of inward-directed rays have little intensity. The opacity of stellar atmospheres is not accurately grey, but this is still a reasonable estimate of the temperature drop. Such a drop may be sufficient to shift substantially the ionization balance, and therefore the opacity. In addition, the Rosseland mean opacity, derived for stellar interiors (in which $\tau_\nu \gtrsim 1$ at all frequencies) is inapplicable in optically thin regions. In the opposite limit, $\tau_\nu \lesssim 1$ at all frequencies, the radiation force is proportional to $\int I_\nu \kappa_\nu \, d\nu$; the arithmetic mean opacity exceeds the Rosseland mean. Strong atomic or ionic lines may now make a large contribution to the force of radiation pressure, and calculations show that in the upper atmospheres of hot luminous stars the net acceleration may be upward.

A simple argument makes it possible to estimate the mass efflux. Suppose the matter is accelerated by radiation pressure in a spectral line of rest frequency $\nu_\circ$. Radiation between $\nu_\circ$ and $\nu_\circ(1 - v/c)$ may be absorbed or scattered by the outflowing wind; the total pressure the radiation field can exert on the matter may be $\sim H_{\nu_\circ}\nu_\circ v/c^2$. Equate this to the momentum efflux rate per unit area $\dot{m}v$ (where $\dot{m}$ is the rate of mass loss per unit area) to obtain the total mass loss rate $\dot{M}$:

$$\begin{aligned} \dot{M} &= 4\pi R^2 \dot{m} \\ &\sim 4\pi R^2 H_{\nu_\circ}\nu_\circ/c^2 \\ &\sim L/c^2, \end{aligned} \tag{1.15.2}$$

where we have approximated $L \equiv \int H_\nu \, d\nu \approx H_{\nu_\circ}\nu_\circ$. This result is an upper bound, because not all of the radiation at the frequencies of the Doppler-shifted line will be absorbed or scattered, and because gravity has been neglected. $\dot{M}$ is independent of v; calculations usually show that v is a few times the stellar surface escape velocity.

If N strong lines contribute to the absorption of radiation, then $\dot{M}$ may be larger by a factor N, which may be $\gg 1$, but not by orders of magnitude.

The mass efflux rate (1.15.2) is small, although readily observable spectroscopically. It is roughly the same as the equivalent mass carried off by the radiation field itself; we know that during a star's life thermonuclear reactions convert less than 1% of its mass to energy. If $L \approx L_E$ then $\dot{M} \lesssim 10^{-9} M/\text{year}$.

A luminous star with a very cool surface (a red supergiant) may lose mass in a related, but more effective way. Above its photosphere the temperature may be cool enough for carbon (and other elements or molecules) to condense into grains; this is probably the origin of interstellar grains. These grains (in particular, those of carbon) are very effective absorbers of visible and near-infrared radiation across the entire spectrum ($\kappa \sim 10^5$ cm^2/gm), so that the pressure of the radiation on the matter may be $\sim \int H_\nu \, d\nu / c$; the Doppler shift factor $v/c \ll 1$ does not enter. Then we obtain

$$\dot{M} \sim \frac{L}{vc}. \tag{1.15.3}$$

For a red supergiant $v \sim \sqrt{GM/R} \sim 30$ km/sec, so this result is $\sim 10^4$ times as large as (1.15.2). The time required to halve M may be as short as $\sim$ 30,000 years. Such a large mass loss rate may change the evolutionary history of the star; for example, it may reduce M below M_{Ch}. Unfortunately, it has not been possible to quantitatively calculate mass loss by this process, although observations indicate it does take place.

The highest estimate of mass loss comes if the energy of the star's radiation may be efficiently used to overcome the gravitational binding energy and to provide kinetic energy, so that

$$\dot{M} \sim \frac{L}{v^2}. \tag{1.15.4}$$

In order for this to occur the radiation must be trapped between an expanding optically thick outflow and the luminous stellar core, and be the working fluid in a heat engine. The required optical depth at all frequencies is $\tau \gtrsim c/v \gg 1$; the acceleration occurs in the stellar interior rather than in the atmosphere. However, such a radiatively accelerated optically thick shell will probably be unstable to convection if it is in hydrostatic equilibrium. Mass loss rates as high as (1.15.4) may be obtained when hydrostatic equilibrium does not apply; for example if L rises significantly above L_E in a time $< t_h$. Such an event resembles an explosion rather than steady mass loss.

Rapid astronomical processes are hard to observe directly, because the fraction of objects undergoing then at any time is inversely proportional to their duration. There is much less direct evidence for mass loss at the rates of (1.15.3) or (1.15.4) than at the slow rate (1.15.2), but the more rapid processes may be important in many objects; the formation of planetary nebulae is a probable example.

1.16 References

Bahcall, J. N., Huebner, W. F., Lubow, S. H., Parker, P. D., and Ulrich, R. K. 1982, *Rev. Mod. Phys.* **54**, 767.

Chandrasekhar, S. 1939, *An Introduction to the Study of Stellar Structure* (Chicago: University of Chicago Press).

Clayton, D. D. 1968, *Principles of Stellar Evolution and Nucleosynthesis* (New York: McGraw-Hill).

Hamada, T., and Salpeter, E. E. 1961, *Ap. J.* **134**, 683.

Harris, M. J., Fowler, W. A., Caughlan, G. R., and Zimmerman, B. A. 1983, *Ann. Rev. Astron. Ap.* **21**, 165.

Mihalas, D. 1978 *Stellar Atmospheres* 2nd ed. (San Francisco: W. H. Freeman).

Rees, M. J. 1966, *Nature* **211**, 468.

Salpeter, E. E. 1964, *Ap. J.* **140**, 796.

Schwarzschild, M. 1958, *Structure and Evolution of the Stars* (Princeton: Princeton University Press).