Quantum Field Theory:
Lecture notes for FY3464 and FY3466 and a bit more

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If you find typos (almost sure, if you read carefully enough), conceptional errors, strange explanations, inconsistencies or have any suggestions, send me please an email!
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Notation and conventions

We use natural units with $\hbar = c = 1$, but keep Newton’s gravitational constant $G_N \neq 1$. Maxwell’s equations are written in the Lorentz-Heaviside version of the cgs system. Thus there is a factor $4\pi$ in the Coulomb law, but not in Maxwell’s equations. Sommerfeld’s fine-structure constant is $\alpha = e^2/(4\pi) \approx 1/137$.

We choose as signature of the metric $-2$, thence the metric tensor in Minkowski space is $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. If not otherwise specified, Einstein’s summation convention is implied.

The d’Alembert or wave operator is $\Box \equiv \partial_\mu \partial^\mu = \partial_t^2 - \Delta$, while the four-dimensional nabla operator has the components $\partial_\mu \equiv \partial/\partial x^\mu = (\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)$.

A boldface letter denotes the components of a three-vector $V = \{V_x, V_y, V_z\} = \{V_i, i = 1, 2, 3\}$ or the three-dimensional part of a contravariant vector with components $V^\mu = \{V^0, V^1, V^2, V^3\} = \{V^0, V\}$; a covariant vector has in Minkowski space the components $V_\mu = (V_0, -V)$.

Vectors and tensors in index free notation are also denoted by boldface letters, $V = V_\mu \partial^\mu$ or $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$.

Greek indices $\alpha, \beta, \ldots$ encompass the range $\alpha = \{0, 1, 2, \ldots D\}$, latin indices $i, j, k, \ldots$ encompass $i = \{1, 2, \ldots D\}$, where $D$ denotes the dimension of the space-time. In chapter 15 latin indices $a, b, c, \ldots$ denote tensor components with respect to the vielbein field $e^a_\mu$.

Our nomenclature for disconnected, connected, and one-particle irreducible (1PI) $n$-point Green functions and their corresponding generating functionals is as follows:

<table>
<thead>
<tr>
<th></th>
<th>Green function</th>
<th>generating functional</th>
</tr>
</thead>
<tbody>
<tr>
<td>disconnected</td>
<td>$G(x_1, \ldots, x_n)$</td>
<td>$Z[J, \ldots]$</td>
</tr>
<tr>
<td>connected</td>
<td>$G(x_1, \ldots, x_n)$</td>
<td>$W[J, \ldots]$</td>
</tr>
<tr>
<td>1PI</td>
<td>$\Gamma(x_1, \ldots, x_n)$</td>
<td>$\Gamma[\phi, \ldots]$</td>
</tr>
</tbody>
</table>

We normalise Dirac spinors as $\bar{u}(p, s) u(p, s) = 2m$.

We use as covariant derivative $D_\mu = \partial_\mu + ig A^a_\mu T^a$ with coupling $g > 0$, field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu$ and generators $T^a$ satisfying $[T^a, T^b] = if^{abc} T^c$ for all gauge groups. Special cases used in the SM are the groups U(1), SU(2) and SU(3) with $g = \{e, g_s, g_u\}$ and $T^a = \{1, r^a/2, \lambda^a/2\}$ in the fundamental representation. In particular, the electric charge of the positron is $q = e > 0$.

Employing dimensional regularisation, we change the dimension of loop integrals from $n = 4$ to $n = 4 - \varepsilon$ or $d = 2\omega = 4 - 2\varepsilon$.

The results of problems marked by ♣ are used later in the text, those marked by ♥ require more efforts and time than average ones.
1. Introduction

This chapter introduces material from classical mechanics and differential geometry. Depending on the background of the reader, the degree of familiarity with these topics may vary considerably. As a hopefully for most easy starter, we review the formulation of classical mechanics using action principles in the first section. We illustrate also the use of the Green function method using the example of harmonic oscillator and recall the mechanics of a relativistic particle. Then we introduce the concept of tensor fields on Riemannian manifolds and the tools necessary to do physics in curved space-times. The major two concepts introduced are “covariant derivative” which describes how a vector is transported in a curved space-time.

1.1. Classical mechanics

We start by recalling briefly the Lagrangian and Hamiltonian formulation of classical mechanics which are both important for the transition to a quantum theory. Either formulation of classical mechanics can be derived using an action principle as starting point.

Variational principles  Fundamental laws of Nature as Newton’s axioms or Maxwell equations were discovered as differential equations. Starting from Leibniz and Euler, physicists realised that one can re-express differential equations in the form of variational principles: In this approach, the evolution of a physical system is described by the extremum of an appropriately chosen functional. Various versions of such variational principles exist, but they have in common that the functionals used have the dimension of “energy times time”, i.e. the functionals have the same dimension as Planck’s constant $\hbar$. A quantity with this dimension is called action $S$. An advantage of using the action as main tool to describe dynamical systems is that this allows us to implement easily both space-time and internal symmetries. For instance, choosing as action a Lorentz scalar leads automatically to relativistically invariant field equations. Moreover, the action $S$ as a scalar quantity summarises economically the information contained typically in a set of various coupled (partial) differential equations.

If the variational principle is formulated as an integral principle, then the functional $S$ will depend on the whole path $q(t)$ described by the system between the considered initial and final time. In the formulation of quantum theory we will pursue, the propagator as probability amplitude for the time evolution of a particle from the point $q(t)$ to the point $q'(t')$ will play a central role. In the usual approach to quantum mechanics, we reinterpret the classical Hamilton function $H(q,p)$ as an operator imposing canonical commutation relations, $[q,p] = i$. In contrast, we will look for a direct connection from the classical action $S[q]$ along the path $[q(t):q'(t')]$ to the transition amplitude $\langle q',t'\mid q,t \rangle$. Thus the use of the action principle will not only allow us an easy discussion of the symmetries of a physical system, but lies also at the heart of the approach to quantum theory we will follow.
1. Introduction

1.1.1. Hamilton’s principle and Lagrange’s equations

A map $F[f(x)]$ from a certain space of functions $f(x)$ into the real or complex numbers is called a functional. We will consider functionals from the space $C^2[a : b]$ of (at least) twice differentiable functions between fixed points $a$ and $b$. More specifically, Hamilton’s principle uses as functional the action $S$ defined by

$$S[q^i] = \int_a^b dt \, L(q^i, \dot{q}^i, t), \tag{1.1}$$

where $L$ is a function of the 2$n$ independent functions $q^i$ and $\dot{q}^i = dq^i/dt$ as well as of the parameter $t$. In classical mechanics, we call $L$ the Lagrange function of the system, $q^i$ are its generalised coordinates, $\dot{q}^i$ the corresponding velocities and $t$ is the time. The extrema $\delta S[q]=0$ of this action give those paths $q(t)$ from $a$ to $b$ which are solutions of the equation of motions for the system described by $L$.

How do we find those paths that extremize the action $S$? First of all, we have to prescribe which variables are kept constant, which are varied and which constraints the variations have to obey. Depending on the variation principle we choose, these conditions and the functional form of the action will differ. Hamilton’s principle corresponds to an infinitesimal variation of the path, $q^i(t) \rightarrow q^i(t) + \delta q^i(t)$ with $\delta q^i(t) = \varepsilon \eta^i(t)$, that keeps the endpoints fixed but is otherwise arbitrary. Moreover, the time $t$ is not varied in Hamilton’s principle, $\delta t=0$ and the scale factor $\varepsilon$ determining the magnitude of the variation $\delta q^i(t)$ is time-independent. Thus varying the action $S$ requires only to calculate the variation $\delta L$ of the Lagrangian $L$. The notation $S[q^i]$ stresses that we consider the action as a functional only of the coordinates $q^i$; the velocities $\dot{q}^i$ are not varied independently because $\varepsilon$ is time-independent, and thus their change $\delta (\dot{q}^i)$ is determined by the change of the coordinates, $\delta (\dot{q}^i) = d/dt(\delta q^i)$.

Following this prescription, the resulting infinitesimal variation $\delta S[q^i]$ of the action,

$$\delta S[q^i] \equiv S[q^i + \delta q^i] - S[q^i] + \mathcal{O}(\varepsilon^2), \tag{1.2}$$

can be obtained by a Taylor expansion of $L(q^i + \delta q^i, \dot{q}^i + \delta \dot{q}^i, t)$ up to linear terms in $\varepsilon$. Performing the expansion, we arrive at

$$\delta S[q^i] = \int_a^b dt \, \delta L(q^i, \dot{q}^i, t) = \int_a^b dt \, \left( \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right), \tag{1.3}$$

where we applied—as always in the following—Einstein’s convention to sum over a repeated index pair. Thus e.g. the first term in the bracket equals

$$\frac{\partial L}{\partial q^i} \delta q^i \equiv \sum_{i=1}^n \frac{\partial L}{\partial q^i} \delta q^i$$

for a system described by $n$ generalised coordinates. We can eliminate the dependent variation $\delta \dot{q}^i$ of the velocities, integrating the second term by parts using $\delta (\dot{q}^i) = d/dt(\delta q^i)$,

$$\delta S[q^i] = \int_a^b dt \left[ \frac{\partial L}{\partial q^i} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}^i} \right) \right] \delta q^i + \left[ \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right]^b_a. \tag{1.4}$$

The boundary term $[\ldots]_a^b$ vanishes, because we required that the variations $\delta q^i$ are zero at the endpoints $a$ and $b$. Since the variations are otherwise arbitrary, the sum in the first bracket

The boundary term $[\ldots]_a^b$ vanishes, because we required that the variations $\delta q^i$ are zero at the endpoints $a$ and $b$. Since the variations are otherwise arbitrary, the sum in the first bracket...
has to be zero for an extremal curve, $\delta S = 0$. Paths that satisfy $\delta S = 0$ are classically allowed. The equations resulting from the condition $\delta S = 0$ are called the (Euler-) Lagrange equations of the action $S$,

$$\frac{\delta S}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0,$$

and give the equations of motion of the system specified by $L$.

Hamilton’s principle is often also called the principle of least action. Note that the last name is a misnomer, since the extremum can be also a maximum or saddle point of the action.

**Example:** Show that the variation $\delta$ and the time-derivative $d/dt$ acting on $q$ commute: We consider a variation $\delta q = q - q_0 = \varepsilon \eta(t)$ with $\varepsilon$ time-independent and denote the original path with $q_0$. Then

$$\frac{dq}{dt} = \dot{q} = \dot{q}_0 + \varepsilon \dot{\eta},$$

or

$$\delta(\dot{q}) \equiv \dot{q} - \dot{q}_0 = \varepsilon \dot{\eta} = \frac{d}{dt}(\delta q),$$

since we keep $\varepsilon$ constant.

The Lagrangian $L$ is not uniquely fixed: Adding a total time-derivative, $L \to L' = L + df(q,t)/dt$, does not change the resulting Lagrange equations,

$$S' = S + \int_a^b dt \frac{df}{dt} = S + f(q(b), t_b) - f(q(a), t_a),$$

since the last two terms vanish varying the action with the restriction of fixed endpoints $a$ and $b$.

**Lagrange function** We illustrate now how one can use symmetries to constrain the possible form a Lagrangian $L$. As example, we consider the case of a free non-relativistic particle with mass $m$ subject to the Galilean principle of relativity. More precisely, we use that the homogeneity of space and time forbids that $L$ depends on $x$ and $t$, while the isotropy of space implies that $L$ depends only on the norm of the velocity vector $v$, but not on its direction. Thus the Lagrange function of a free particle can be only a function of $v^2$,

$$L = L(v^2).$$

Let us consider two inertial frames moving with the infinitesimal velocity $\varepsilon$ relative to each other. Then a Galilean transformation connects the velocities measured in the two frames as $v' = v + \varepsilon$. The Galilean principle of relativity requires that the laws of motion have the same form in both frames, and thus the Lagrangians can differ only by a total time-derivative. Expanding the difference $\delta L$ in $\varepsilon$ gives with $\delta v^2 = 2v\varepsilon$

$$\delta L = \frac{\partial L}{\partial v^2} \delta v^2 = 2v\varepsilon \frac{\partial L}{\partial v^2}.$$

The difference $\delta L$ has to be a total time-derivative. Since $v = \dot{q}$, the derivative term $\partial L/\partial v^2$ has to be independent of $v$. Hence, $L \propto v^2$ and, to be consistent with usual notation, we call the proportionality constant $m/2$, and the total expression kinetic energy $T$,

$$L = T = \frac{1}{2}mv^2.$$
1. Introduction

For a system of non-interacting particles, the Lagrange function $L$ is additive, $L = \sum_a \frac{1}{2} m_a v_a^2$. If there are interactions (assumed for simplicity to depend only on the coordinates), then we subtract a function $V(r_1, r_2, \ldots)$ called potential energy. One confirms readily that this choice for $L$ reproduces Newton’s law of motion.

Energy  The Lagrangian of a closed system depends, because of the homogeneity of time, not on time. Its total time derivative is

$$\frac{dL}{dt} = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i + \frac{\partial L}{\partial q^i} \ddot{q}^i. \tag{1.9}$$

Replacing $\partial L/\partial q^i$ by $(d/dt)\partial L/\partial \dot{q}^i$, it follows

$$\frac{dL}{dt} = \dot{q}^i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i = \frac{d}{dt} \left( \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} \right). \tag{1.10}$$

Hence the quantity

$$E \equiv \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \tag{1.11}$$

remains constant during the evolution of a closed system. This holds also more generally, e.g. in the presence of static external fields, as long as the Lagrangian is not time-dependent.

We have still to show that $E$ coincides indeed with the usual definition of energy. Using as Lagrange function $L = T(q, \dot{q}) - V(q)$, where the kinetic energy $T$ is quadratic in the velocities, we have

$$\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = \dot{q}^i \frac{\partial T}{\partial \dot{q}^i} = 2T \tag{1.12}$$

and thus $E = 2T - L = T + V$.

Conservation law’s  In a general way, we can derive the connection between a symmetry of the Lagrangian and a corresponding conservation law as follows: Let us assume that under a change of coordinates $q^i \rightarrow q^i + \delta q^i$, the Lagrangian changes as total time derivative,

$$L \rightarrow L + \delta L = L + \frac{d\delta F}{dt}. \tag{1.13}$$

In this case, the equation of motions are unchanged and the coordinate change $q^i \rightarrow q^i + \delta q^i$ is a symmetry of the Lagrangian. But the change $d\delta F/dt$ has to equal $\delta L$ induced by the variation $\delta q^i$,

$$\frac{\partial L}{\partial \dot{q}^i} \delta q^i + \frac{\partial L}{\partial q^i} \delta q^i - \frac{d\delta F}{dt} = 0. \tag{1.14}$$

Replacing again $\partial L/\partial q^i$ by $(d/dt)\partial L/\partial \dot{q}^i$ and applying the product rule gives as conserved quantity

$$Q = \frac{\partial L}{\partial \dot{q}^i} \delta q^i - \delta F. \tag{1.15}$$

Thus any continuous symmetry of a Lagrangian system results in a conserved quantity. In particular, energy conservation follows for a system invariant under time-translations with $\delta q^i = \dot{q}^i \delta t$. Other conservation laws are discussed in problem [1.3].
1.1. Classical mechanics

1.1.2. Palatini’s formalism and Hamilton’s equations

Legendre transformation and the Hamilton function In the Lagrange formalism, we describe a system by specifying its generalized coordinates and velocities using the Lagrangian, \( L = L(q^i, \dot{q}^i, t) \). An alternative is to use generalized coordinates and their canonically conjugated momenta \( p_i \) defined as

\[
p_i = \frac{\partial L}{\partial \dot{q}^i}.
\]

The passage from \( \{q^i, \dot{q}^i\} \) to \( \{q^i, p_i\} \) is a special case of a Legendre transformation. Starting from the Lagrangian \( L \) we define a new function \( H(q^i, p_i, t) \) called Hamiltonian or Hamilton function via

\[
H(q^i, p_i, t) = p_i \dot{q}^i - L(q^i, \dot{q}^i, t).
\]

Here we assume that we can invert the definition (1.16) and are thus able to substitute velocities \( \dot{q}^i \) by momenta \( p_i \) in the Lagrangian \( L \).

The physical meaning of the Hamiltonian \( H \) follows immediately comparing its defining equation with the one for the energy \( E \). Thus the numerical value of the Hamiltonian equals the energy of a dynamical system; we insist however that \( H \) is expressed as function of coordinates and their conjugated momenta.

A coordinate \( q_i \) that does not appear explicitly in \( L \) is called cyclic. The Lagrange equations imply then \( \partial L/\partial \dot{q}^i = \text{const.} \), so that the corresponding canonically conjugated momentum \( p_i = \partial L/\partial \dot{q}^i \) is conserved.

Palatini’s formalism and Hamilton’s equations Previously, we considered the action \( S \) as a functional only of \( q^i \). Then the variation of the velocities \( \dot{q}^i \) is not independent and we arrive at \( n \) second order differential equations for the coordinates \( q^i \). An alternative approach is to allow independent variations of the coordinates \( q^i \) and of the velocities \( \dot{q}^i \). We trade the latter against the momenta \( p_i = \partial L/\partial \dot{q}^i \) and rewrite the action as

\[
S[q^i, p_i] = \int_a^b dt \left[ p_i \dot{q}^i - H(q^i, p_i, t) \right].
\]

The independent variation of coordinates \( q^i \) and momenta \( p_i \) gives

\[
\delta S[q^i, p_i] = \int_a^b dt \left[ p_i \delta \dot{q}^i + \dot{q}^i \delta p_i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i \right].
\]

The first term can be partially integrated, and the resulting boundary terms vanishes by assumption. Collecting then the \( \delta q^i \) and \( \delta p_i \) terms and requiring that the variation is zero, we obtain

\[
0 = \delta S[q^i, p_i] = \int_a^b dt \left[ - \left( \dot{p}_i + \frac{\partial H}{\partial q^i} \right) \delta q^i + \left( \dot{q}^i - \frac{\partial H}{\partial p_i} \right) \delta p_i \right].
\]

As the variations \( \delta q^i \) and \( \delta p_i \) are independent, their coefficients in the round brackets have to vanish separately. Thus we obtain in this formalism directly Hamilton’s equations,

\[
\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.
\]

The concept of a Legendre transformation should be familiar from thermodynamics.
This approach can be applied also to field theories, treating e.g. the vector potential $A^\mu$ and the field-strength $F^{\mu\nu}$ in electrodynamics as independent variables.

Consider now an observable $O = O(q^i, p_i, t)$. Its time-dependence is given by

$$\frac{dO}{dt} = \frac{\partial O}{\partial q^i} \dot{q}^i + \frac{\partial O}{\partial p_i} \dot{p}_i = \frac{\partial O}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial O}{\partial p_i} \frac{\partial H}{\partial q^i} + \frac{\partial O}{\partial t},$$  \hspace{1cm} (1.22)

where we used Hamilton’s equations. If we define the Poisson brackets $\{A, B\}$ between two observables $A$ and $B$ as

$$\{A, B\} = \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i},$$  \hspace{1cm} (1.23)

then we can rewrite Eq. (1.22) as

$$\frac{dO}{dt} = \{O, H\} + \frac{\partial O}{\partial t}.$$  \hspace{1cm} (1.24)

Note that we have obtained a formal correspondence between classical and quantum mechanics: The time-evolution of an observable $O$ in the Heisenberg picture is given by the same equation as in classical mechanics, if the Poisson bracket is changed against a commutator.

Liouville’s theorem We consider now the evolution of a many-particle system in $6n$-dimensional phase space $\omega_i \equiv (q^i, p_i)$ with $i = 1, \ldots, 3n$. The phase space density $f(q^i, p_i, t)$ determines the probability to find the system in the state $\omega_i = (q^i, p_i)$ at time $t$. Conservation of probability leads to a conservation law valid for $f$,

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \omega_i} (f \dot{\omega}_i) = 0.$$  \hspace{1cm} (1.26)

We use first Hamilton’s equations (1.21) to replace $\dot{\omega}$,

$$\frac{\partial}{\partial \omega_i} (f \dot{\omega}_i) = \frac{\partial}{\partial q^i} (f \dot{q}^i) + \frac{\partial}{\partial p_i} (f \dot{p}_i) = \frac{\partial f}{\partial q^i} \left( f \frac{\partial H}{\partial p_i} \right) - \frac{\partial f}{\partial p_i} \left( f \frac{\partial H}{\partial q^i} \right),$$  \hspace{1cm} (1.27)

and then that the mixed second derivatives of $H$ commute, $\partial q^i \partial p_i = \partial p_i \partial q^i$,

$$\frac{\partial}{\partial \omega_i} (f \dot{\omega}_i) = \frac{\partial f}{\partial q^i} \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q^i} = \{f, H\}$$

$$= q^i \frac{\partial f}{\partial q^i} + \dot{p}_i \frac{\partial f}{\partial p_i}.$$  \hspace{1cm} (1.28)

Inserting these results into the conservation law (1.26), Liouville’s theorem follows

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i} \dot{q}^i + \frac{\partial f}{\partial p_i} \dot{p}_i = 0.$$  \hspace{1cm} (1.30)

Thus the time evolution of a Hamiltonian system in phase-space is divergence-free. In analogy to the dynamics of a fluid, one rephrases this often as “a Hamiltonian flow is incompressible.”
1.1.3. Green functions and the response method

We can test the internal properties of a physical system, if we impose an external force \( J(t) \) on it and compare its measured effects to its calculated response. If the system is described by linear differential equations, then the superposition principle is valid: We can reconstruct the solution \( x(t) \) for an arbitrary applied external force \( J(t) \), if we know the response to a normalised delta-function-like kick \( J(t) = \delta (t - t') \). Mathematically, this corresponds to the knowledge of the Green function \( G(t - t') \) for the differential equation \( D(t)x(t) = J(t) \) describing the system. Even if the system is described by a non-linear differential equation, we can often use a linear approximation in case of a sufficiently small external force \( J(t) \). Therefore the Green function method is extremely useful and we will apply it extensively discussing quantum field theories.

We illustrate this method for the example of the harmonic oscillator which is the prototype for a quadratic, and thus exactly solvable, action. In classical physics, causality implies that the knowledge of the external force \( J(t') \) at times \( t' < t \) is sufficient to determine the solution \( x(t) \) at time \( t \). We define two Green functions \( G \) and \( G_R \) by

\[
x(t) = \int_{-\infty}^{t} \text{d}t' G(t - t') J(t') = \int_{-\infty}^{\infty} \text{d}t' G_R(t - t') J(t'),
\]

where the retarded Green function \( G_R \) satisfies \( G_R(t - t') = G(t - t') \delta(t - t') \). This definition is motivated by the trivial relation \( J(t) = \int \text{d}t' \delta(t - t') J(t') \): An arbitrary force \( J(t) \) can be seen as a superposition of delta functions \( \delta(t - t') \) with weight \( J(t') \). If the Green function \( G(t - t') \) determines the response of the system to a delta function-like force, then we should obtain the solution \( x(t) \) integrating \( G(t - t') \) with the weight \( J(t) \).

We convert the equation of motion \( m \ddot{x} + m\omega^2 x = J \) of a forced harmonic oscillator into the form \( D(t)x(t) = J(t) \) by writing

\[
D(t)x(t) \equiv m \left( \frac{d^2}{dt^2} + \omega^2 \right) x(t) = J(t).
\]

Inserting (1.31) into (1.32) gives

\[
\int_{-\infty}^{\infty} \text{d}t' D(t) G_R(t - t') J(t') = J(t).
\]

For an arbitrary external force \( J(t) \), this relation can be only valid if

\[
D(t) G_R(t - t') = \delta(t - t').
\]

Thus a Green function \( G(t - t') \) is the inverse of its defining differential operator \( D(t) \). Performing a Fourier transformation,

\[
G(t - t') = \int \frac{d\Omega}{2\pi} G(\Omega) e^{-i\Omega(t-t')} \quad \text{and} \quad \delta(t - t') = \int \frac{d\Omega}{2\pi} e^{-i\Omega(t-t')},
\]

the differential equation (1.34) is transformed into an algebraic one,

\[
G_R(\Omega) = \frac{1}{m \omega^2 - \Omega^2}.
\]
1. Introduction

For the back-transformation with \( \tau = t - t' \),

\[
G_R(\tau) = \int \frac{d\Omega}{2\pi m} \frac{e^{-i\Omega \tau}}{\omega^2 - \Omega^2},
\]

we have to specify how the poles at \( \Omega^2 = \omega^2 \) are avoided. We will use Cauchy’s residue theorem,

\[
\oint dz f(z) = 2\pi i \sum \text{res} \ f(z),
\]

to calculate the integral. This requires to close the integration contour adding a path which gives a vanishing contribution to the integral. This is achieved, when the integrand \( G(\Omega)e^{-i\Omega \tau} \) vanishes fast enough along the added path. Thus we have to choose for positive \( \tau \) the contour \( C_+ \) in the lower plane, \( e^{-i\Omega \tau} = e^{-|\Im(\Omega)| \tau} \to 0 \) for \( \Im(\Omega) \to -\infty \), while we have to close the contour in the upper plane for negative \( \tau \). If we want to obtain the retarded Green function \( G_R(\tau) \) which vanishes for \( \tau > 0 \), we have to shift therefore the poles \( \Omega_{1/2} = \pm \omega \) into the lower plane by adding a small negative imaginary part, \( \Omega_{1/2} \to \Omega_{1/2} = \pm \omega - i\varepsilon \), or

\[
G_R(\tau) = -\frac{1}{2\pi m} \int d\Omega \frac{e^{-i\Omega \tau}}{(\Omega - \omega - i\varepsilon)(\Omega + \omega + i\varepsilon)},
\]

as result for the retarded Green function of the forced harmonic oscillator.

We can now obtain a particular solution solving (1.31). For instance, choosing \( J(t) = \delta(t - t') \), results in

\[
x(t) = \frac{1}{m} \frac{\sin(\omega \tau)}{\omega} \vartheta(\tau).
\]

Thus the oscillator was at rest for \( t < 0 \), got a kick at \( t = 0 \), and oscillates according \( x(t) \) afterwards. Note that the following two points: First, the fact that the kick proceeds the movement is the result of our choice of the retarded (or causal) Green function. Second, the particular solution (1.41) for a oscillator initially at rest can generalised by adding the solution to the homogeneous equation \( \ddot{x} + \omega^2 x = 0 \).
1.1. Classical mechanics

1.1.4. Relativistic particle

In special relativity, we replace the Galilean transformations as symmetry group of space and time by Lorentz transformations. The latter are all those coordinate transformations \( x^\mu \rightarrow \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu \) that keep the squared distance

\[
s_{12}^2 \equiv (t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2
\]  

(1.42)

between two space-time events \( x_1^\mu \) and \( x_2^\mu \) invariant. The squared distance of two infinitesimally close space-time events is called line-element. In Minkowski space, it is given by

\[
ds^2 = dt^2 - dx^2 - dy^2 - dz^2
\]  

(1.43)

using a Cartesian inertial frame. We can interprete the line-element \( ds^2 \) as a scalar product, if we introduce the metric tensor \( \eta_{\mu\nu} \) with elements

\[
\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]  

(1.44)

and a scalar product as

\[
a_\mu b^\mu = \eta_{\mu\nu} a^\mu b^\nu.
\]  

(1.45)

Since the metric \( \eta_{\mu\nu} \) is indefinite, the norm of a vector \( a^\mu \) can be

\[
a_\mu a^\mu > 0, \quad \text{time-like},
\]

(1.46)

\[
a_\mu a^\mu = 0, \quad \text{light-like or null-vector},
\]

(1.47)

\[
a_\mu a^\mu < 0, \quad \text{space-like}.
\]

(1.48)

The cone of all light-like vectors starting from a point \( P \) is called \textit{lightcone}, cf. Fig. 1.2. The time-like region inside the light-cone consists of two parts, past and future. Only events inside the past light-cone can influence the physics at point \( P \), while \( P \) can influence only his future light-cone. The line \( x = x(\sigma) \) describing the position of an observer is called \textit{worldline}. The proper-time \( \tau \) is the time displayed by a clock moving with the observer. With our conventions—negative signature of the metric and \( c = 1 \)—the proper-time elapsed between two space-time events equals the integrated line-element between them,

\[
\tau_{12} = \int_1^2 ds = \int_1^2 d\tau = \int_1^2 [dt^2 - (dx^2 + dy^2 + dz^2)]^{1/2}
\]

(1.49)

\[
= \int_1^2 [\eta_{\mu\nu} dx^\mu dx^\nu]^{1/2}.
\]  

(1.50)

If we parameterise the worldline traced by an observer passing through 1 and 2 by the parameter \( \sigma \), e.g. such that \( \sigma(\tau = 1) = 0 \) and \( \sigma(\tau = 2) = 1 \), then

\[
\tau_{12} = \int_0^1 d\sigma \left[ \eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right]^{1/2}.
\]  

(1.51)

Note that \( \tau_{12} \) is invariant under a reparameterisation \( \tilde{\sigma} = f(\sigma) \).
1. Introduction

Figure 1.2.: Lightcone at the point $x$ generated by light-like vectors. Contained in the light-cone are the time-like vectors, outside the space-like ones.

The only invariant differential we have at our disposal to form an action for a free point-like particle is the line-element, or equivalently the proper-time,

$$S_0 = \alpha \int_a^b ds = \alpha \int_a^b d\sigma \frac{ds}{d\sigma}$$  \hspace{1cm} \text{(1.52)}

with $L = \alpha ds/d\sigma$. We check now if this choice which implies the Lagrangian

$$L = \alpha \frac{d\tau}{d\sigma} = \alpha \left[ \eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right]^{1/2}$$  \hspace{1cm} \text{(1.53)}

for a free particle is sensible: The action has the correct non-relativistic limit,

$$S_0 = \alpha \int_a^b ds = \alpha \int_a^b dt \sqrt{1 - v^2} = \int_a^b dt \left( -m + \frac{1}{2} mv^2 + \mathcal{O}(v^4) \right),$$  \hspace{1cm} \text{(1.54)}

if we set $\alpha = -m$. The mass $m$ corresponds to a potential energy in the non-relativistic limit and has therefore a negative sign in the Lagrangian. The time $t$ enters the relativistic Lagrangian in a Lorentz invariant way as one of the dynamical variables, $x^\mu = (t, \mathbf{x})$, while $\sigma$ assumes now $t$’s purpose to parameterise the trajectory, $x^\mu(\sigma)$. Since a moving clock goes faster than a clock at rest, solutions of this Lagrangian maximize the action.

The Lagrange equations are

$$\frac{d}{d\sigma} \frac{\partial L}{\partial (dx^\alpha/d\sigma)} = \frac{\partial L}{\partial x^\alpha}.$$  \hspace{1cm} \text{(1.55)}

Consider e.g. the $x^1$ component, then

$$\frac{d}{d\sigma} \frac{\partial L}{\partial (dx^1/d\sigma)} = \frac{d}{d\sigma} \left( \frac{1}{L} \frac{dx^1}{d\sigma} \right) = 0.$$  \hspace{1cm} \text{(1.56)}

1.1. Classical mechanics

Since \( L = \frac{d\tau}{d\sigma} \), Newton’s law follows for the \( x^1 \) coordinate after multiplication with \( \frac{d\sigma}{d\tau} \),

\[
\frac{d^2 x^1}{d\tau^2} = 0 ,
\]

and similar for the other coordinates. An equivalent, but often more convenient form for the Lagrangian of a free particle is

\[
L = -m\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} ,
\]

where we set \( \dot{x} = \frac{dx}{d\sigma} \). If there are no interactions (except gravity), we can neglect the mass \( m \) of the particle and one often sets \( m \to -1 \). Since the Lagrangian of a free particle does not depend explicitly on the evolution parameter \( \sigma \), there exists at least one conserved quantity. This conservation law, \( H = m \), expresses the fact that the tangent vector \( \dot{x}^{\mu} \) has a constant norm.

Next we want to add an interaction term \( S_{em} \) between a charged particle and an electromagnetic field. The simplest possible action is to integrate the potential \( A_{\mu} \) along the world-line \( x^{\mu}(\sigma) \) of the particle,\n
\[
S_{em} = -q \int dx^{\mu} A_{\mu}(x) = -q \int d\sigma \frac{dx^{\mu}}{d\sigma} A_{\mu}(x) .
\]

In the last expression, we can view \( q\dot{x}^{\mu} \) as the current \( j^{\mu} \) induced by the particle and thus the interaction has the form \( j^{\mu} A_{\mu} \). Any candidate for \( S_{em} \) should be invariant under a gauge transformation of the potential,

\[
A_{\mu}(x) \to A_{\mu}(x) + \partial_{\mu}\Lambda(x) .
\]

This is the case, since the induced change in the action,

\[
\delta\Lambda S_{em} = -q \int_{1}^{2} d\sigma \frac{dx^{\mu}}{d\sigma} \left[ \partial_{\mu}(\Lambda(x)) - q\partial^{\lambda} A_{\lambda}(x) \right] = -q\int_{1}^{2} d\Lambda = -q[\Lambda(2) - \Lambda(1)] ,
\]

depends only on the endpoints. Thus the variation of this contribution to the action vanishes, \( \delta(\delta\Lambda S_{em}) = 0 \), keeping the endpoints fixed.

Assuming that the Lagrangian is additive,

\[
L = L_0 + L_{em} = -m[\eta_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}]^{1/2} - q\frac{dx^{\mu}}{d\sigma} A_{\mu}(x)
\]

the Lagrange equations give now

\[
\frac{d}{d\sigma} \left[ \frac{mdx_{\alpha}/d\sigma}{[\eta_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}]^{1/2}} + qA_{\alpha} \right] = q \frac{dx^{\lambda}}{d\sigma} \frac{\partial A_{\lambda}(x)}{\partial x^{\alpha}} .
\]

Performing then the differentiation of \( A(x(\sigma)) \) with respect to \( \sigma \) and moving it to the RHS, we find

\[
m \frac{d}{d\sigma} \left[ \frac{dx_{\alpha}/d\sigma}{d\tau/d\sigma} \right] = q \left( \frac{dx^{\lambda}}{d\sigma} \frac{\partial A_{\lambda}(x)}{\partial x^{\alpha}} - \frac{dx^{\lambda}}{d\sigma} \frac{\partial A_{\alpha}(x)}{\partial x^{\lambda}} \right) = q \frac{dx^{\lambda}}{d\sigma} F_{\alpha\lambda} ,
\]

where we introduced the electromagnetic field-strength tensor \( F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \). Choosing \( \sigma = \tau \) we obtain the covariant version of the Lorentz equation,

\[
m \frac{d^2 x^{\alpha}/d\tau^2}{d\tau/d\sigma} = q F_{\alpha\lambda} u^{\lambda} .
\]

You should work through problem 1.4, if this equation and the covariant formulation of the Maxwell equations are not familiar to you.
1. Introduction

1.2. Differential geometry

We develop in this section the techniques required to do analysis in spaces which look only locally as $\mathbb{R}^n$ or Minkowski space $\mathbb{R}^{1,3}$. Perhaps the most important application of these methods is Einstein’s theory of general relativity, where Minkowski space is replaced by a curved space-time. However, most of the mathematical structures we introduce have also a close analogue in gauge theories which are theories we will use later on to describe electroweak and strong interactions.

**Equivalence principle**  
The idea underlying the equivalence principle emerged in the 16th century, when among others Galileo Galilei found experimentally that the acceleration $g$ of a test mass in a gravitational field is universal. Because of this universality, the gravitating mass $m_g$ and the inertial mass $m_i$ are identical in classical mechanics. While $m_i = m_g$ can be achieved for one material always by a convenient choice of units, there should be in general deviations for test bodies with differing compositions. Current limits for departures from universal gravitational attraction for different materials are however extremely tight, $|\Delta g_i/g| < 10^{-12}$.

As a result, gravity has compared to the three other known fundamental interactions the unique property that it can be switched-off locally: Inside a freely falling elevator, one does not feel any gravitational effects except tidal forces. The latter arise if the gravitational field is non-uniform and tries to deform the elevator. Inside a sufficiently small freely falling system, also tidal effects play no role. Einstein promoted the equivalence of inertial and gravitating mass to the postulate of the “strong equivalence principle:” In a small enough region around the center of a freely falling coordinate system all physics is described by the laws of special relativity.

In general relativity, the gravitational force of Newton’s theory that accelerates particles in an Euclidean space is replaced by a curved space-time in which particles move force-free along geodesic lines. In particular, photons move still as in special relativity along curves satisfying $ds^2 = 0$, while all effects of gravity are now encoded in the non-Euclidean geometry of space-time which is determined by the line-element $ds^2$ or the metric tensor $g_{\mu\nu}$,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \quad (1.66)$$

Switching on a gravitational field, the metric tensor $g_{\mu\nu}$ can be transformed only locally by a coordinate change into the form $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Thus we should develop the tools necessary to do analysis on a curved manifold $\mathcal{M}$ which geometry is described by the metric tensor $g_{\mu\nu}$.

1.2.1. Manifolds and tensor fields

**Manifolds**  
A manifold $\mathcal{M}$ is any set that can be continuously parametrized. The number of independent parameters needed to specify uniquely any point of $\mathcal{M}$ is its dimension $n$, the parameters $x = (x^1, \ldots, x^n)$ are called coordinates. Locally, a manifold with dimension $n$ can be approximated by $\mathbb{R}^n$. Examples are Lie groups, the phase space $(q^i, p_i)$ of classical mechanics with dimension $2n$, and space-time.

We require the manifold to be smooth: the transitions from one set of coordinates to another one, $x^i = f(\tilde{x}^1, \ldots, \tilde{x}^n)$, should be $C^\infty$. In general, it is impossible to cover all $\mathcal{M}$ with one coordinate system that is well-defined on all $\mathcal{M}$. An example are spherical coordinate $(\theta, \phi)$...
1.2. Differential geometry on a sphere $S^2$, where $\phi$ is ill-defined at the poles. Instead one has to cover the manifold with patches of different coordinates that partially overlap.

**Vector fields** A vector field $V(x^\mu)$ on (a subset $\mathcal{S}$ of) $M$ is a set of vectors associating to each space-time point $x^\mu \in \mathcal{S}$ exactly one vector. The paradigm for such a vector field is the four-velocity $u(\tau) = \frac{dx}{d\tau}$ which is the tangent vector to the world-line $x(\tau)$ of a particle. Since the differential equation $\frac{dx}{d\sigma} = X(\sigma)$ has locally always a solution, we can find for any given $X$ a curve $x(\sigma)$ which has $X$ as tangent vector. Although the definition $u(\tau) = \frac{dx}{d\tau}$ coincides with the one familiar from Minkowski space, there an important difference: In a general manifold, we can not imagine a vector $V$ as an “arrow” $\overrightarrow{PP'}$ pointing from a certain point $P$ to another point $P'$ of the manifold. Instead, the vectors $V$ generated by all smooth curves through $P$ span a $n$-dimensional vector space at the point $P$ called tangent space $T_P$.

We can visualise the tangent space for the case of a two-dimensional manifold embedded in $\mathbb{R}^3$: At any point $P$, the tangent vectors lie in a plane $\mathbb{R}^2$ which we can associate with $T_P$. In general, $T_P \neq T_{P'}$ and we cannot simply move a vector $V(x^\mu)$ to another point $P'$ of the manifold. Therefore we cannot differentiate a vector field without introducing an additional mathematical structure which allows us to transport a vector from one tangent space to another.

If we want to decompose the vector $V(x^\mu)$ into components $V^\nu(x^\mu)$, we have to introduce a basis $e_\mu$ in the tangent space. There are two natural choices for such a basis: First, we could use four Cartesian basis vectors as in a Cartesian inertial system in Minkowski space. We will follows this approach later, when we discuss gravity as a gauge theory in chapter 15. Now, we will use the more conventional approach and use as basis vectors the tangential vectors along the coordinate lines $x^\mu$ in $M$,

$$e_\mu = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu. \quad (1.67)$$

Note that the index $\mu$ with value $i$ in $e_\mu$ denotes the $i$th basis vector $e_\mu = (0, \ldots, 1, \ldots 0)$, with an one at the $i$th position, not a component. Using this basis, a vector can be decomposed as

$$V = V^\mu e_\mu = V^\mu \partial_\mu. \quad (1.68)$$

A coordinate change

$$x^\mu = f(\tilde{x}^1, \ldots, \tilde{x}^n), \quad (1.69)$$

or more briefly $x^\mu = x^\mu(\tilde{x}^\nu)$ changes the basis vectors as

$$e_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\nu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{e}_\nu. \quad (1.70)$$

Therefore the vector $V$ will be invariant under general coordinate transformations,

$$V = V^\mu \partial_\mu = \tilde{V}^\mu \tilde{\partial}_\mu = \tilde{V}, \quad (1.71)$$

if its components transform opposite to the basis vectors $e_\mu = \partial_\mu$, or

$$V^\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{V}^\nu. \quad (1.72)$$
1. Introduction

Covectors or 1-forms  In quantum mechanics, we use Dirac’s bracket notation to associate to each vector \( |a⟩ \) a dual vector \( ⟨a| \) and to introduce a scalar product \( ⟨a| b⟩ \). If the vectors \( |n⟩ \) form a basis, then the dual basis \( ⟨n| \) is defined by \( ⟨n| n'⟩ = \delta_{nn'} \).

Similarly, we define a basis \( e^\mu \) dual to the basis \( e_\mu \) in \( T_P \) by

\[
e^\mu (e_\nu) = \delta^\mu_\nu.
\]

This basis can be used to form a new vector space \( T^*_P \) called the cotangent space which is dual to \( T_P \). Its elements \( \omega \) are called covectors or one-forms,

\[
\omega = \omega_\mu e^\mu.
\]

Combining a vector and an one-form, we obtain a map into the real numbers,

\[
\omega(V) = \omega_\mu V^\nu e^\mu (e_\nu) = \omega_\mu V^\mu.
\]

The last equality shows that we can calculate \( \omega(V) \) in component form without reference to the basis vectors. In order to simplify notation, we will use therefore in the future simply \( \omega_\mu V^\mu \); we also write \( e^\mu e_\mu \) instead of \( e^\mu (e_\nu) \).

Using a coordinate basis, the duality condition (1.73) is obviously satisfied, if we choose \( e_\mu = dx^\mu \). Then the 1-form becomes

\[
\omega = \omega_\mu dx^\mu.
\]

Thus the familiar “infinitesimals" \( dx^\mu \) are actually the finite basis vectors of the cotangent space \( T^*_P \).

We require again that the transformation of the components \( \omega_\mu \) of a covector cancels the transformation of the basis vectors,

\[
\omega_\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{\omega}_\nu.
\]

This condition guarantees that the covector itself is an invariant object, since

\[
\omega = \omega_\mu dx^\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{\omega}_\nu \frac{\partial x^\rho}{\partial \tilde{x}^\sigma} d\tilde{x}^\sigma = \tilde{\omega}_\mu d\tilde{x}^\mu = \tilde{\omega}.
\]

Covariant and contravariant tensors  Next we want to generalise the concept of vectors and covector. We call a vector \( \mathbf{X} \) also a contravariant tensor of rank one, while we call a covector also a covariant vector or covariant tensor of rank one. A general tensor of rank \((n,m)\) is a multilinear map

\[
T = T^{\alpha_1,...,\nu}_{\alpha,...,\beta} \frac{\partial}{\partial x^\alpha} \otimes \cdots \otimes \frac{\partial}{\partial x^\nu} \otimes dx^{\alpha_1} \otimes \cdots \otimes dx^{\beta}
\]

which components transforms as

\[
\tilde{T}^{\mu_1,...,\nu}_{\alpha_1,...,\beta}(\tilde{x}) = \frac{\partial \tilde{x}^{\mu_1}}{\partial x^\nu} \cdots \frac{\partial \tilde{x}^{\mu_{\nu}}}{\partial x^\nu} \frac{\partial x^{\alpha_1}}{\partial \tilde{x}^{\gamma}} \cdots \frac{\partial x^{\beta}}{\partial \tilde{x}^{\gamma}} T^{\rho_1,...,\gamma}_{\gamma,...,\delta}(x)
\]

under a coordinate change.
1.2 Differential geometry

**Metric tensor** A Riemannian manifold is a differentiable manifold containing as additional structure a symmetric tensor field $g_{\mu\nu}$ which allows us to measure distances and angles. We define the scalar product of two vectors $a(x)$ and $b(x)$ which have the coordinates $a^\mu$ and $b^\mu$ in a certain basis $e_\mu$ as

$$ a \cdot b = (a^\mu e_\mu) \cdot (b^\nu e_\nu) = (e_\mu \cdot e_\nu)a^\mu b^\nu = g_{\mu\nu}a^\mu b^\nu. \quad (1.81) $$

Thus we can evaluate the scalar product between any two vectors, if we know the symmetric matrix $g_{\mu\nu}$ composed out of the $N^2$ products of the basis vectors,

$$ g_{\mu\nu}(x) = e_\mu(x) \cdot e_\nu(x), \quad (1.82) $$
at any point $x$ of the manifold. This symmetric matrix $g_{\mu\nu}$ is called *metric tensor*.

In the same way, we define for the dual basis $e^\mu$ the metric $g^{\mu\nu}$ via

$$ g^{\mu\nu} = e^\mu \cdot e^\nu. \quad (1.83) $$

A comparison with Eq. (1.75) shows that the metric $g^{\mu\nu}$ maps covariant vectors $X_\mu$ into contravariants vectors $X^\mu$, while $g_{\mu\nu}$ provides a map into the opposite direction. In the same way, we can use the metric tensor to raise and lower indices of any tensor.

Next we want to determine the relation of $g^{\mu\nu}$ with $g_{\mu\nu}$. We multiply $e^\mu$ with $e_\nu = g_{\mu\nu}e^\nu$, obtaining

$$ \delta^\mu_\rho = e^\rho \cdot e_\mu = e^\rho \cdot g_{\mu\nu}e^\nu = g^{\rho\nu}g_{\mu\nu} \quad (1.84) $$
or

$$ \delta^\mu_\rho = g_{\mu\nu}g^{\nu\rho}. \quad (1.85) $$

Thus the components of the covariant and the contravariant metric tensors, $g_{\mu\nu}$ and $g^{\mu\nu}$, are inverse matrices of each other. Moreover, the mixed metric tensor of rank $(1,1)$ is given by the Kronecker delta, $g^{\nu\mu} = \delta^\nu_\mu$. Note that the trace $\text{tr}(g_{\mu\nu})$ of the metric tensor is therefore not $1 - 1 - 1 - 1 = -2$, but

$$ \text{tr}(g_{\mu\nu}) = g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4, \quad (1.86) $$
because we have to contract an upper and a lower index.

### 1.2.2 Covariant derivative and the geodesic equation

**Covariant derivative** In an inertial system in Minkowski space, taking the partial derivative $\partial_\mu$ maps a tensor of rank $(n,m)$ into a tensor of rank $(n,m+1)$. Additionally, this map obeys linearity and the Leibniz product rule. We will see that the partial derivative in a curved space does not satisfy in general anymore these rules. Therefore, we have to introduce a modified derivative which we call covariant derivative.

We start by considering the gradient $\partial_\mu \phi$ of a scalar $\phi$. Since a scalar by definition does not depend on the coordinate system, $\phi(x) = \tilde{\phi}(\tilde{x})$, its gradient transforms as

$$ \partial_\mu \phi \rightarrow \tilde{\partial}_\mu \tilde{\phi} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \partial_\nu \phi. \quad (1.87) $$

Thus the gradient is a covariant vector. Similarly, the derivative of a vector $V$ transforms as a tensor,

$$ \partial_\mu V \rightarrow \tilde{\partial}_\mu \tilde{V} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \partial_\nu V, \quad (1.88) $$
because $V$ is an invariant quantity. If we consider however its components $V^\mu = e^\mu \cdot V$, then the moving coordinate basis in curved space-time, $\partial_\mu e^\nu \neq 0$, leads to an additional term in the derivative,
\[ \partial_\mu V^\nu = e^\nu \cdot (\partial_\mu V) + V \cdot (\partial_\mu e^\nu). \] (1.89)
The term $e^\nu \cdot (\partial_\mu V)$ transforms as a tensor, since both $e^\nu$ and $\partial_\mu V$ are tensors. This implies that the combination of the two remaining terms has to transform as tensor too, which we define as covariant derivative
\[ \nabla_\mu V^\nu \equiv e^\nu \cdot (\partial_\mu V) = \partial_\mu V^\nu - V \cdot (\partial_\mu e^\nu). \] (1.90)
The first equality tells us that we can view the covariant derivative $\nabla_\mu V^\nu$ as the projection of $\partial_\mu V$ onto the direction $e^\nu$.

We expand now the partial derivatives of the basis vectors as a linear combination of the basis vectors,
\[ \partial_\rho e^\mu = -\Gamma^\mu_{\rho\sigma} e^\sigma. \] (1.91)

The $n^3$ numbers $\Gamma^\mu_{\rho\sigma}$ are called (affine) connection coefficients or symbols, in order to stress that they are not the components of a tensor. You are asked to derive the transformation properties of the connection coefficients in problem \[ \text{[1.10]} \] Introducing this expansion into (1.90) we can rewrite the covariant derivative of a vector field as
\[ \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho. \] (1.92)

Using $\nabla_\sigma \phi = \partial_\sigma \phi$ and requiring that the usual Leibniz rule is valid for $\phi = X_\mu X^\mu$ leads to
\[ \nabla_\sigma X_\mu = \partial_\sigma X_\mu - \Gamma^\nu_{\mu\rho} X_\nu \] (1.93)
and to
\[ \partial_\rho e_\mu = \Gamma^\sigma_{\mu\rho} e^\sigma. \] (1.94)

For a general tensor, the covariant derivative is defined by the same reasoning as
\[ \nabla_\sigma T^\mu_{\nu...} = \partial_\sigma T^\mu_{\nu...} + \Gamma^\rho_{\mu\sigma} T^\mu_{\rho...} + \ldots - \Gamma^\rho_{\nu\sigma} T^\mu_{\rho...} - \ldots \] (1.95)
Note that it is the last index of the connection coefficients that is the same as the index of the covariant derivative. The plus sign goes together with upper (superscripts), the minus with lower indices.

**Parallel transport** We say a tensor $T$ is parallel transported along the curve $x(\sigma)$, if its components $T^\mu_{\nu...}$ stay constant. In flat space, this means simply
\[ \frac{d}{d\sigma} T^\mu_{\nu...} = \frac{dx^\alpha}{d\sigma} \partial_\alpha T^\mu_{\nu...} = 0. \] (1.96)
In curved space, we have to replace the normal derivative by a covariant one. We define the directional covariant derivative along $x(\sigma)$ as
\[ \frac{D}{d\sigma} = \frac{dx^\alpha}{d\sigma} \nabla_\alpha. \] (1.97)

Then a tensor is parallel transported along the curve $x(\sigma)$, if
\[ \frac{D}{d\sigma} T^\mu_{\nu...} = \frac{dx^\alpha}{d\sigma} \nabla_\alpha T^\mu_{\nu...} = 0. \] (1.98)
1.2. Differential geometry

**Metric compatibility** Relations like \(ds^2 = g_{\mu\nu}dx^\mu dx^\nu\) or \(g_{\mu\nu}p^\mu p^\nu = m^2\) become invariant under parallel transport only, if the metric tensor is covariantly constant,

\[
\nabla_\sigma g_{\mu\nu} = \nabla_\sigma g^{\mu\nu} = 0. \quad (1.99)
\]

A connection satisfying Eq. (1.99) is called metric compatible and leaves lengths and angles invariant under parallel transport. This requirement guarantees that we can introduce locally in the whole space-time Cartesian inertial coordinate systems where the laws of special relativity are valid. Moreover, these local inertial systems can be consistently connected by parallel transport using an affine connection satisfying the constraint (1.99).

Note that we already build in this constraint into our definition of the covariant derivative: If the length of a vector would not be conserved under parallel transport, then we should differentiate in (1.89) also the scalar product in \(V^\mu = e^\mu \cdot V\), leading to additional terms in (1.90).

**Geodesic equation** Which one of the infinitely many affine connection should we use to calculate covariant derivatives on a general space-time? Ultimately, the combined action for gravity and matter should select the correct connection—an approach we resume in Chapter 15. For the moment, we use a simple workaround which does not require the knowledge of the action of gravity: In flat space, we know that the solution to the equation of motions of a free particle is a straight line. Such a path is characterized by two properties: It is the shortest curve between the considered initial and final point, and it is the curve whose tangent vectors remains constant if they are parallel transported along it. Both conditions can be generalised to curved space and the curves satisfying either one of them are called geodesics.

Using the definition of a geodesics as the “straightest” line on a manifold requires as mathematical structure only the possibility to parallel transport a tensor and thus the existence of an affine connection. In contrast, the concept of an “extremal” (shortest or longest) line between two points on a manifold relies on the existence of a metric. Requiring that these two definitions agree fixes uniquely the connection to be used in the covariant derivative.

We start by defining geodesics as the “straightest” line or an autoparallel curve on a manifold—the case which is almost trivial: The tangent vector along the path \(x(\sigma)\) is \(u^\mu = dx^\mu/d\sigma\). Then the requirement (1.98) of parallel transport for \(u^\mu\) becomes

\[
\frac{D}{d\sigma} dx^\mu = \frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\sigma} \frac{dx^\sigma}{d\sigma} = 0. \quad (1.100)
\]

Introducing the short-hand \(\dot{x}^\mu = dx^\mu/d\sigma\), we obtain the geodesic equation in its usual form,

\[
\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0. \quad (1.101)
\]

Note that the antisymmetric part of the connection \(\Gamma^\mu_{\rho\sigma}\) drops out of the geodesic equation, because \(\dot{x}^\rho \dot{x}^\sigma\) is symmetric: Contracting a symmetric tensor \(S_{\mu\nu}\) with an antisymmetric tensor \(A_{\mu\nu}\), we find

\[
S_{\mu\nu} A^{\mu\nu} = -S_{\mu\nu} A^{\nu\mu} = -S_{\nu\mu} A^{\nu\mu} = -S_{\mu\nu} A^{\mu\nu}, \quad (1.102)
\]

where we used first the antisymmetry of \(A^{\mu\nu}\), then the symmetry of \(S_{\mu\nu}\), and finally exchanged the dummy summation indices. Thus the contraction is zero, \(S_{\mu\nu} A^{\mu\nu} = 0\).
Next we derive the defining equation for a geodesic as the extremal curve between two points on a manifold. The Lagrangian of a free particle in Minkowski space, Eq. (1.58), is generalized to a curved space-time manifold with the metric tensor \( g_{\mu\nu} \) by replacing \( \eta_{\mu\nu} \) with 

\[ L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \]  

(1.103)

The Lagrange equations are 

\[ \frac{d}{d\sigma} \frac{\partial L}{\partial (\dot{x}^\lambda)} - \frac{\partial L}{\partial x^\lambda} = 0. \]  

(1.104)

Only the metric tensor \( g_{\mu\nu} \) depends on \( x^\mu \) and thus \( \partial L/\partial x^\lambda = g_{\mu\nu,\lambda} \dot{x}^\mu \dot{x}^\nu \). Here we introduced also the short-hand notation \( g_{\mu\nu,\lambda} = \partial_\lambda g_{\mu\nu} \) for partial derivatives. Now we use \( \partial \dot{x}^\mu/\partial \dot{x}^\nu = \delta^\mu_\nu \) and apply the chain rule for \( g_{\mu\nu}(x(\sigma)) \), obtaining first 

\[ g_{\mu\nu,\lambda} \dot{x}^\mu \dot{x}^\nu = 2 \frac{d}{d\sigma} (g_{\mu\lambda} \dot{x}^\mu) = 2(g_{\mu\lambda,\nu} \dot{x}^\mu \dot{x}^\nu + g_{\mu\lambda} \ddot{x}^\mu) \]  

(1.105)

and then 

\[ g_{\mu\lambda} \ddot{x}^\mu + \frac{1}{2}(2g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda}) \dot{x}^\mu \dot{x}^\nu = 0. \]  

(1.106)

Next we rewrite the second term as 

\[ 2g_{\lambda\mu,\nu} \dot{x}^\nu \dot{x}^\nu = (g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu}) \dot{x}^\nu \dot{x}^\nu, \]  

(1.107)

multiply everything by \( g^{\kappa\mu} \) and arrive at our desired result, 

\[ \ddot{x}^\kappa + \frac{1}{2}g^{\kappa\lambda}(g_{\mu\lambda,\nu} + g_{\mu\nu,\lambda}) \dot{x}^\mu \dot{x}^\nu = \ddot{x}^\kappa + \{^{\kappa}\}_{\nu\lambda} \dot{x}^\mu \dot{x}^\nu = 0. \]  

(1.108)

Here we defined in the last step the Christoffel symbols 

\[ \{^{\mu}\}_{\nu\lambda} = \frac{1}{2}g^{\mu\kappa} (\partial_\nu g_{\kappa\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\kappa g_{\nu\lambda}). \]  

(1.109)

They are also called Levi-Civita or Riemannian connection. Comparison with Eq. (1.101) shows that our two geodesic equations agree, if we choose as connection the Christoffel symbols. Moreover, the Christoffel symbol is symmetric in its two lower indices and, as we will show next, compatible to the metric tensor.

We define

\[ \Gamma_{\mu\nu,\lambda} = g_{\mu\kappa} \Gamma_{\nu\lambda}^\kappa. \]  

(1.110)

Thus \( \Gamma_{\mu\nu,\lambda} \) is symmetric in the last two indices. Then it follows 

\[ \Gamma_{\mu\nu,\lambda} = \frac{1}{2} (\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda}). \]  

(1.111)

Adding \( 2\Gamma_{\mu\nu,\lambda} \) and \( 2\Gamma_{\nu\mu,\lambda} \) gives 

\[ 2(\Gamma_{\mu\nu,\lambda} + \Gamma_{\nu\mu,\lambda}) = \partial_\kappa g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda} + \partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\kappa g_{\nu\lambda} = 2\partial_\lambda g_{\mu\nu}. \]  

(1.112)

\[ 2\partial_\lambda g_{\mu\nu} \]

(1.113)

\[ ^2\text{We showed that the metric tensor can be used to raise or to lower tensor indices, but the connection } \Gamma \text{ is not a tensor.} \]
1.2. Differential geometry

or

\[ \partial_\lambda g_{\mu\nu} = \Gamma_{\mu\nu\lambda} \cdot \tag{1.14} \]

Applying the general rule for covariant derivatives, Eq. (1.95), to the metric,

\[ \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa_{\mu\nu\kappa} - \Gamma^\kappa_{\nu\lambda\kappa} = \partial_\lambda g_{\mu\nu} - \Gamma_{\mu\lambda\nu} - \Gamma_{\nu\lambda\mu}, \tag{1.115} \]

and inserting Eq. (1.114) shows that

\[ \nabla_\lambda g_{\mu\nu} = \nabla_\lambda g_{\mu\nu} = 0 \tag{1.116} \]

Hence \( \nabla_\lambda \) commutes with contracting indices,

\[ \nabla_\lambda (X^\mu X_\mu) = \nabla_\lambda (g_{\mu\nu} X^\mu X_\nu) = g_{\mu\nu} \nabla_\lambda (X^\mu X_\nu) \tag{1.117} \]

and conserves the norm of vectors as announced. Thus the Levi-Civita connection is symmetric and compatible with the metric. These two properties specify uniquely the connection.

The fact that we obtained as connection the Christoffel symbols when we derived the geodesic equation for a classical spin-less point particle justifies the use of a torsionless connection which is metric compatible: Although e.g. a star consists of a collection of individual particles carrying spin \( s_i \), its total spin sums up to zero, \( \sum_i s_i \approx 0 \), because the \( s_i \) are uncorrelated. Thus we can describe macroscopic matter in general relativity as a classical spinless point particle (or fluid if its extension is important) leading to a symmetric connection. In the remainder of this section, we will use always as affine connection the Levi-Civita connection. Following standard practise, we will denote these connection coefficients also with \( \Gamma^a_{\; 
abla \mu \nu} \).

**Example: Sphere \( S^2 \).** Calculate the Christoffel symbols of the two-dimensional unit sphere \( S^2 \).

The line-element of the two-dimensional unit sphere \( S^2 \) is given by \( ds^2 = d\vartheta^2 + \sin^2 \vartheta d\phi^2 \). A faster alternative to the definition (1.109) of the Christoffel coefficients is the use of the geodesic equation:

From the Lagrange function

\[ L = g_{ab} \dot{x}^a \dot{x}^b = \dot{\vartheta}^2 + \sin^2 \vartheta \dot{\phi}^2 \]

we find

\[ \frac{\partial L}{\partial \phi} = 0 \quad , \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = 2 \sin^2 \vartheta \dot{\phi} + 4 \cos \vartheta \sin \vartheta \dot{\vartheta} \dot{\phi} \]

\[ \frac{\partial L}{\partial \dot{\vartheta}} = 2 \cos \vartheta \sin \vartheta \dot{\phi}^2 \quad , \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\vartheta}} \right) = 2 \dot{\vartheta} \]

and thus the Lagrange equations are

\[ \ddot{\phi} + 2 \cot \vartheta \dot{\vartheta} \dot{\phi} = 0 \quad \text{and} \quad \ddot{\vartheta} - \cos \vartheta \sin \vartheta \dot{\vartheta}^2 = 0. \]

Comparing with the geodesic equation \( \ddot{x}^\alpha + \Gamma^\alpha_{\; \mu \nu} \dot{x}^\mu \dot{x}^\nu = 0 \), we can read off the non-vanishing Christoffel symbols as \( \Gamma^\phi_{\; \vartheta \phi} = \Gamma^\phi_{\; \phi \vartheta} = \cot \vartheta \) and \( \Gamma^\vartheta_{\; \phi \vartheta} = - \cos \vartheta \sin \vartheta \). (Note that 2 \cot \vartheta = \Gamma^\phi_{\; \vartheta \phi} + \Gamma^\phi_{\; \phi \vartheta}.)

We can use also the Hamiltonian formulation for a relativistic particle. From the Lagrangian

\[ L = \frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu \]

we determine first the conjugated momenta \( p_\mu = \partial L / \partial \dot{x}^\mu = \dot{x}^\mu \) and perform then a Legendre transformation,

\[ H(x^\mu, p_\mu, \tau) = p_\mu \dot{x}^\mu - L(x^\mu, \dot{x}^\mu, \tau) = \frac{1}{2} g^{\mu \nu} p_\mu p_\nu. \tag{1.118} \]

\(^3\)The curious student may wonder if orbital angular momentum leads to torsion: The answer is no, because one cannot define an orbital angular momentum density which transforms properly as a tensor, cf. Eq. (3.24f).
Hamilton equations give then
\[ \dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = g^{\mu \nu} p_\nu \]  
(1.119)
and
\[ \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2} g^{\alpha \beta} \partial_{x^\mu} p_\alpha p_\beta . \]  
(1.120)

This is a useful alternative to the standard geodesic equation: First, it makes clear that the momentum component \( p_\mu \) is conserved, if the metric tensor is independent of the coordinate \( x^\mu \). Second, we can calculate \( \dot{p}_\mu \) directly from the metric tensor, without knowing the Christoffel symbols. Combining the Eqs. (1.119) and (1.120), one can rederive the standard form of the geodesic equation, cf. problem 1.9.

1.2.3. Integration and Gauss’ theorem

Having defined the covariant derivative of an arbitrary tensor field, it is natural to ask how the inverse, the integral over a tensor field, can be defined. The short answer is that this is in general impossible: Integrating a tensor field requires to sum tensors at different points in an invariant way, which is only possible for scalars.

If we attempt to generalise an integral like
\[ I = \int d^4 x \, \phi(x) \]
valid in an Cartesian inertial frame \( x^\mu \) in Minkowski space to a general space-time with coordinates \( \tilde{x} \), we have to take into account that the Jacobian determinant \( J = \text{det}(\partial \tilde{x}^\mu / \partial x^\rho) \) of the coordinate transformation can deviate from one. We can express the Jacobian \( J \) by the determinant \( g \equiv \text{det}(g_{\mu \nu}) \) of the metric tensor as follows: Applying the transformation law of the metric tensor,
\[ \tilde{g}^{\mu \nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} g^{\rho \sigma}(x) \]
(1.121)
to the case where the \( x^\rho \) are inertial coordinates, we obtain with \( g = \text{det}(g_{\mu \nu}) = \text{det}(\eta_{\mu \nu}) = -1 \) that
\[ \text{det}(\tilde{g}) = J^2 \text{det}(g) = -J^2 \]
(1.122)
or \( J = \sqrt{|g|} \). Thus \( I = \int d^4 x \sqrt{|g|} \phi \) is an invariant definition of an integral over a scalar field which agrees for inertial coordinates with the one known from special relativity. Now we choose as scalar \( \phi \) the divergence of a vector field, \( \phi = \nabla_\mu X^\mu \), or
\[ I = \int d^4 x \sqrt{|g|} \nabla_\mu X^\mu . \]
(1.123)

Our aim is to derive a generalised Gauss’ theorem. Let us recall that this theorem allows us to convert a \( n \) dimensional volume integral over the divergence of a vector field into a \( n - 1 \) dimensional surface integral of the vector field,
\[ \int_\Omega d^4 x \partial_\mu X^\mu = \int_{\partial \Omega} d S_\mu X^\mu . \]
(1.124)
The only way how (1.124) may be reconciled with (1.123) is to hope that the contracted Christoffel symbols can be expressed as a partial derivative of \( \sqrt{|g|} \) in such a way that the unwanted terms in (1.123) disappear. In order to check this possibility, we calculate therefore now the partial derivative of the metric determinant \( g \).
1.2. Differential geometry

**Useful formula for derivatives** We start considering the variation of a general matrix $M$ with elements $m_{ij}(x)$ under an infinitesimal change of the coordinates, $\delta x^\mu = \varepsilon x^\mu$. It is convenient to look at the change of $\ln \det M$,

$$
\delta \ln \det M \equiv \ln \det(M + \delta M) - \ln \det(M) \\
= \ln \det[M^{-1}(M + \delta M)] = \ln[1 + M^{-1}\delta M] = \\
= [1 + \ln(1 + \varepsilon)] + \mathcal{O}(\varepsilon^2) = \text{tr}(M^{-1}\delta M) + \mathcal{O}(\varepsilon^2). \tag{1.125}
$$

In the last step, we used $\ln(1 + \varepsilon) = \varepsilon + \mathcal{O}(\varepsilon^2)$. Expressing now both the LHS and the RHS as $\delta f = \partial_\mu f \delta x^\mu$ and comparing then the coefficients of $\delta x^\mu$ gives

$$
\partial_\mu \ln \det M = \text{tr}(M^{-1}\partial_\mu M).
$$

Applied to derivatives of $\sqrt{|g|}$, we obtain

$$
\frac{1}{2}g^{\mu\nu}\partial_\lambda g_{\mu\nu} = \frac{1}{2}\partial_\lambda \ln g = \frac{1}{\sqrt{|g|}}\partial_\lambda (\sqrt{|g|}). \tag{1.127}
$$

This expression coincides with contracted Christoffel symbols,

$$
\Gamma^\nu_{\mu\nu} = \frac{1}{2}g^{\mu\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) = \frac{1}{2}g^{\mu\nu}\partial_\gamma g_{\mu\nu} = \frac{1}{2}\partial_\nu \ln g = \frac{1}{\sqrt{|g|}}\partial_\nu (\sqrt{|g|}). \tag{1.128}
$$

Next we consider the divergence of a vector field,

$$
\nabla_\mu X^\mu = \partial_\mu X^\mu + \Gamma^\nu_{\mu\lambda}X^\lambda = \partial_\mu X^\mu + \frac{1}{\sqrt{|g|}}(\partial_\mu \sqrt{|g|})X^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu (\sqrt{|g|}X^\mu), \tag{1.129}
$$

and of an anti-symmetic tensor of rank 2,

$$
\nabla_\mu A^{\mu\nu} = \partial_\mu A^{\mu\nu} + \Gamma^\mu_{\lambda\mu}A^{\lambda\nu} + \Gamma^\nu_{\lambda\mu}A^{\mu\lambda} = \partial_\mu A^{\mu\nu}. \tag{1.130}
$$

In the latter case, the Christoffel symbols cancel. This generalises to completely anti-symmetric tensors of all orders. In contrast, we find for a symmetric tensor of rank 2,

$$
\nabla_\mu S^{\mu\nu} = \partial_\mu S^{\mu\nu} + \Gamma^\mu_{\lambda\mu}S^{\nu\lambda} + \Gamma^\nu_{\lambda\mu}S^{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\mu (\sqrt{|g|}S^{\mu\nu}) + \Gamma^\nu_{\lambda\mu}S^{\mu\lambda}. \tag{1.131}
$$

We can express in the second term the Christoffel symbol as derivative of the metric tensor,

$$
\nabla_\mu S^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu (\sqrt{|g|}S^{\mu\nu}) - \frac{1}{2}(\partial_\nu g_{\mu\lambda})S^{\mu\lambda}. \tag{1.132}
$$

Thus we can perform the covariant derivative of $S^{\mu\nu}$ without the need to know the Christoffel symbols.

**Gauss’ theorem** for the divergence of a vector field follows now directly from Eq. (1.129),

$$
\int_\Omega d^4x \sqrt{|g|} \nabla_\mu X^\mu = \int_\Omega d^4x \partial_\mu (\sqrt{|g|}X^\mu) = \int_{\partial\Omega} dS_\mu \sqrt{|g|}X^\mu. \tag{1.133}
$$

If the fields $X^\mu$ vanishes on the boundary, terms like $\nabla_\mu X^\mu$ can be dropped in the action. Similarly, this equation allows us to derive global conservation laws from $\nabla_\mu X^\mu = 0$ in the same way as in Minkowski space. In contrast, the divergence (1.132) of a symmetric tensor of rank two contains an additional term $(\partial_\nu g_{\mu\lambda})S^{\mu\lambda}$ which prohibits the use of Gauss’ theorem.
1. Introduction

Summary

Lagrange’s and Hamilton’s equations follow extremizing the action $S[q^i] = \int_a^b dt \ L(q^i, \dot{q}^i, t)$ and $S[q^i, p_i] = \int_a^b dt \ [p_i \dot{q}^i - H(q^i, p_i, t)]$, respectively.

In a curved space-time we require a connection to compare vectors at different points. The unique connection which is symmetric and compatible with the metric is the Levi-Civita connection.

Further reading

Volume 1 of the Landau-Lifshitz’s series treats classical mechanics in concise way. [Car03] contains a clear introduction to differential geometry on a level accessible for physicists.

Problems

1.1 Higher derivatives.

Find the Lagrange equations for a Lagrange function containing higher derivatives, $L = L(q^i, \dot{q}^i, \ddot{q}^i, \ldots)$.

1.2 Oscillator with friction.

Consider an one-dimensional system described by the Lagrangian

$$L = e^{2 \alpha t} \left[ \frac{1}{2} m \dot{q}^2 - V(q) \right].$$

a.) Show that the equation of motion corresponds to an oscillator with friction term.

b.) Derive the energy lost per time $dE/dt$ of the oscillator, with $E = \frac{1}{2} m \dot{q}^2 + V(q)$.

c.) Show that the result in b.) agrees with the one obtained from the Lagrange equations of the first kind,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i,$$

where the generalised forces $Q^i$ perform the work $\delta A = Q_i \delta q^i$.

d.) Derive the Green function for a harmonic oscillator with friction.

1.3 Conservation laws.

Discuss the symmetries of the Galilean transformations and the resulting conservation laws, following the example of time translation and energy conservation.

1.4 Electromagnetism.

Compare Eq. (1.65) to the usual $F = q(E + v \times B)$ and derive thereby the elements of the field-strength tensor $F_{\mu \nu}$. What is the meaning of the zero component of the Lorentz force?

1.5 Little group

Find the Lorentz transformations $\Lambda^\nu_\mu$ which keep the momentum of a particle invariant, $\Lambda^\nu_\mu p^\mu = p^\nu$, for a) massive particle and b) massless particle, considering an infinitesimal transformation $\omega_{\mu \nu}$. By how many parameters is $\omega_{\mu \nu}$ determined in the two cases?

1.6 Inertial coordinates.

The equivalence principle implies that we can find in a curved space-time locally coordinates with $\ddot{x}^\mu = 0$ and $g_{\mu \nu} = \eta_{\mu \nu}$. Show that this requirement holds in a Riemannian manifold choosing as connection the Christoffel symbols.

1.7 Lie bracket.

Show that the commutator $[V, W]$ of two vector fields $V = V^\mu \partial_\mu$ and $W = W^\mu \partial_\mu$ is again a vector field.

1.8 Torsion

Derive the transformation properties of a connection. Show that the difference of two connections is a tensor.

1.9 Geodesic equation from $H$.

Show that the Eqs. (1.119) and (1.20) imply the
1.2. Differential geometry

standard form of the geodesic equation.

1.10 Affine parameter.

1.11 Hamiltonian of relativistic particle.
1.bis Quantum mechanics

We introduce Feynman’s path integral as an alternative to the standard operator approach to quantum mechanics. Most of our discussion of quantum fields will be based on this approach, and thus becoming familiar with this technique using the simpler case of quantum mechanics is of central importance. Instead of using the path integral or the propagator directly, we will use as basic tool the ground-state persistence amplitude \( \langle 0, \infty | 0, -\infty \rangle \). This quantity is the probability amplitude that a system under the influence of an external force \( J \) stays in its ground-state. Since we can apply an arbitrary force \( J \), it contains all information about the system. Moreover, it is relatively easy to calculate and therefore it will serve us as main tool to study quantum field theories later on.

1.1. Reminder of the operator approach

A classical system described by a Hamiltonian \( H(q^i, p_i, t) \) can be quantised promoting \( q^i \) and \( p_i \) to operators which satisfy the canonical commutation relations \( [q^i, p_j] = i \delta_{ij} \). The latter are the formal expression of Heisenberg’s uncertainty relation. Apart from ordering ambiguities, the Hamilton operator \( H(q^i, p_i, t) \) can be directly read from the Hamiltonian \( H(q^i, p_i, t) \).

General principles A physical system in a pure state is fully described by a probability amplitude

\[
\psi(a, t) = \langle a | \psi(t) \rangle \in \mathbb{C},
\]

where \( \{a\} \) is a set of quantum numbers specifying the system and the states \( |\psi(t)\rangle \) form a complex Hilbert space. The probability to find the specific value \( a_* \) in a measurement is given by \( p(a_*) = |\psi(a_*, t)|^2 \). The possible values \( a_* \) are the eigenvalues of hermitian operators \( A \) and the eigenvectors \( |a\rangle \) form an orthogonal, complete basis. In Dirac’s bra-ket notation,

\[
A|a\rangle = a|a\rangle, \quad \langle a|a'\rangle = \delta (a - a'), \quad \int da|a\rangle \langle a| = 1,
\]

In general, operators do not commute. Their commutation relations can be obtained by the replacement \( \{A, B\} \rightarrow i[A, B] \) in (1.23).

The state of a particle moving in one dimension in a potential \( V(q) \) can be described either by the eigenstates of the position operator \( \hat{q} \) or of the momentum operator \( \hat{p} \). Both form a complete, orthonormal basis, and they are connected by a Fourier transformation which we choose to be asymmetric. This asymmetry is reflected in the completeness relation of the states,

\[
\int dq |q\rangle \langle q| = 1 \quad \text{and} \quad \int \frac{dp}{2\pi} |p\rangle \langle p| = 1.
\]

When there is the danger of an ambiguity, operators will be written with a “hat”; otherwise we drop the hat.
1.1. Reminder of the operator approach

**Time-evolution** Since the states $|\psi(t)\rangle$ form a complex Hilbert space, the superposition principle is valid: If $\psi_1$ and $\psi_2$ are possible states of the system, then also

$$\psi(\alpha, t) = c_1 \psi_1(\alpha, t) + c_2 \psi_2(\alpha, t), \quad c_i \in \mathbb{C}.$$  \hfill (1.4)

In quantum mechanics, a stronger version of this principle holds which states that if $\psi_1(\alpha, t)$ and $\psi_2(\alpha, t)$ are describing the possible time-evolution of the system, then it does also the superposed state $\psi(\alpha, t)$. This implies that the time-evolution is described by a linear, homogeneous differential equation. Choosing it as first-order in time, we can write the evolution equation as

$$i\partial_t |\psi(\alpha, t)\rangle = D|\psi(\alpha, t)\rangle,$$  \hfill (1.5)

where the differential operator $D$ on the RHS has to be still determined.

We call the operator describing the evolution of a state from $\psi(t)$ to $\psi(t')$ the time-evolution operator $U(t', t)$. This operator is unitary, $U^{-1} = U^\dagger$, in order to conserve probability and forms a group, $U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1)$ with $U(t, t) = 1$. For an infinitesimal time step $\delta t$,

$$|\psi(t + \delta t)\rangle = U(t + \delta t, t)|\psi(t)\rangle,$$  \hfill (1.6)

we can set with $U(t, t) = 1$

$$U(t + \delta t, t) = 1 - iH\delta t.$$  \hfill (1.7)

Here we introduced the generator of infinitesimal time-translations $H$. The analogue to classical mechanics suggests that $H$ is the operator version of the classical Hamilton function $H(q, p)$. Hence

$$\frac{|\psi(t + \delta t)\rangle - |\psi(t)\rangle}{\delta t} = -iH |\psi(t)\rangle.$$  \hfill (1.8)

Comparing (1.5) and (1.8) reveals that the generator of infinitesimal time-translations $H$ is equal to the operator $D$ on the RHS of (1.5). We call a time-evolution equation of this type for an arbitrary Hamiltonian $H$ Schrödinger equation.

Plugging $\psi(t) = U(t, 0)|\psi(0)\rangle$ in the Schrödinger equation gives

$$\left[\frac{\partial U(t, 0)}{\partial t} - HU(t, 0)\right]|\psi(0)\rangle = 0.$$  \hfill (1.9)

Since this equation is valid for an arbitrary state $|\psi(0)\rangle$, we can rewrite it as an operator equation,

$$i\partial_t U(t', t) = HU(t', t).$$  \hfill (1.10)

Integrating gives as formal solution

$$U(t', t) = 1 - i \int_t^{t'} dt'' H(t'')U(t'', t).$$  \hfill (1.11)

If $H$ is time-independent, we obtain simply by iteration

$$U(t, t') = \exp(-iH(t - t')).$$  \hfill (1.12)
Propagator We insert the solution of $U$ for a time-independent $H$ into $|\psi(t')\rangle = U(t', t)|\psi(t)\rangle$ and multiply from the left with $\langle q'|$,

$$\psi(q', t') = \langle q'| \psi(t') \rangle = \langle q'| \exp[-iH(t' - t)] |\psi(t)\rangle.$$  \hspace{1cm} (1.13)

Then we insert $1 = \int d^3q |q\rangle \langle q|$, 

$$\psi(q', t') = \int d^3q \langle q'| \exp[-iH(t' - t)] |q\rangle \langle q| \psi(t) \rangle = \int d^3q \ K(q', t'; q, t) \psi(q, t).$$  \hspace{1cm} (1.14)

In the last step we introduced the propagator or Green function $K$ in its coordinate representation,

$$K(q', t'; q, t) = \langle q'| \exp[-iH(t' - t)] |q\rangle.$$  \hspace{1cm} (1.15)

The Green function $K$ equals the probability amplitude for the propagation between two space-time points; $K(q', t'; q, t)$ is therefore also called more specifically two-point Green function. We can express the propagator $K$ by the solutions of the Schrödinger equation, $\psi_n(q, t) = \langle q|n(t)\rangle = \langle q|n\rangle \exp(-iE_nt)$ as 

$$K(q', t'; q, t) = \sum_{n, n'} \delta_{n, n'} \langle q'| n \rangle \exp(-iH(t' - t)) \langle n'| n \rangle \langle n| q \rangle = \sum_n \psi_n(q') \psi_n^*(q) \exp(-iE_n(t' - t)).$$  \hspace{1cm} (1.16)

Let us compute the propagator of a free particle in one dimension, described by the Hamiltonian $H = \frac{p^2}{2m}$. We write with $\tau = t' - t$

$$K(q', t'; q, t) = \langle q'| \exp(-iH\tau) |q\rangle = \langle q'| \exp(-i\tau \frac{p^2}{2m}) \int \frac{dp}{2\pi} |p\rangle \langle p| q \rangle = \int \frac{dp}{2\pi} \exp(-i\tau \frac{p^2}{2m}) \langle q'| p \rangle \langle p| q \rangle = \int \frac{dp}{2\pi} \exp(-i\tau \frac{p^2}{2m} + i(q' - q)p).$$

The integral is Gaussian, if we add an infinitesimal factor $\exp(-\varepsilon p^2)$ to the integrand to ensure the convergence of the integral. Thus the physical value of the energy $E = \frac{p^2}{2m}$ seen as a complex variable is approached from the negative imaginary plane, $E \to E - i\varepsilon$. Taking afterwards the limit $\varepsilon \to 0$, we obtain

$$K(q', t'; q, t) = \left( \frac{m}{2\pi i\tau} \right)^{1/2} \exp[im(q' - q)^2 / 2\tau].$$  \hspace{1cm} (1.17)

\textbf{Example:} Calculate the three Gaussian integrals $A = \int dx \exp(-x^2/2)$, $B = \int dx \exp(-ax^2/2 + bx)$, and $C = \int dx \cdots dx_n \exp(-xAx^2/2 + Jx)$ for a symmetric matrix $A$.

a.) We square the integral and calculate then $A^2$ introducing polar coordinates, $r^2 = x^2 + y^2$,

$$A^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp(- (x^2 + y^2)/2) = 2\pi \int_0^{\infty} dr \ re^{-r^2/2} = 2\pi \int_0^{\infty} dt \ e^{-t} = 2\pi,$$

where we substituted $t = r^2/2$ in the last step. Thus the result for the basic Gaussian integral is $A = \sqrt{2\pi}$. All others solvable variants of Gaussian integrals can be reduced to this result.

b.) We complete the square in the exponent, 

$$-\frac{a}{2} \left( x^2 - \frac{2b}{a} x \right) = -\frac{a}{2} \left( x - \frac{b}{a} \right)^2 + \frac{b^2}{2a}. $$
and shift then the integration variable to \( x' = x - b/a \). The result is

\[
B = \int_{-\infty}^{\infty} dx \exp(-ax^2/2 + bx) = e^{b^2/2a} \int_{-\infty}^{\infty} dx' \exp(-ax'^2/2) = \sqrt{\frac{2\pi}{a}} e^{b^2/2a}.
\]  
(1.18)

c.) The notation \( xAx = x^T Ax \) and \( bx = b^T x \) means now matrix multiplication. We complete again the square,

\[
(X - A^{-1}J)^T A(X - A^{-1}J) = X^T AX - X^T A A^{-1} J - J^T A^{-1} AX + J^T A^{-1} AA^{-1} J
= X^T AX - 2J^T X + J^T A^{-1} J
\]
and shift the integration vector, \( X' = X - A^{-1} J \). Then we obtain

\[
C = \int dx_1 \cdots dx_n \exp(-X^T AX/2 + J^T x) = \exp(J^T A^{-1} J/2) \int dx'_1 \cdots dx'_n \exp(-X'^T AX'/2).
\]

Since the matrix \( A \) is symmetric, we can diagonalize \( A \) via an orthogonal transformation \( D = O^T AO \). This corresponds to a rotation of the integration variables, \( Y = OX' \). The Jacobian of this transformation is one, and thus the result is

\[
C = \exp(J^T A^{-1} J/2) \int dy_1 \cdots dy_n \prod_{i=1}^{n} \exp(-a_i y_i^2/2)
= \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(\frac{1}{2} J^T A^{-1} J\right).
\]  
(1.19)

In the last step we expressed the product of eigenvalues \( a_i \) as the determinant of the matrix \( A \).

### 1.2. Path integrals in quantum mechanics

In problem 1.14 you are asked to calculate the classical action of a free particle and of a harmonic oscillator and to compare them to the corresponding propagators found in quantum mechanics. Surprisingly, you will find that in both cases the propagator can be written as \( K(q', t'; q, t) = N \exp(iS) \) where \( S \) is the classical action along the path \([q': q]\) and \( N \) a normalisation constant. This suggests that we can reformulate quantum mechanics, replacing the standard operator formalism used to evaluate the propagator (1.15) “somehow” by the classical action.

To get an idea how to proceed, we look at the famous double-slit experiment: According to the superposition principle, the amplitude \( A \) for a particle to move from the source at \( q_1 \) to the detector at \( q_2 \) is the sum of the amplitudes \( A_i \) for the two possible paths,

\[
A = K(q_2, t_2; q_1, t_1) = \sum_{\text{paths}} A_i.
\]  
(1.20)

Clearly, we could add in a Gedankenexperiment more and more screens between \( q_1 \) and \( q_2 \), increasing at the same time the number of holes. Although we replace in this way continuous space-time by a discrete lattice, the differences between these two descriptions should vanish for sufficiently small spacing \( \tau \). Moreover, for \( \tau \to 0 \), we can expand \( U(\tau) = \exp(-iH\tau) \sim 1 - iH\tau \). Applying then \( H = \hat{p}^2/(2m) + V(\hat{q}) \) to eigenfunctions \(|q\rangle\) of \( V(\hat{q}) \) and \(|p\rangle\) of \( \hat{p}^2 \), we can replace the operator \( H \) by its eigenvalues. In this way, we hope to express the propagator as a sum over paths, where the individual amplitudes \( A_i \) contain only classical quantities.
Figure 1.1.: Left: The double slit experiment. Right: The propagator $K(q_N, \tau; q_0, 0)$ expressed as a sum over all $N$-legged paths.

We will apply now this idea a particle moving in one dimension in a potential $V(q)$. The transition amplitude $A$ for the evolution from the state $|q, 0\rangle$ to the state $|q', t'\rangle$ is

$$A \equiv K(q', t'; q, 0) = \langle q' | e^{-iHt'} | q \rangle.$$  (1.21)

This amplitude equals the matrix-element of the propagator $K$ for the evolution from the initial space-time point $q(0)$ to the final point $q'(t')$.

Let us split the time evolution into two smaller steps, writing $e^{-iHt'} = e^{-iH(t'-t_1)}e^{-iHt_1}$. The amplitude becomes

$$A = \langle q' | e^{-iH(t'-t_1)}e^{-iHt_1} | q \rangle.$$  

Inserting unity expressed as a sum over the position eigenstates gives

$$A = \langle q' | e^{-iH(t'-t_1)} \int dq_1 |q_1\rangle \langle q_1|e^{-iHt_1} | q \rangle = \int dq_1 K(q', t'; q_1, t_1)K(q_1, t_1; q, 0).$$  (1.22)

This formula expresses simply the group property, $U(t', 0) = U(t', t_1)U(t_1, 0)$, of the time-evolution operator $U$ evaluated in the basis of the continuous variable $q$. More physically, we can view this equation as an expression of the quantum mechanical rule for combining amplitudes: If the same initial and final states can be connected by various ways, the amplitudes for each of these processes should be added. A particle propagating from $q$ to $q'$ must be somewhere at the intermediate time $t_1$. Labelling this intermediate position as $q_1$, we compute the amplitude for propagation via the point $q_1$ as the product of the two propagators in Eq. (1.22) and integrate over all possible intermediate positions $q_1$.

We continue to divide the time interval $t'$ into a large number $N$ of time intervals of duration $\tau = t'/N$, with the final aim of performing the limit of infinitesimal small time steps $\tau$. Then the propagator becomes

$$A = \langle q' | e^{-iH\tau} \underbrace{e^{-iH\tau} \cdots e^{-iH\tau}}_{N \text{ times}} | q \rangle.$$  (1.23)
We insert again a complete set of states \(|q\rangle_i\) between each exponential, obtaining
\[
A = \int dq_1 \cdots dq_{N-1} \langle q' | e^{-iH\tau} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\tau} | q_{N-2} \rangle \cdots \langle q_1 | e^{-iH\tau} | q_0 \rangle
\]
\[
\equiv \int dq_1 \cdots dq_{N-1} K_{q_N,q_{N-1}} K_{q_{N-1},q_{N-2}} \cdots K_{q_2,q_1} K_{q_1,q_0},
\]
where we have defined \(q_0 = q\) and \(q_N = q'\). Note that these initial and final positions are fixed and therefore are not integrated over. Figure 1.1 illustrates that we can view the amplitude as sum over the amplitudes for all possible paths,
\[
A = \sum_{\text{paths}} A_{\text{path}},
\]
where
\[
\sum_{\text{paths}} = \int dq_1 \cdots dq_{N-1}, \quad A_{\text{path}} = K_{q_N,q_{N-1}} K_{q_{N-1},q_{N-2}} \cdots K_{q_2,q_1} K_{q_1,q_0}.
\]
Let us look at the last expression in detail. We can expand the exponential in each propagator \(K_{q_{j+1},q_j} = \langle q_{j+1} | e^{-iH\tau} | q_j \rangle\) for a single sub-interval, because \(\tau\) is small,
\[
K_{q_{j+1},q_j} = \langle q_{j+1} | \left(1 - iH\tau - \frac{1}{2}H^2\tau^2 + \cdots \right) | q_j \rangle
= \langle q_{j+1} | q_j \rangle - i\tau \langle q_{j+1} | H | q_j \rangle + \mathcal{O}(\tau^2).
\]
The first term is a delta function, which we can express as
\[
\langle q_{j+1} | q_j \rangle = \delta(q_{j+1} - q_j) = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1} - q_j)}.
\]
In the second term of (1.25), we insert a complete set of momentum eigenstates between \(H\) and \(|q_j\rangle\). This gives
\[
- \tau \int \frac{dp_j}{2\pi} \left(\frac{p_j^2}{2m} + V(q)\right) \frac{dp_j}{2\pi} \langle p_j | q_j \rangle
= - \tau \int \frac{dp_j}{2\pi} \left(\frac{p_j^2}{2m} + V(q_{j+1})\right) \langle q_{j+1} | p_j \rangle \langle p_j | q_j \rangle
= - \tau \int \frac{dp_j}{2\pi} \left(\frac{p_j^2}{2m} + V(q_{j+1})\right) e^{ip_j(q_{j+1} - q_j)},
\]
where we used \(\langle q | p \rangle = \exp(ipq)\). In the first line, we view the operator \(\hat{p}\) as operating to the right, while \(V(q)\) operates to the left.

The expression (1.25) is not symmetric in \(q_j\) and \(q_{j+1}\). The reason for this asymmetry is that we could insert the factor 1 either to the right or to the left of the Hamiltonian \(H\) in the second term of (1.25). In the latter case, we would have obtained \(p_{j+1}\) and \(V(q_j)\) in (1.25). Since the difference \([V(q_{j+1}) - V(q_j)]\tau \sim V'(q_j)(q_{j+1} - q_j)\tau \sim V'(q_j)\hat{q}_j \tau^2\) is of order \(\tau^2\), the
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ordering problem should not matter in the continuum limit which we will take eventually. Combining (1.20) and (1.27), the propagator for the step \( q_j \to q_{j+1} \) is

\[
K_{q_{j+1}, q_j} = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1} - q_j)} \left[ 1 - i\tau \left( \frac{p_j^2}{2m} + V(q_j) \right) + O(\tau^2) \right].
\]  

(1.28)

Since we work only at \( O(\tau) \), we can exponentiate the factor in the square bracket,

\[
1 - i\tau H(p_j, q_j) + O(\tau^2) = e^{-i\tau H(p_j, q_j)}.
\]

(1.29)

Next we rewrite the exponent in the first factor using \( \dot{q}_j = (q_{j+1} - q_j)/\tau \), such that we can factor out the time-intervall \( \tau \). The amplitude \( A_{\text{path}} \) consists of \( N \) such factors. Combining them, we obtain

\[
A_{\text{path}} = \prod_{j=0}^{N-1} \int \frac{dp_j}{2\pi} \exp i\tau \sum_{j=0}^{N-1} [p_j \dot{q}_j - H(p_j, q_j)].
\]

(1.30)

We recognise the argument of the exponential as the discrete approximation of the action \( S[q, p] \) in the Palatini form of a path passing through the points \( q_0 = q, q_1, \cdots, q_{N-1}, q_N = q' \). The propagator becomes then

\[
K = \int dq_1 \cdots dq_{N-1} A_{\text{path}}
\]

\[
= \prod_{j=1}^{N-1} \int dq_j \prod_{j=0}^{N-1} \int \frac{dp_j}{2\pi} \exp i\tau \sum_{j=0}^{N-1} [p_j \dot{q}_j - H(p_j, q_j)].
\]

(1.31)

Note that there is one momentum integral for each of the \( N \) intervals, while there is one position integral for each of the \( N - 1 \) intermediate positions.

If \( N \to \infty \), this approximates an integral over all functions \( p(t), q(t) \) consistent with the boundary conditions \( q(0) = q, q(t') = q' \). We adopt the notation \( \mathcal{D}p\mathcal{D}q \) for the functional or path integral over all functions \( p(t) \) and \( q(t) \),

\[
K \equiv \int \mathcal{D}p(t)\mathcal{D}q(t) e^{iS[q,p]} = \int \mathcal{D}p(t)\mathcal{D}q(t) \exp \left( i \int_0^t dt (p\dot{q} - H(p, q)) \right).
\]

(1.32)

This result is known as the path integral in phase-space. It allows us to obtain for any classical system which can be described by a Hamiltonian the corresponding quantum dynamics.

If the Hamiltonian is of the form \( H = \frac{p^2}{2m} + V(q) \), as we have assumed in our derivation, we can carry out the quadratic momentum integrals in (1.31). We can rewrite this expression as

\[
K = \prod_{j=1}^{N-1} \int dq_j \exp -i\tau \sum_{j=0}^{N-1} V(q_j) \prod_{j=0}^{N-1} \int \frac{dp_j}{2\pi} \exp i\tau \sum_{j=0}^{N-1} (p_j \dot{q}_j - p_j^2/2m).
\]

The \( p \) integrals are all uncoupled Gaussians. One such integral gives

\[
\int \frac{dp}{2\pi} e^{i\tau(p\dot{q} - p^2/2m)} = \sqrt{\frac{m}{2\pi i\tau}} e^{i\tau q^2/2},
\]

(1.33)

where we added again an infinitesimal factor \( \exp(-\varepsilon p^2) \) to the integrand.
The propagator becomes

\[ K = \prod_{j=1}^{N-1} \int dq_j \exp \left( -i \tau \sum_{j=0}^{N-1} V(q_j) \prod_{j=0}^{N-1} \left( \sqrt{\frac{m}{2\pi i \tau}} \exp i \frac{m q_j^2}{2} \right) \right) \]

\[ = \left( \frac{m}{2\pi i \tau} \right)^{N/2} \prod_{j=1}^{N-1} \int dq_j \exp \left( i \sum_{j=0}^{N-1} \frac{m q_j^2}{2} - V(q_j) \right). \] (1.34)

The argument of the exponential is again a discrete approximation of the action \( S[q] \) of a path passing through the points \( q_0 = q, q_1, \ldots, q_{N-1}, q_N = q' \), but now seen as functional of only the coordinate \( q \). As above, we can write this in the more compact form

\[ K = \langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q(t) e^{iS[q]} = \int \mathcal{D}q(t) \exp \left( i \int_{t_i}^{t_f} dt L(q, \dot{q}) \right), \] (1.35)

where the integration includes all paths satisfying the boundary condition

\[ q(t_i) = q_i, \quad q(t_f) = q_f. \] (1.36)

This is the main result of this section, and is known as the path integral in configuration space. It will serve us as starting point discussing quantum field theories, although it is less general than the previous path integral in phase space.

Knowing the path integral and thus the propagator is sufficient to solve scattering problems in quantum mechanics. In a relativistic theory, the particle number during the course of a scattering process is however not fixed, since energy can be converted into matter. In order to prepare us for such more complex problems, we will generalise in the next section the path integral to a generating functional for \( n \)-point Green functions. For instance, 4-point Green functions will be the essential ingredient to calculate \( 2 \to 2 \) scattering processes.

### 1.3. Generating functional for Green functions

Having re-expressed the transition amplitude \( \langle q_f, t_f | q_i, t_i \rangle \) of a quantum mechanical system as a path integral, we want generalize first this result to the matrix elements of an arbitrary potential \( V(q) \) between energy eigenstates \(|n\rangle \) and \(|n'\rangle \). For all practical purposes, we can assume that we can expand \( V(q) \) as a power-series in \( q \); thus it is sufficient to consider the matrix elements \( \langle n'| q^m | n \rangle \). In quantum field theory, the initial and final states are generally free particles which are described mathematically as harmonic oscillators. In this case, we are able to reconstruct all excited states \(|n\rangle \) from the ground-state,

\[ |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \]

Therefore it will be sufficient to study matrix elements between the ground-state \(|0\rangle \). Aim of this subsection is to define a generating functional for these quantities.
Time-ordered products of operators and the path integral. In a first step, we try to include the operator $q^m$ into the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle$. We can reinterpret our result for the path integral as follows,

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q(t) \ 1 \ e^{iS[q(t)]}.$$  (1.37)

Thus we can see the LHS as matrix element of the unit operator $1$, while the RHS corresponds to the path integral average of the classical function $f(q) = 1$. Now we want to generalize this trivial statement to two operators $\hat{A}(q(t_a))$ and $\hat{B}(q(t_b))$ given in the Heisenberg picture. In evaluating

$$\int \mathcal{D}q(t) \ A(q(t_a))B(q(t_b)) \ e^{iS[q(t)]} = \langle q_f, t_f | q_i, t_i \rangle,$$  (1.38)

we go back to Eq. (1.24) and choose $t_a$ and $t_b$ as two of the intermediate times $t_i$,

$$= \left\{ \begin{array}{ll}
\int dq_1 \cdots dq_{N-1} \cdots \langle q_{a+1}, t_{a+1} | \hat{A} | q_a, t_a \rangle \cdots \langle q_{b+1}, t_{b+1} | \hat{B} | q_b, t_b \rangle \cdots & \text{for } t_a > t_b. \\
\int dq_1 \cdots dq_{N-1} \cdots \langle q_{b+1}, t_{b+1} | \hat{B} | q_b, t_b \rangle \cdots \langle q_{a+1}, t_{a+1} | \hat{A} | q_a, t_a \rangle \cdots & \text{for } t_a < t_b.
\end{array} \right.$$  (1.39)

Since the time along a classical path increases, the matrix-elements of the operators $\hat{A}(t_a)$ and $\hat{B}(t_b)$ are also ordered with time increasing from the right to the left. If we define the time-ordered product of two operators as

$$T\{A(x_1)B(x_2)\} = A(x_1)B(x_2)\theta(t_1 - t_2) + B(x_2)A(x_1)\theta(t_2 - t_1),$$  (1.40)

then the path integral average of the classical quantities $A(t_a)$ and $B(t_b)$ corresponds to the matrix-element of the time-ordered product of these two operators,

$$\langle q_f, t_f | T\{\hat{A}(t_a)\hat{B}(t_b)\} | q_i, t_i \rangle = \int \mathcal{D}q(t) \ A(t_a)B(t_b) \ e^{iS[q(t)]},$$  (1.41)

and similar for more than two operators.

External sources. We want to include in our formalism the possibility that we can change the state of our system applying an external driving force or source term $J(t)$. In quantum mechanics, we could imagine e.g. a harmonic oscillator in the ground-state $|0\rangle$, making a transition under the influence of an external force $J$ to the state $|n\rangle$ at time $t$ and back to the ground-state $|0\rangle$ at time $t' > t$. Including such transitions, we can mimick the relativistic process of creation and annihilation of a particle as follows: We identify the vacuum (i.e. the state containing zero real particles) with the ground-state of quantum mechanical system, and the creation and annihilation of $n$ particle with the (de-) excitation of the $n$th energy level by an external source $J$.

Schwinger realized that adding a coupling to an external source, viz.

$$L \rightarrow L + J(t)q(t),$$  (1.42)

will lead to an efficient way to calculate the matrix elements of an arbitrary polynomial of operators $q(t_n) \cdots q(t_1)$: If the source $J(t)$ would be a simple number instead of a time-dependent function in the augmented path-integral

$$\langle q_f, t_f | q_i, t_i \rangle_J \equiv \int \mathcal{D}q(t)e^{i \int_t^{t_f} dt \langle L + J \rangle},$$  (1.43)
then we could obtain $\langle q_f, t_f | q^n | q_i, t_i \rangle_J$ simply by differentiating $\langle q_f, t_f | q_i, t_i \rangle_J$ m-times with respect to $J$. However, the LHS is a functional of $J(t)$ and thus we need to perform instead a functional derivative with respect to $J(t)$. We define the concept of a functional derivative by analogy with the normal differentiation of functions,

$$\frac{\partial}{\partial x^k} 1 = 0, \quad \frac{\partial x^l}{\partial x^k} = \delta^l_k, \quad \frac{\partial}{\partial x^k}[F(x)G(x)] = F(x)\frac{\partial G(x)}{\partial x^k} + G(x)\frac{\partial F(x)}{\partial x^k}. \quad (1.44)$$

For a continuous index, the Kronecker delta becomes a delta function,

$$\frac{\delta}{\delta J(x)} 1 = 0, \quad \frac{\delta J(x)}{\delta J(x')} = \delta(x - x'), \quad (1.45)$$

and the Leibniz rule becomes

$$\frac{\delta}{\delta J(x)} \{ F[J]G[J] \} = F[J]\frac{\delta G[J]}{\delta J(x)} + G[J]\frac{\delta F[J]}{\delta J(x)}. \quad (1.46)$$

Now we are able to differentiate $\langle q_f, t_f | q_i, t_i \rangle_J$ with respect to the source $J$. Starting from

$$\frac{1}{\delta J(t_1)} \int Dq(t) e^{i \int_{-\infty}^{\infty} dt J(t)q(t)} = i \int Dq(t) q(t_1) e^{i \int_{-\infty}^{\infty} dt J(t)q(t)}. \quad (1.47)$$

we obtain

$$\langle q_f, t_f | T \{ q(t_1) \cdots q(t_n) \} | q_i, t_i \rangle = (-i)^n \frac{\delta^n}{\delta J(t_1) \cdots \delta J(t_n)} \langle q_f, t_f | q_i, t_i \rangle_J \bigg|_{J=0}. \quad (1.48)$$

Thus the source $J(t)$ is a convenient tool to obtain the functions $q(t_1) \cdots q(t_n)$ in front of $\exp(iS)$. Having performed the functional derivatives, we set the source $J(t)$ to zero, coming back to the usual path integral.

**Ground-state persistence amplitude** As last step, we want to convert the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle_J$ into the probability amplitude that a system which was in the ground-state $|0\rangle$ at $t_i \to -\infty$ remains there at $t_f \to \infty$ in the presence of the source $J(t)$. Inserting a complete set of energy eigenstates, $1 = \sum_n |n\rangle \langle n|$, into the propagator, we obtain

$$\langle q', t' | q, t \rangle = \sum_n \psi_n(q') \overline{\psi}_n(q) \exp(-iE_n(t' - t)). \quad (1.49)$$

We can isolate the ground-state $n = 0$ by adding either to the energies $E_n$ or to the time difference $\tau = (t' - t)$ a small negative imaginary part. In this case, all terms would be exponentially damped in the limit $\tau \to \infty$, and the ground-state as state with the smallest energy would more and more dominate the sum. Alternatively, we can add a term $+i\xi q^2$ to the Lagrangian. Instead of adding an infinitesimal small negative imaginary part to the time, we can do a somewhat more drastic change, rotating the time axis by 90 degrees in the complex time plane, $\tau \to -i\tau$. This procedure called Wick rotation corresponds to the transition from Minkowski to Euclidean space,

$$x^2 = t^2 - x^2 \to x_E^2 = -[t^2 + x^2],$$

resulting in an exponentially damped path integral.
Finally, we have only to connect our results, Eqs. (1.41) and (1.48), we obtained so far. Adding a coupling to an external source \( J(t) \) and a damping factor \(+i\varepsilon q^2\) to the Lagrangian gives us the ground-state persistence amplitude

\[
Z[J] \equiv \langle 0, \infty | 0, -\infty \rangle_J = \int Dq(t) e^{\int_{-\infty}^{\infty} dt (L + Jq + i\varepsilon q^2)}
\] (1.50)

in the presence of a classical source \( J \). Taking derivatives w.r.t. the external sources \( J \), and setting them afterwards to zero, we obtain

\[
\left. \frac{\delta^n Z[J]}{\delta J(t_1) \cdots \delta J(t_n)} \right|_{J=0} = i^n \int Dq(t) q(t_1) \cdots q(t_n) e^{\int_{-\infty}^{\infty} dt (L + i\varepsilon q^2)}. \] (1.51)

The RHS corresponds to the path integral in Eqs. (1.41), except for the \( i\varepsilon q^2 \) factor. But this factor damps in the limit of large \( t \) everything except the ground state. Thus we found that \( Z[J] \) is the generating functional of the expectation value of the time-ordered product of operators \( \hat{q}(t_i) \),

\[
(-i)^n \left. \frac{\delta^n Z[J]}{\delta J(t_1) \cdots \delta J(t_n)} \right|_{J=0} = \langle 0, \infty | T\{\hat{q}(t_1) \cdots \hat{q}(t_n)\}|0, -\infty \rangle = G(t_1, \ldots, t_n). \] (1.52)

In the last step, we defined also the \( n \)-point Green function \( G(t_1, \ldots, t_n) \). These functions will be the main building block we will use to perform calculations in quantum field theory, and the formula corresponding to Eq. (1.52) will be our master formula in field theory.

### 1.4. Oscillator as an 1–dimensional field theory

**Canonical quantisation** An one-dimensional harmonic oscillator can be viewed as a free quantum field theory in one time and zero space dimensions. In order to stress this equivalence, we rescale the usual Lagrangian

\[
L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2,
\] (1.53)

where \( m \) is the mass of the oscillator and \( \omega \) its frequency as

\[
\phi(t) \equiv \sqrt{m} x(t).
\] (1.54)

We call the variable \( \phi(t) \) a “scalar field,” and the Lagrangian now reads

\[
L(\phi, \dot{\phi}) = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \omega^2 \phi^2.
\] (1.55)

After the rescaling, the kinetic term \( \dot{\phi}^2 \) is “canonically normalised,” i.e. it has the dimensionless coefficient 1/2.

We derive the corresponding Hamiltonian, determining first the conjugate momentum \( \pi \) as

\[
\pi(t) = \partial L/\partial \dot{\phi} = \dot{\phi}(t).
\]

Thus the (classical) Hamiltonian follows as

\[
H(\phi, \pi) = \frac{1}{2} \pi^2 + \frac{1}{2} \omega^2 \phi^2.
\] (1.56)
The transition to quantum mechanics is performed by promoting $\phi$ and $\pi$ to operators which satisfy the canonical commutation relations $[\phi, \pi] = i$.

The harmonic oscillator is solved most efficiently introducing creation and annihilation operators, $a^\dagger$ and $a$. They are defined by

$$
\phi = \frac{1}{\sqrt{2\omega}} \left( a^\dagger + a \right), \quad \text{and} \quad \pi = i \sqrt{\frac{\omega}{2}} \left( a^\dagger - a \right),
$$

(1.57)

and satisfy $[a, a^\dagger] = 1$. The Hamiltonian follows as

$$
H = \frac{\omega}{2} \left( aa^\dagger + a^\dagger a \right) = \left( aa^\dagger + \frac{1}{2} \right) \omega.
$$

(1.58)

We interprete $N \equiv aa^\dagger$ as the number operator, counting the number of quanta having each the energy $\omega$ in a given state $|n\rangle$.

We now work in the Heisenberg picture where the operators are time dependent. The time evolution of the operator $a(t)$ can be found from the Heisenberg equation,

$$
i \frac{da}{dt} = [a, H] = \omega a,
$$

(1.59)

from which we deduce that

$$
a(t) = a(0) e^{-i\omega t} = a_0 e^{-i\omega t}.
$$

(1.60)

As a consequence, the field operator can be expressed in terms of the creation and annihilation operators as

$$
\phi(t) = \frac{1}{\sqrt{2\omega}} \left( a_0 e^{-i\omega t} + a_0^\dagger e^{i\omega t} \right). \quad (1.61)
$$

**Path integral approach** We solve now the same problem, the rescaled Lagrangian (1.55), in the path integral approach. Using this method, it is convenient to include a coupling to an external force $J$. Moreover, it is also useful to recall how we solved the classical forced oscillator in section 1.1.3 applying the Green function method.

Let us define the effective action $S_{\text{eff}}$ as the sum of the classical action $S$, the coupling to the external force $J$ and a small negative imaginary part to make the path integral well-defined,

$$
S_{\text{eff}} = S + \int dt \left( J\phi + i\varepsilon\phi^2 \right) = \int dt \left[ \frac{1}{2} \phi^2 - \frac{1}{2} \omega^2 \phi^2 + J\phi + i\varepsilon\phi^2 \right].
$$

(1.62)

This function is the integrand of the path integral. We start our work by massaging $S_{\text{eff}}$ into a form such that the path integral can be easily performed. The first two terms in $S_{\text{eff}}$ can be viewed again as the action of differential operator $D(t)$ on $\phi^2(t)$,

$$
D(t)\phi^2(t) = \frac{1}{2} \left( \frac{d^2}{dt^2} - \omega^2 \right) \phi^2(t).
$$

We can evaluate this operator, if we go to Fourier space,

$$
\phi(t) = \int \frac{dE}{2\pi} e^{-iEt} \phi(E) \quad \text{and} \quad J(t) = \int \frac{dE}{2\pi} e^{-iEt} J(E).
$$

(1.63)
To keep the action real, we have to write all bilinear quantities as $\phi(E)\phi(-E')$, etc. Then only the phases dependent on time, and the $t$ integration gives a factor $2\pi\delta(E - E')$, expressing energy conservation,

$$S_{\text{eff}} = \frac{1}{2} \int \frac{dE}{2\pi} \left[ \phi(E)(E^2 - \omega^2 + i\varepsilon)\phi(-E) + J(E)x(-E) + J(-E)x(E) \right]$$

(1.64)

In the path integral, this expression corresponds to a Gaussian integral of the type of (1.18), where we should “complete the square.” Shifting the integration variable to $\tilde{\phi}(E) = \phi(E) + \frac{J(E)}{E^2 - \omega^2 + i\varepsilon}$,

we obtain

$$S_{\text{eff}} = \frac{1}{2} \int \frac{dE}{2\pi} \left[ \tilde{\phi}(E)(E^2 - \omega^2 + i\varepsilon)\tilde{\phi}(-E) - J(E)\frac{1}{E^2 - \omega^2 + i\varepsilon}J(-E) \right].$$

(1.65)

Here we see that the “damping rule” for the path integral makes also energy integral over the denominator well-defined. The physical interpretation of this way of shifting the poles—which differs from our treatment of the retarded Green function in the classical case—will be postponed to the next chapter, where we will discuss this issue in detail.

We are now in the position to evaluate the generating functional $Z[J]$. The path integral measure is invariant under a simple shift of the integration variable, $D\tilde{\phi} = D\phi$, and we omit the tilde from now on. Furthermore, the second term in $S_{\text{eff}}$ does not depend on $\phi$ and can be factored out,

$$Z[J] = \exp \left( -\frac{i}{2} \int \frac{dE}{2\pi} J(E)\frac{1}{E^2 - \omega^2 + i\varepsilon}J(-E) \right)$$

(1.66)

$$\times \int D\phi \ exp \left( \frac{i}{2} \int \frac{dE}{2\pi} \left[ \phi(E)(E^2 - \omega^2 + i\varepsilon)\phi(-E) \right] \right)$$

(1.67)

Setting the external force to zero, $J = 0$, the first factor becomes one and the generating functional $Z[0]$ becomes equal to the path integral in the second line. But for $J = 0$, the oscillator remains in the ground-state and thus $Z[0] = (0, \infty|0, -\infty) = 1$. Therefore

$$Z[J] = \exp \left( \frac{i}{2} \int \frac{dE}{2\pi} J(E)\frac{1}{E^2 - \omega^2 + i\varepsilon}J(-E) \right)$$

(1.68)

Inserting the Fourier transformed quantities, we arrive at

$$Z[J] = \exp \left( \frac{i}{2} \int dt \int dt' J(t)G(t - t')J(t') \right).$$

(1.69)

where the Feynman propagator

$$G_F(t - t') = \int \frac{dE}{2\pi} e^{-iE(t-t')}\frac{1}{E^2 - \omega^2 + i\varepsilon}$$

(1.70)

differs from the retarded propagator $G_R$ (1.38) by the position of its poles.
This result allows us to calculate arbitrary matrix elements between oscillator states. For instance, we obtain the expectation value $\langle 0 | \phi^2 | 0 \rangle$ from

$$\langle 0 | T \{ \phi(t_2) \phi(t_1) \} | 0 \rangle = (-i)^2 \frac{\delta^2 Z[J]}{\delta J(t_1) \delta J(t_n)} \bigg|_{J = 0} = i G_F(t_2 - t_1) = \frac{1}{2\omega} \exp[i \omega |t_2 - t_1|].$$ (1.71)

Here, we used in the last step the explicit expression for $G_F$ which you should check in problem 1.6. Taking the limit $t_2 \searrow t_1$ and replacing $\phi^2 \rightarrow mx^2$, we reproduce the standard result $\langle 0 | x^2 | 0 \rangle = 1/(2m\omega)$. Matrix elements between excited states $|n\rangle = (n!)^{-1/2} (a^\dagger)^n | 0 \rangle$ are obtained by expressing the creation operator $a^\dagger$ using $\pi(t) = \dot{\phi}(t)$ as

$$a^\dagger = \sqrt{\frac{\omega}{2}} \left( 1 - \frac{i}{\omega \, dt} \right) \phi(t).$$ (1.72)

### Summary

Using Feynman’s path integral approach, we can express a transition amplitude as a sum over all paths, $\langle q_f, t_f | q_i, t_i \rangle = \int Dq(t) \exp(iS[q])$. Adding a linear coupling to an external source $J$ and a damping term to the Lagrangian, we obtain the ground-state persistence amplitude $\langle 0, \infty | 0, -\infty \rangle J$. This quantity serves as the generating functional for $n$-point Green functions $G(t_1, \ldots, t_n)$ which are the time-ordered vacuum expectation values of operators $\hat{q}_1, \ldots, \hat{q}_n$. 

### Further reading

Our presentation of the path integral follows the one of [Mac99] which contains as well as [Das06] useful additional material. Reading original papers like Feynman’s presentation of the path integral in [Fey48] can be still very informative, particularly about the sometime surprising initial motivations of the authors.

### Problems

1. **Classical action.**

   Calculate the classical action $S[q]$ for a free particle and an harmonic oscillator. Compare the results with the expression for the propagator $K = \langle x', t' | x, t \rangle = N \exp(i\phi)$ of the corresponding quantum mechanical system and express both $\phi$ and $N$ through the action $S$.

2. **Propagator as Green function.**

   Show that the Green function or propagator $K(x', t'; x, t) = \langle x' | \exp[-iH(t' - t)] | x \rangle$ of the Schrödinger equation is the inverse of the differential operator $(i\partial_t - H)$.

3. **Statistical mechanics.**

   Derive the connection between the partition function $Z = \text{Tr} \exp[-\beta H]$ of statistical mechanics and the path integral in Euclidean time.

4. **Feynman propagator.**

   Find the explicit expression for the Feynman propagator used in (1.71) from a.) its definition as time-ordered product of fields $\phi$, and b.)
evaluating (1.70) using Cauchy’s theorem.

1.5 Matrix elements from $Z[J]$. Evaluate the matrix element $\langle 0 | \phi^2 | 1 \rangle$ of a harmonic oscillator from $Z[J]$.

1.6 Classical driven oscillator.
Consider an harmonic oscillator satisfying
\[
\ddot{q}(t) + \omega^2 q(t) = 0, \quad \text{for} \quad t < 0 \quad \text{and} \quad t > T
\]
\[
\ddot{q}(t) - \Omega^2 q(t) = 0, \quad \text{for} \quad 0 < t < T
\]
where $\omega$ and $\Omega$ are real constants. a.) Show that for $q(t) = A_1 \sin(\omega t)$ for $t < 0$ and $\Omega T \gg 1$, the solution $q(t) = A_2 \sin(\omega_0 t + \alpha)$ with $\alpha = \text{const.}$ satisfies
\[
A_2 \approx \frac{1}{2} \left(1 + \frac{\omega^2}{\Omega^2}\right)^{1/2} \exp(\Omega T).
\]
b.) If the oscillator was in the ground-state at $t < 0$, how many quanta are created? Find the explicit expression for the Feynman propagator used in (1.71) from its definition as time-ordered product of fields $\phi$. 

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2. Scalar fields

We extend in this chapter the path integral approach from quantum mechanics to the simplest field theory, a real scalar field $\phi(x)$. In particular, we will introduce the generating functional $Z[J] = \langle 0| \exp \int d^4x \phi(x) J(x) | 0 \rangle$ of $n$-point Green functions as our main tool to calculate the time-ordered vacuum expectation value of a product of fields $\phi(x_1) \cdots \phi(x_n)$. After having gained some experience with the case of a free field in sections 2.1–2.4 we will use perturbation theory to discuss a scalar field with a $\lambda \phi^4$ self-interaction in the last two sections.

2.1. Lagrange formalism and path integrals for fields

A field is a map which associates to each space-time point $x$ a $k$-tupel of values $\phi_a(x)$, $a = 1, \ldots, k$. The space of field values $\phi_a(x)$ can be characterised by its transformation properties under Poincaré transformations and internal symmetry groups. The latter are in practically all physical applications Lie groups like $\mathbb{R}$, $\text{U}(1)$, $\text{SU}(n)$ or $\text{SO}(n)$. Except for a real scalar field $\phi$, these fields have several components. Thus we have to generalise Hamilton’s principle to a collection of fields $\phi_a(x)$, where the index $a$ includes all internal as well as space-time indices. Moreover, the Lagrangian for a field $\phi_a(x)$ will contain not only time but also space derivatives.

To ensure Lorentz invariance, we consider a scalar Lagrange density $L$ that may, analogously to $L(q, \dot{q})$, depend on the fields $\phi_a$ and their first derivatives $\partial_\mu \phi_a$. We include no explicit time-dependence, since “everything” should be explained by the fields $\phi_a$ and their interactions. The Lagrangian $L(\phi_a, \partial_\mu \phi_a)$ is obtained by integrating the density $L$ over a given space volume $V$. The action $S$ is thus the four-dimensional integral

$$ S[\phi_a] = \int_a^b dt L(\phi_a, \partial_\mu \phi_a) = \int_\Omega d^4x L(\phi_a, \partial_\mu \phi_a) \tag{2.1} $$

with $\Omega = V \times [t_a : t_b]$.

A variation $\delta \phi_a(x)$ of the fields leads to a variation of the action,

$$ \delta S = \int_\Omega d^4x \left[ \frac{\partial L}{\partial \phi_a} \delta \phi_a + \frac{\partial L}{\partial (\partial_\mu \phi_a)} \delta (\partial_\mu \phi_a) \right] \tag{2.2} $$

where we have to sum over field components ($a = 1, \ldots, k$) and the Lorentz index $\mu = 0, \ldots, 3$. The correspondence $q(t) \rightarrow \phi(x)$ implies that the parameter $\varepsilon$ in Hamilton’s principle depends not on $x$. We can therefore eliminate again the variation of the field gradients $\partial_\mu \phi^a$ by a partial integration using Gauß’ theorem,

$$ \delta S = \int_\Omega d^4x \left[ \frac{\partial L}{\partial \phi_a} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi_a)} \right) \right] \delta \phi_a = 0. \tag{2.3} $$

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2. Scalar fields

The surface term vanishes, since we require that the variation is zero on the boundary $\partial \Omega$. Thus the Lagrange equations for the fields $\phi^a$ are

$$\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \right) = 0.$$  \hfill (2.4)

If the Lagrange density $\mathcal{L}$ is changed by a four-dimensional divergence, $\delta \mathcal{L} = \partial_\mu K^\mu$, and surface terms can be dropped, the same equations of motion result. Note also that it is in most applications in field theory more efficient to perform directly the variation $\delta \phi^a$ of the action $S[\phi^a]$ than to use the Lagrange equations.

**Path integrals for fields** The path integrals becomes now a functional integral over the $n$ fields $\phi^a$,

$$K = \int D\phi_1 \cdots D\phi_n e^{iS[\phi]} = \int D\phi_1 \cdots D\phi_n e^{i \int_\Omega d^4x \mathcal{L}(\phi^a, \partial_\mu \phi^a)}$$ \hfill (2.5)

A major problem we have to address later is that the $n$ fields $\phi^a$ are often not independent: For instance, in electrodynamics all potentials $A^\mu$ connected by a gauge transformation describe the same physical situation. This redundancy makes the path integral ill-defined. In such cases, we need to factor out these gauge degrees of freedom from the integration measure $DA_0 \cdots DA_3$.

We start therefore with the simplest case of a scalar field $\phi$, discussing first the case of a free field and then the case of a scalar field with self-interactions.

### 2.2. Generating functional for a scalar field

**Lagrangian** The Klein-Gordon equation as a relativistic wave equation can be derived analogously to the (free) Schrödinger equation,

$$i \partial_t \psi = H_0 \psi,$$ \hfill (2.6)

which is obtained by substituting

$$\omega \rightarrow i \partial_t \quad k \rightarrow -i \nabla_x$$ \hfill (2.7)

in the non-relativistic energy-momentum relation $\omega = k^2/(2m)$. With the same replacements, the relativistic $\omega^2 = m^2 + k^2$ becomes

$$(\Box + m^2) \phi = 0 \quad \text{with} \quad \Box = \eta_{\mu\nu} \partial^\mu \partial^\nu = \partial_\mu \partial^\mu.$$ \hfill (2.8)

The relativistic energy-momentum relation implies that the solutions to the free Klein-Gordon equation consist of plane-waves with positive and negative energies $\pm \sqrt{k^2 + m^2}$. For the stability of a quantum system it is essential that its energy eigenvalues are bounded from below. Otherwise, we could generate e.g. in a scattering process $\phi + \phi \rightarrow n\phi$ an arbitrarily high number of $\phi$ particles with sufficiently low energy, and no stable form of matter could exist. Interpreting the Klein-Gordon equation as a relativistic wave equation for a single particle can be therefore not fully satisfactory, since the energy of its solutions is not bounded from below.
2.2. Generating functional for a scalar field

How do we guess the correct Lagrange density $L$? Plane waves can be seen as a collection of coupled harmonic oscillators at each space-time point. The correspondence $\dot{q} \leftrightarrow \partial_\mu \phi$ means that the “kinetic” field energy is quadratic in the field derivatives. Relativistic invariance implies that the Lagrangian is a scalar, leaving as the only two possible terms containing derivatives

$$\eta_{\mu\nu} (\partial^\mu \phi)(\partial^\nu \phi) \quad \text{and} \quad \phi \Box \phi.$$  

Using the action principle to derive the equation of motions, we can however drop boundary terms performing partial integrations. Thus these two terms are equivalent, up to a minus sign. The Klein-Gordon equation $\Box \phi = -m^2 \phi$ suggests that the mass term is also quadratic in the field $\phi$. Therefore we try as Lagrange density

$$L = \frac{1}{2} \eta_{\mu\nu} (\partial^\mu \phi)(\partial^\nu \phi) - \frac{1}{2} m^2 \phi^2 \equiv \frac{1}{2} \eta_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} m^2 \phi^2. \quad (2.9)$$

From now, we will drop the parenthesis around $(\partial^\mu \phi)$ and it should be understood from the context that the derivative $\partial^\mu$ acts only on the first field $\phi$. Even shorter alternative notations are $(\partial^\mu \phi)^2$ and finally the concise $(\partial \phi)^2$. With the short-cut $\phi,\alpha \equiv \partial^\alpha \phi$ and

$$\frac{\partial}{\partial \phi,\alpha} (\eta^{\mu\nu} \phi,\mu \phi,\nu) = \eta^{\mu\nu} (\delta^\alpha_{\mu} \phi,\nu + \delta^\alpha_{\nu} \phi,\mu) = \eta^{\alpha\nu} \phi,\nu + \eta^{\mu\alpha} \phi,\mu = 2 \phi^\alpha, \quad (2.10)$$

the Lagrange equation becomes

$$\frac{\partial L}{\partial \phi} - \partial_\alpha \left( \frac{\partial L}{\partial \phi,\alpha} \right) = -m^2 \phi - \partial^\alpha \phi,\alpha = 0. \quad (2.11)$$

Thus the Lagrange density $(2.9)$ leads indeed to the Klein-Gordon equation. We can understand the relative sign in the Lagrangian splitting the relativistic kinetic energy into the “proper” kinetic energy $(\partial_t \phi)^2/2$ and the gradient energy density $(\nabla \phi)^2/2$,

$$L = \frac{1}{2} \phi^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2. \quad (2.12)$$

The last two terms correspond to a potential energy and carry therefore the opposite sign of the first one.

Instead of guessing, we can derive the correct Lagrangian $L$ as follows: We multiply the free field equation for $\phi$ by a variation $\delta \phi$ that vanishes at the endpoints $t_a$ and $t_b$. Then we integrate over $\Omega = V \times [t_a : t_b]$, perform a partial integration of the kinetic energy, use the Leibniz rule and ask that the variation vanishes,

$$A \int_\Omega \, d^4 x \, \delta \phi (\Box + m^2) \phi = A \int_\Omega \, d^4 x \left[ -\delta (\partial_\mu \phi) \partial^\mu \phi + \delta \phi m^2 \right] =$$

$$= A \int_\Omega \, d^4 x \delta \left[ -\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \phi^2 m^2 \right] = 0. \quad (2.14)$$

The term in the square brackets agrees with our guess $(2.9)$, taking into account that the source-free field equation fixes the Lagrangian only up to the overall factor $A$. In analogy with a quantum mechanical oscillator, we want that the coefficients of the two terms are $\pm 1/2$ and thus we set $|A| = 1$.

We can determine the correct overall sign of $L$ by calculating the energy density $\rho$ of the scalar field and requiring that it is bounded from below and stable against small perturbations.
2. Scalar fields

We identify the energy density $\rho$ of the scalar field with its Hamiltonian density $\mathcal{H}$, and use the connection between the Lagrangian and the Hamiltonian known from classical mechanics.

The transition from a system with a finite number of degrees of freedom to one with an infinite number of degrees of freedom proceeds as follows,

$$
p_i = \frac{\partial L}{\partial \dot{q}^i} \quad \Rightarrow \quad \pi_a = \frac{\partial L}{\partial \dot{\phi}_a},$$

(2.15)

$$
H = p_i \dot{q}^i - L \Rightarrow \mathcal{H} = \sum_a \pi_a \dot{\phi}_a - L.
$$

(2.16)

The canonically conjugated momentum $\pi$ of a real scalar field is

$$
\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}.
$$

(2.17)

Thus the Hamilton density is

$$
\mathcal{H} = \pi \dot{\phi} - L = \pi^2 - L = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \geq 0
$$

(2.18)

and thus obviously positive definite. Moreover, generating fluctuations $\delta \phi$ costs energy and thus the system is stable against small perturbations. Hence the transition from a single particle interpretation of the Klein-Gordon equation to a field theoretic interpretation is sufficient to cure the problem of the negative energy solutions. Note also that we could subtract a constant $\rho_0$ from the Langragian which would drop out of the equation of motions. From Eq. (2.18) we see that such a constant corresponds to a uniform energy density in space. Such a term would give a contribution to the cosmological constant in the gravitational field equation, but would be otherwise unobservable.

Next we generalise the Lagrangian by subtracting a polynomial in the fields, $V(\phi)$, subject to the stability constraint discussed above. Hence the potential should be bounded from below, and we can expand it around its minimum at $\phi \equiv v$,

$$
\left. \frac{dV}{d\phi} \right|_{\phi=v} = 0, \quad \left. \frac{d^2V}{d\phi^2} \right|_{\phi=v} \equiv m^2 > 0.
$$

(2.19)

The term $V''(v)$ acts as mass term, while we will see that terms $\phi^n$ with $n \geq 3$ generate interactions between $n$ particles, as expected from the analogy of a quantum field to coupled quantum mechanical oscillators discussed in problem ??.

Generating functional In quantum mechanics a particle cannot be created or destroyed and thus the main question to ask in a scattering problem is how likely it is that a particle moves from $x$ to $x'$. In quantum field theory, we address a more complex problem: We want to calculate how likely it is that a particle is created by an external source $J(x)$, propagates from $x$ to $x'$, disappearing there in an interaction with another source $J(x')$. As we have already seen in the previous chapter, such questions can be conviniently addressed considering the functional $Z[J]$, where we add a coupling between the field and an external source,

$$
Z[J] = \langle 0 + |0-\rangle_J = \mathcal{N} \int D\phi \exp i \int_{\Omega} d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J \phi \right).
$$

(2.20)
To ensure the convergence of the integral, we add an infinitesimal small imaginary part to the squared mass of the particle, \( m^2 \rightarrow m^2 - i\varepsilon \). Next we perform one integration by part of the first term, exploiting the fact that the boundary term vanishes,

\[
\mathcal{Z}\left[ J \right] = \mathcal{N} \int \mathcal{D}\phi \exp i \int_\Omega d^4x \left( -\frac{1}{2} \phi (\Box + m^2)\phi + J\phi \right).
\]

The first two terms, \( A = -(\Box + m^2)/2 \), are quadratic and symmetric in the field \( \phi \),

\[
-\frac{1}{2} \int d^4x \phi(x)(\Box + m^2)\phi(x) = \int d^4xd^4x' \phi(x)A(x,x')\delta(x - x')\phi(x').
\]

Note that the operator \( A \) is local, \( A(x,x') \propto \delta(x - x') \): Since special relativity forbids action at a distance, non-local terms like \( \phi(x')(\Box + m^2)\phi(x) \) should not appear in a relativistic Lagrangian.

If we discretise continuous space-time \( x^\mu \) into a lattice, we can use Eq. (1.19) to perform the path integral,

\[
\mathcal{Z}\left[ J \right] = \mathcal{N}\left( \frac{(2\pi)^N}{\det[A]} \right)^{1/2} \exp \left( -\frac{1}{2} iJ A^{-1}J \right) = \mathcal{N} Z[0] \exp(iW[J]).
\]

The pre-factor of the exponential function does not depend on \( J \) and is thus given by \( \mathcal{N} Z[0] = (0 + |0\rangle) \). The vacuum should be stable and normalised to one in the absence of sources, \( (0 + |0\rangle) = 1 \). Therefore we should ensure that \( Z[J] \) is properly normalised what implies that \( \mathcal{N}^{-1} = Z[0] \).

In the last step of Eq. (2.23), we defined a new functional \( W[J] \) that depends only quadratically on the source \( J \); therefore it should be much easier to calculate than \( Z[J] \). Going for \( N \rightarrow \infty \) back to continuous space-time, the matrix multiplications become integrations,

\[
\mathcal{Z}\left[ J \right] = \exp(iW[J]) = \exp \left( -\frac{i}{2} \int d^4x d^4x' J(x)A^{-1}(x,x')J(x') \right)
\]

and

\[
W[J] = -\frac{1}{2} \int d^4x d^4x' J(x)A^{-1}(x,x')J(x').
\]

**Propagator** In order to evaluate the functional \( W[J] \) we have to find the inverse \( \Delta(x,x') \equiv A^{-1}(x,x') \) of the differential operator \( A \), defined by

\[
-(\Box + m^2) \Delta(x,x') = \delta(x - x').
\]

Because of translation invariance, the Green function \( \Delta(x,x') \) can depend only on the difference \( x - x' \). Therefore it is advantageous to perform a Fourier transformation and to go to momentum space,

\[
-\int \frac{d^4k}{(2\pi)^4} (\Box + m^2) \Delta(k)e^{-ik(x-x')} = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')},
\]

or

\[
\Delta_F(k) = \frac{1}{k^2 - m^2 + i\varepsilon},
\]

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2. Scalar fields

where the pole at $k^2 = m^2$ is avoided by the $i\varepsilon$ prescription. Thus the $m^2 \rightarrow m^2 - i\varepsilon$ prescription introduced to ensure the convergence of the path integral tells us also how to handle the poles of the Green function. The index $F$ specifies that the propagator $\Delta_F$ is the Green function obtained with the $m^2 - i\varepsilon$ prescription proposed by Feynman. (Some authors use instead $DF$ for the propagator of massive bosons and $\Delta_F$ for the propagator of massless bosons.)

Note that the four components $k^\mu$ are independent, i.e. that $\Delta(k)$ describes the propagation of a virtual particle that has—in contrast to a real or external particle—not to be on “mass-shell,” in general

$$k_0 \neq \pm \omega_k \equiv \sqrt{k^2 + m^2 - i\varepsilon}.$$  

We can evaluate the $k_0$ integral in the coordinate representation of $\Delta_F(x - x')$ explicitly,

$$\Delta_F(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - k^2 - m^2 + i\varepsilon}$$  

$$= \int \frac{d^3k}{(2\pi)^3} \int dk_0 \frac{e^{-ik_0(t-t')}e^{ik(x-x')}}{2\pi (k_0 - \omega_k + i\varepsilon)(k_0 + \omega_k - i\varepsilon)},$$  

using Cauchy’s theorem. The integrand has two simple poles at $+\omega_k - i\varepsilon$ and $-\omega_k + i\varepsilon$, cf. Fig. 2.1. For negative $\tau = t - t'$, we can close the integration contour $C_+$ on the upper half-plane, including the pole at $-\omega_k$,

$$\int dk_0 \frac{e^{-ik_0\tau}}{(k_0 - \omega_k + i\varepsilon)(k_0 + \omega_k - i\varepsilon)} = 2\pi i \text{res}_{-\omega_k} = 2\pi i \frac{e^{i\omega_k\tau}}{-2\omega_k} \quad \text{for} \quad \tau < 0. \quad (2.31)$$

For positive $\tau$, we have to choose the contour $C_-$ in the lower plane, picking up $2\pi i \frac{e^{-i\omega_k\tau}}{2\omega_k}$ and an additional minus sign since the contour is clockwise. Combining both results, we obtain

$$i\Delta_F(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ e^{-i\omega_k t} \vartheta(x^0) + e^{i\omega_k t} \vartheta(-x^0) \right] e^{ikx}, \quad (2.32)$$

or after shifting the integration variable $k \rightarrow -k$ in the second term,

$$i\Delta_F(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ e^{-i(\omega_k t - kx)} \vartheta(x^0) + e^{i(\omega_k t - kx)} \vartheta(-x^0) \right]. \quad (2.33)$$

Comparing this expression with the spectral representation (1.16) of the non-relativistic propagator, we note three important properties of the relativistic propagator: First, the propagator contains positive and negative frequencies, as expected from the existence of solutions to the Klein-Gordon equation with positive and negative frequencies. Second, positive frequencies propagate forward in time, while negative frequencies propagate backward. If we imagine that the propagating particle carries a conserved charge, then we can associate the positive frequencies to the propagation of a particle (with charge $q$) and the negative frequencies to the propagation of an anti-particle (with charge $-q$). In this way, the resulting current is frame-independent. Third, the normalisation of plane waves include a factor $1/\sqrt{2\omega_k}$, or

$$\langle k|k' \rangle = 2\omega_k (2\pi)^3 \delta(k - k'), \quad (2.34)$$

to ensure a relativistically invariant normalisation. In order to check this statement, you should work through problem 2.3.
2.2. Generating functional for a scalar field

\[ W[J] = -\frac{1}{2} \int d^4x d^4x' J(x) \Delta_F(x - x') J(x'). \] (2.35)

We are now in the position to evaluate the generating functional in Fourier space. Inserting the Fourier transformations for the propagator as well as for the sources \( J \) gives

\[ W[J] = -\frac{1}{2} \int d^4x d^4x' \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} J(k)^* e^{ikx} \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon} J(k') e^{-ik'x'}. \] (2.36)
Exchanging the integration order and performing the space-time integrations leads to the conservation of the four-momenta entering and leaving the two interaction points, \((2\pi)^3 \delta(k - k')\delta(k - k')\): The source \(J(k)\) produce a scalar particle with momentum \(k\), and thus only the Fourier component \(k\) of the scalar propagator contributes. This is a very general behavior, based solely on the translation invariance of the free particle states we are using. In the final step, we cancel two of the three momentum integrations with the two momentum delta functions and are left with

\[
W[J] = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J(k) .
\] (2.37)

The closer the particles emitted by the source \(J(k)\) are on-shell, \(k^2 \to m^2\), the larger is their contribution to \(W[J]\). For \(k^2 = m^2\), the propagator diverges finally. Reason for this unphysical result is that our formalism assumes that the exchanged particle is stable, and a real particle can thus travel an infinite distance. If we would take the finite life-time \(\tau_{1/2}\) of the exchanged particle into account, the infinitesimal \(i\varepsilon\) would be replaced by a finite quantity determined by \(\tau_{1/2}\).

The functional \(W[J]\) has the same structure as the one for the harmonic oscillator found in the last chapter. We will see that it contains, as in 1-dimensional case, all information about a free scalar field, not only about its ground state.

**Attractive Yukawa potential by scalar exchange** From our macroscopic experience, we know the two cases of electromagnetism, where equal electric charges repel each other, and of gravity where two masses attract each other. The first physics question we want to answer with our newly developed formalism is if the scalar field falls into the category of a fundamentally attractive or repulsive interaction.

In order to address this question, we consider two static point charges as external sources, \(J = J_1(x_1) + J_2(x_2)\) with \(J_i = \delta(x - x_i)\), in \(W[J]\). Multiplying out the terms in \(J(x)\Delta_F(x - x')J(x')\) gives four contributions, \(W_{ij} \propto J_i J_j\): The terms \(W_{11}[J]\) and \(W_{22}[J]\) correspond to the emission and re-absorption of the particle by the same source \(J_i\). They are examples for self-interactions that we neglect for the moment. The interaction between two different charges is given by

\[
W_{12}[J] = W_{21}[J] = -\frac{1}{2} \int d^4x \int d^4x' \frac{d^4k}{(2\pi)^4} J_1(x) \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon} J_2(x').
\] (2.38)

Performing one of the two time integrals, e.g. the one over \(t'\), gives \(2\pi\delta(k_0)\). Hence our assumption of static sources implies that the virtual particle carries zero energy and is space-like, \(k^2 = -k^2 < 0\). Eliminating then the \(k_0\) integral with the help of the delta function, we obtain next

\[
W_{12}[J] = \frac{1}{2} \int dt \frac{d^3k}{(2\pi)^3} \frac{e^{ikr}}{k^2 + m^2}
\] (2.40)

with \(r = x_1 - x_2\). The denominator is always positive, and we can therefore omit the \(i\varepsilon\). Before we can go on, we have to make sense out of the infinite time integral: Looking at

\[
Z[J] = \langle 0 | \exp(-iH[J]\tau) | 0 \rangle = \exp(iW[J]),
\] (2.41)
we see that \( W[J] = -E \tau \), where \( \tau = t - t' \) is the considered time interval. Hence the potential energy \( V \) of two static point charges separated by the distance \( r \) is

\[
V = -(W_{12} + W_{21})/\tau = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{ikr}}{k^2 + m^2} = -\frac{e^{-mr}}{4\pi r} < 0.
\]

(2.42)

Thus the potential energy \( V \) of two equal charges is reduced by the exchange of a scalar particle, which means that the scalar force between them is attractive. If the exchanged particle is massive, the range of the force is of order \( 1/m \). These two observations were the basic motivation for Yukawa to suggest pion-exchange as model for the nuclear force. Note also that we obtain in the limit \( m \to 0 \) a \( 1/r \) potential as in Newton’s and Coulomb’s law. Thus we learned the important fact that the only two known forces of infinite range, the electromagnetic and the gravitational force, are transmitted by massless particles, the photon and the graviton, respectively.

The result \( V \propto 1/r \) for \( m = 0 \) and \( n = 4 \) space-time dimensions, or more generally \( V \propto 1/r^{n-3} \) for \( n \geq 4 \), follows from simple dimensional analysis. For \( m = 0 \), the only remaining dimensionfull parameter after the integration over \( k \) is the distance \( r \). As the potential energy \( V \) has the dimension \([V] = m^{n-3}\), it has to scale as \( r^{-n+3} \).

### 2.3. Green functions for a free scalar field

We have used up to now the definition of a Green function as the inverse of the corresponding differential operator. This definition is sufficiently general for quantum mechanics, where we want to describe the propagation of a single particle from \( x \) to \( x' \). In quantum field theory, we are interested in more complicated processes where many particles can be created and destroyed. For instance, two protons scattering at the LHC in Geneva can produce dozens of particles in the final state. Such processes are described by \( n \)-point functions Green’s functions \( G(x_1, \ldots, x_n) \). Moreover, we will introduce two types of Green functions, namely disconnected \( n \)-point functions \( G(x_1, \ldots, x_n) \) and connected \( n \)-point functions \( \tilde{G}(x_1, \ldots, x_n) \).

Consider the expansion of the exponential in Eq. (2.79),

\[
Z[J] = \exp(iW[J]) = \sum_{n=0}^{\infty} \frac{i^n}{n!} W^n = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n G_n(x_1, \ldots, x_n) J(x_1) \cdots J(x_n) .
\]

(2.43)

For the moment, we assume that \( Z[J] \) is normalised properly so that \( Z[0] = 1 \). We will discuss the normalisation later in the case of an interacting theory. The RHS serves as definition of the disconnected \( n \)-point Greens function \( G(x_1, \ldots, x_n) \). They can be calculated as the functional derivatives of \( Z[J] \),

\[
G(x_1, \ldots, x_n) = \left. \frac{1}{i} \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} Z[J] \right|_{J=0} .
\]

(2.44)

For \( n = 2 \) we should rederive the Feynman propagator. Starting from

\[
\frac{1}{i} \frac{\delta Z[J]}{\delta J(x)} = \frac{1}{i} \frac{\delta}{\delta J(x)} \exp \left( -\frac{i}{2} \int d^4x_1 d^4x_2 J(x_1) \Delta_F(x_1 - x_2) J(x_2) \right)
\]

\[
= -\int d^4x_1 \Delta_F(x - x_1) J(x_1) \exp(iW[J]) ,
\]

(2.45)
we obtain
\[ Z[J] = \frac{1}{i \delta J(x)} \frac{1}{i \delta J(y)} \left( \int d^4 x_1 \Delta_F(x - x_1)J(x_1) \right) \left( \int d^4 x_1 \Delta_F(y - x_1)J(x_1) \right) \exp(iW[J]). \] (2.46)

Setting \( J = 0 \) gives the desired result for the 2-point function as
\[ G(x, y) = i\Delta_F(x - y). \] (2.47)

It is straightforward to continue: Another functional derivative gives the 3-point function,
\[ \frac{\delta}{i \delta J(x_1)} \frac{\delta}{i \delta J(x_2)} \frac{\delta}{i \delta J(x_3)} Z[J] = \]
\[- \left( \int d^4 x \Delta_F(x_1 - x)J(x) \right) \left( \int d^4 x \Delta_F(x_2 - x)J(x) \right) \left( \int d^4 x \Delta_F(x_3 - x)J(x) \right) \exp(iW[J]) \]
\[- i\Delta_F(x_2 - x_3) \int d^4 x \Delta_F(x_1 - x_3)J(x) \exp(iW[J]) \]
\[- i\Delta_F(x_2 - x_1) \int d^4 x \Delta_F(x_3 - x)J(x) \exp(iW[J]) \]
\[- i\Delta_F(x_3 - x_1) \int d^4 x \Delta_F(x_2 - x)J(x) \exp(iW[J]). \] (2.48)

For \( n \) odd, we obtain always a source \( J \) in the pre-factor because \( W[J] \) is an even polynomial in \( J \). Hence all odd \( n \)-point functions are zero. We continue with the 4-point function: After taking another derivative and setting \( J = 0 \), only terms linear in \( J \) of (2.48) contribute and thus
\[ G(x_1, x_2, x_3, x_4) = - \left[ \Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3) \Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4) \Delta_F(x_2 - x_3) \right]. \] (2.49)

We see that the 4-point function is the sum of all permutations of the product of two 2-point functions. For instance, the first term \( \Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) \) in the 4-point function describes the independent propagation of a scalar particle from \( x_1 \) to \( x_2 \) and of another one from \( x_3 \) to \( x_4 \). Thus our approach leads indeed to a many-particle theory. Since we did not include interactions, particles are propagating independently and the \( n \)-point function factorises into products of two 2-point functions. Thus the functional \( Z[J] \) generates disconnected Green functions. The statement that the \( n \)-point function is the sum of all permutations of the product of two 2-point functions holds for all \( n \) and is called “Wick’s theorem”.

We consider next the connected \( n \)-point functions \( G(x_1, \ldots, x_n) \). Their generating functional is \( W[J] \),
\[ G(x_1, \ldots, x_n) = \frac{1}{i^n \delta J(x_1) \cdots \delta J(x_n)} \left. iW[J] \right|_{J=0}. \] (2.50)

For a free theory, \( W \) is quadratic in the sources \( J \). Hence, all connected \( n \)-point functions \( G(x_1, \ldots, x_n) \) with \( n > 2 \) vanish and the only non-zero one is the two-point function with
\[ G(x, y) = i\Delta_F(x - y) = G(x, y). \] (2.51)
2.3. Green functions for a free scalar field

To summarise: There exists only one non-zero connected n-point function in a free theory which is determined by the Feynman propagator, $G(x, y) = i \Delta_F(x - y)$. All non-zero disconnected n-point functions can be obtained by permuting the product of $n/2$ two-point functions (“Wick’s theorem”). Hence any higher-order Green’s function can be constructed out of a single building block, the Feynman propagator.

In perturbation theory, we will recast the interacting theory – loosely speaking – in “interaction vertices times free propagators”. This enables us to derive simple Feynman rules that tell us how one constructs an arbitrary Green function out of vertices and propagators.

Causality and the Feynman propagator We expect that a relativistic field theory automatically implements the requirement of causality: No signal using our $\phi$ particles as carrier should travel with a speed larger than the one of light. On the other hand, the Feynman propagator $i \Delta_F(x_1 - x_2)$ does not vanish outside the light-cone, but goes only exponentially to zero, $i \Delta_F(x_1 - x_2) \propto \exp(-m|x_1 - x_2|)$ for space-like distances $(x_1 - x_2)^2 < 0$, cf. problem 2.3. One may wonder, if this means that the uncertainty principle makes the light-cone “fuzzy” and thus the axiom of special relativity that no signal can be transmitted with $v > c$ is violated on scales smaller $\lesssim 1/m$.

This problem is addressed best following the usual approach of quantum mechanics: We consider the classical field $\phi(x)$ as operator $\hat{\phi}(x)$ and ask when a measurement of $\hat{\phi}(x)$ influences $\hat{\phi}(x')$. We have shown that the Feynman propagator equals the 2-point Green function. From our discussion in Sec. 1.3 we know that a n-point Green function corresponds to the vacuum expectation value of the time-ordered product of n field operators. Combining all this gives

$$G(x_1, x_2) = \frac{1}{(2\pi)^3 2\omega_k} \int \frac{d^3k}{Z[J]} \exp(i \omega_k t - k \cdot x) \langle 0 | T \{ \hat{\phi}(x_1) \hat{\phi}(x_2) \} | 0 \rangle = i \Delta_F(x_1 - x_2).$$

The property $\Delta_F(x - y) = \Delta_F(y - x)$ implies that the field operators $\hat{\phi}(x_1)$ and $\hat{\phi}(x_2)$ commute,

$$\langle 0 | T \{ \hat{\phi}(x_1) \hat{\phi}(x_2) \} | 0 \rangle = \langle 0 | \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle \delta(t_1 - t_2) + \langle 0 | \hat{\phi}(x_2) \hat{\phi}(x_1) | 0 \rangle \delta(t_2 - t_1).$$

Using the analogy of a free quantum field to an infinite set of oscillators, we try to express the field operator $\hat{\phi}(x)$ through annihilation and creation operators. Comparing

$$\langle 0 | T \{ \hat{\phi}(x) \hat{\phi}(0) \} | 0 \rangle = i \Delta_F(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ e^{-i(\omega_k t - k \cdot x)} \hat{a}(x) + e^{i(\omega_k t - k \cdot x)} \hat{a}^\dagger(-x) \right]$$

with the Fourier decomposition of a classical, free scalar field,

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[ a(k)e^{-ikx} + a^\dagger(k)e^{+ikx} \right],$$

suggests to reinterpret the classical Fourier coefficients $a(k)$ and $a^\dagger(k)$ as annihilation and creation operators,

$$a^\dagger(k) | 0 \rangle = | k \rangle, \quad a(k) | k' \rangle = (2\pi)^3 2\omega_k \delta(k - k'), \quad \text{and} \quad a(k) | 0 \rangle = 0.$$
relations\n
we rederive the RHS of Eq. (2.52). Moreover, these relations lead to canonical commutation relations between the field $\hat{\phi}$ and its canonically conjugated momentum density $\hat{\pi} = \dot{\hat{\phi}}$ at equal times,

$$\langle \hat{\phi}(x, t), \hat{\pi}(x', t) \rangle = i \delta(x - x') . \tag{2.58a}$$

$$[\hat{\phi}(x, t), \hat{\phi}(x', t)] = [\hat{\pi}(x, t), \hat{\pi}(x', t)] = 0 . \tag{2.58b}$$

You may compare the (missing) details of this derivation to problem ??, where the analogous calculation is done in $1 + 0$ dimensions.

We come back to the question if the commutator of two fields vanishes for space-like separation. We evaluate first

$$[\hat{\phi}(x), \hat{\phi}(x')] = \int d^3k d^3k' \left[ a(k)e^{-ikx} + a^\dagger(k)e^{+ikx}, a(k')e^{-ik'x'} + a^\dagger(k')e^{+ik'x'} \right] =$$

$$\int d^3k \left( e^{-ik(x-x')} - e^{+ik(x-x')} \right) = \Delta(x - x') - \Delta(x' - x) \equiv D(x - x') . \tag{2.59}$$

For equal times, $t = t'$, exchanging the dummy variable $k \rightarrow -k$ in the second term shows that the contribution from positive and negative energies cancel. Thus the equal-time commutator of two fields is zero, as claimed in \textcolor{red}{(2.58b)}.

For space-like distances, $(x - x')^2 < 0$, we can find a Lorentz transformation which changes the ordering of the space-time events,

$$x - x' \rightarrow -(x - x') .$$

(This is impossible for time-like separated events, since otherwise different observers could not agree on the flow of time.) Since the Green function $\Delta(x_1 - x_2)$ is Lorentz invariant, the value of the scalar function $D(x_1 - x_2)$ has to be the same in all inertial frames. But for space-like distances, we can transform $D(x)$ into $-D(x)$, and therefore $D(x)$ has to vanish if $x$ is outside the light-cone of $x'$ and vice versa. Thus we have shown that also the commutator of two space-like separated fields vanishes,

$$[\hat{\phi}(x), \hat{\phi}(x')] = 0 \quad \text{for } (x - x')^2 < 0 , \tag{2.60}$$

which is the condition for causality.

How do we reconcile this result with the fact that the the Feynman propagator does not vanish outside the light-cone? There are two main differences between these two quantities: First, $[\hat{\phi}(x), \hat{\phi}(x')]$ is an operator, while the Feynman propagator $i\Delta(x_1 - x_2)$ is a vacuum expectation value. But the quantum vacuum fluctuates, and these fluctuations are correlated also on space-like distances, similar to the ERP correlations in quantum mechanics. Second, in $[\hat{\phi}(x), \hat{\phi}(x')]$ we subtract the contribution of positive and negative frequencies, while we add them in the Feynman propagator. As a result, the contributions from a particle travelling the distance $x$ and from an anti-particle travelling the distance $-x$ cancel in the commutator, while they add up in the Feynman propagator. Since causality relies on the cancelation between positive and negative energy modes in $[\hat{\phi}(x), \hat{\phi}(x')]$, we conclude that a relativistic quantum theory has to incorporate anti-particles.
2.4. Vacuum energy and the Casimir effect

**Vacuum energy**  We now aim at calculating the energy of the vacuum state of a free scalar quantum field. The energy density $\rho$ of the field $\phi$ is given by the vacuum expectation value of its Hamiltonian density $\mathcal{H}$,

$$
\rho = \langle 0 | \mathcal{H} | 0 \rangle = \rho_0 + \frac{1}{2} \langle 0 | \pi^2 + (\nabla \phi)^2 + m^2 \phi^2 | 0 \rangle = \rho_0 + \rho_1 .
$$

(2.61)

Here we added the constant energy density $\rho_0$ to (2.18) and used that the vacuum is normalised, $\langle 0 | 0 \rangle = 1$. For the calculation of $\rho_1$, we can recycle our result for the propagator of a scalar field by considering $\phi^2(x)$ as the limit of two fields at nearby points,

$$
\langle 0 | \phi(x') \phi(x) | 0 \rangle_x \rightarrow \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x'-x)} |_{x' \rightarrow x} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} .
$$

(2.62)

We perform first the differentiation in $\langle \pi^2 \rangle = \langle \phi^2 \rangle$ and $\langle (\nabla \phi)^2 \rangle$ and send then $x' \rightarrow x$. Thus $\pi^2$ and $(\nabla \phi)^2$ add a $\omega_k^2$ and $k^2$ term, respectively,

$$
\rho = \langle 0 | \mathcal{H} | 0 \rangle = \rho_0 + \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ \frac{1}{2} (\omega_k^2 + k^2 + m^2) \right] = \rho_0 + \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{1}{2} \omega_k .
$$

(2.63)

If we insert $\hbar$ and $c$ into this expression, we see that $\rho_0$ as a classical contribution to the energy density of the vacuum is $\propto \hbar^0$, while the second term $\rho_1 \propto \hbar \omega_k / V$ as a quantum correction is linear in $\hbar$. The total energy density $\rho$ of the vacuum state of a free scalar field has a very intuitive interpretation: Additionally to the classical energy density $\rho_0$, it sums up the zero-point energies of all individual modes $k$ of a free field. Despite its simplicity, we cannot make sense out of this result: Since both the density of modes and their energy increases with $|k|$, the integral diverges. This is the first example that momentum integrals in quantum field theories are often ill-defined and require some care and cure. In problem 2.3 you should rederive our result for $\rho_1$ expressing the field $\phi$ by annihilation and creation operators, to convince yourself that this divergence is not caused by the point-splitting method employed in Eq. 2.62.

Let us now consider the case that the Hamiltonian (2.18) describes the physics correctly only up to the energy scale $\Lambda$, while the modes with $|k| \gtrsim \Lambda$ do not contribute to $\rho_1$. Such a possibility exists e.g. in supersymmetric theories where the contributions of different particle types cancel each other above the scale $\Lambda_{\text{SUSY}}$ where supersymmetry is broken. Integrating the contribution to the vacuum energy density by field modes up to the cutoff scale $\Lambda$, we find

$$
\rho_1 = \int_0^\Lambda \frac{dk}{2\pi^2} \frac{k^2}{2\omega_k} \sim \Lambda^4
$$

(2.64)

in the limit $\Lambda \gg m$. Since only the total energy density $\rho$ is observable, the unknown $\rho_0$ can be always chosen such that $\rho_0 + \rho_1$ agrees with observations, even if $|\rho_0|, |\rho_1| \gg |\rho|$. Nevertheless, the strong sensitivity of $\rho_1$ to the value of the cutoff scale $\Lambda$ is puzzling for two reasons: First, cosmological observations determine the total vacuum energy density $\rho_\Lambda$ to which all types of fields contribute as $\rho_\Lambda \sim (\text{meV})^4$. On the other hand, accelerator experiments give no indications that a cancellation mechanism as supersymmetry works at energy below few TeV. Thus we should expect at least $\rho_\Lambda \sim (\Lambda_{\text{SUSY}})^4 \gtrsim (\text{few TeV})^4$, which is 60 orders of magnitude larger than observed, if there is not a strong cancellation of the various contributions to $\rho_\Lambda$. 

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taking place. Second, the behavior $\rho_1 \sim \Lambda^4$ implies that all scalar particles with mass $m < \Lambda$ contribute equally to $\rho$. In other words, we cannot calculate $\rho$ without knowing the physics at energy scales much larger than those accessible experimentally to us. Such a behavior should be truly exceptional, otherwise nothing like chemistry or solid state physics would be possible before knowing the “theory of everything”.

**Casimir effect**  Although we cannot calculate ambiguously the vacuum energy, we can determine the energy difference of different vacua. As a concrete example, we consider now the Casimir effect. Between two parallel, perfectly conducting plates at distance $d$ electromagnetic waves have the form $\sin(n\pi x/d)$ with discrete energy $\omega_n = n\pi/d$. Thus the vacuum energy inside the box of volume $dL_yL_z$ is

$$E = \sum_{n=1}^{\infty} \int \frac{dk_ydk_z}{(2\pi)^2L_yL_z} \sqrt{\left(\frac{n\pi}{d}\right)^2 + k_y^2 + k_z^2}. \quad (2.65)$$

Clearly, the change from a continuous to a discrete spectrum of energy eigenvalues has not improved the convergence. Compared to the case of a massless scalar field, there is an additional factor two due to the two spin degrees of freedom of the photon. Moreover, we use a box normalisation in the perpendicular direction.

To simplify the calculations, we consider a $1+1$ dimensional system of two plates separated by the distance $d$. Then the energy density $\rho = E/d$ of a massless scalar field inside the plates is

$$\rho(d) = \frac{\pi}{2d^2} \sum_{n=1}^{\infty} n. \quad (2.66)$$

Next we introduce a cutoff function $f(a) = \exp(-an\pi/d)$ which suppresses the high-energy modes,

$$\rho(d) \to \rho(a,d) = \frac{\pi}{2d^2} \sum_{n=1}^{\infty} ne^{-an\pi/d}. \quad (2.67)$$

This procedure is called *regularisation*: For $a > 0$, we obtain a well-defined mathematical sum which we can manipulate following the usual rules of analysis, while we recover for $a \to 0$ our old divergent result. We have chosen as argument of the exponential $an\pi/d$, because the physically relevant quantities are the energy levels $\omega_n = n\pi/d$ of the system. Now we can evaluate the regularised sum, rewriting it as a geometrical sum,

$$\rho(a,d) = \frac{\pi}{2d^2} \sum_{n=1}^{\infty} ne^{-an\pi/d} = -\frac{1}{2d} \frac{\partial}{\partial a} \sum_{n=0}^{\infty} e^{-an\pi/d} \quad (2.68)$$

$$= -\frac{1}{2d} \frac{\partial}{\partial a} \frac{1}{1 - e^{-a\pi/d}} = \frac{\pi}{2d^2 (1 - e^{-a\pi/d})^2}. \quad (2.69)$$

Next we use $e^x(1 - e^{-x})^2 = 4\sinh^2(x/2)$ and expand $\rho(a,d)$ for small $a$ in a Laurent series,

$$\rho(a,d) = \frac{\pi}{8d^2} \frac{1}{\sinh^2(a\pi/2d)} = \frac{1}{2\pi a^2} - \frac{\pi}{24d^2} + O(a^2d^{-4}). \quad (2.70)$$

Note that we isolated the divergence into a term which does not depend on the distance $d$ of the plates. Thus the divergence cancels in the difference of the vacuum energy with and without plates,

$$\rho_{\text{Cas}}(d) \equiv \lim_{a \to 0} \left[ \rho(a,d) - \rho(a,d \to \infty) \right] = -\frac{\pi}{24d^2}. \quad (2.71)$$
This final step in order to obtain a finite result is called renormalisation. One can verify that the result is not only independent of the cutoff parameter $a$, but also from the shape of the cutoff function $f(a)$. In contrast, the individual terms in Eq. (2.70) may depend on the cutoff function.

The quantity measured in actual experiments is the force $F$ with which the plates attract (or repel) each other. This force is given by

$$-F = \frac{\partial E}{\partial d} = \frac{\partial (d\rho_{\text{Cas}})}{\partial d} = \frac{\pi}{24d^2}.$$ (2.72)

Thus two parallel plates attract each other. The experimentally relevant case of electromagnetic waves between two parallel plates in 3+1 dimensions can be calculated analogously. The theoretical prediction has been confirmed with a precision on the 1% level.

We conclude this discussion of the Casimir effect with two remarks:

1. If the main motivation for the introduction of the cutoff function would be physics (e.g. “X-rays do not see perfectly conducting plates,” then we would have to worry about the exact form of the cutoff function $f(a)$: the transition from $f = 1$ (low frequencies) to $f = 0$ (high frequencies) would have a physical meaning and should show up in the result.

2. We shall come back to the vacuum energy in Section 2.6, examining the connection to vacuum diagrams generated by $Z[0]$ and the cosmological constant.

2.5. Perturbation theory for interacting fields

We know already from quantum mechanics that adding an anharmonic term to an oscillator forces us to use either perturbative or numerical methods. The same happens in field theory: No analytic solution for a realistic interacting theory is at present known in $d = 4$. Therefore we will develop in the remaining part of this chapter perturbative methods.

General formalism The Lagrange density $\mathcal{L}$ in the functional $Z[J]$ for the scalar field considered up to now was at most quadratic in the fields and its derivatives. On one hand, this allowed us to evaluate the path integral, while on the other hand this means that the field has no interactions: Two wave packets described by the free propagator just pass each other without interaction, as the superposition principle prescribes. As next step we add therefore an interaction term $\mathcal{L}_I$ to the free Lagrangian $\mathcal{L}_0$, i.e. we set $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$. Then the generating functional $Z[J]$ for a real scalar field $\phi$ with mass $m$ becomes

$$Z[J] = N \int \mathcal{D}\phi \exp i \int d^4x \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 + \mathcal{L}_I + J\phi \right),$$ (2.73)

where we introduced also a normalization constant $N$. Since the vacuum state should be normalised to one, $\langle 0 + |0- \rangle = 1$, this normalization constant is determined also for an interacting theory by

$$N^{-1} = Z[0] = \int \mathcal{D}\phi \exp i \mathcal{S}.$$ (2.74)

We assume that the interaction term $\mathcal{L}_I$ is a polynomial $\mathcal{P}(\phi)$ in the field $\phi$ and contains an expansion parameter $\lambda$ which is small in the considered kinematic regime, $\mathcal{L}_I = \lambda \mathcal{P}(\phi)$ with $\lambda \ll 1$. This suggests to use perturbation theory, i.e. to expand

$$\exp i \int d^4 x \mathcal{L}_I(\phi) = 1 + i \lambda \int d^4 x \mathcal{P}(\phi(x)) + \frac{(i\lambda)^2}{2!} \int d^4 x_1 d^4 x_2 \mathcal{P}(\phi(x_1)) \mathcal{P}(\phi(x_2)) + \ldots$$ (2.75)
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Since
\[ i\phi(x)e^{i\int d^4x'(L_0 + J\phi)} = \frac{\delta}{\delta J(x)} e^{i\int d^4x'(L_0 + J\phi)}, \]  
we can perform the replacement
\[ L_1(\phi(x)) \rightarrow \frac{1}{i} \frac{\delta}{\delta J(x)}. \]  

Then the interaction \( L_1 \) does not depend longer on \( \phi \) and can be pulled out of the functional integral,
\[
Z[J] = N \exp i \int d^4x L_1 \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) \int D\phi \exp i \int d^4x (L_0 + J\phi) = N \exp i \int d^4x L_1 \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) e^{iW_0[J]}.
\]  

Here we denoted the free functionals discussed earlier with \( Z_0 \) and \( W_0 \). Their solution was given in Eq. (2.23),
\[
Z_0[J] = Z_0[0] \exp \left( -\frac{i}{2} \int d^4x d^4x' J(x) \Delta_F(x - x') J(x') \right) = Z_0[0] \sum_{n=0}^{\infty} \frac{i^n}{n!} W_0^n.
\]  

Perturbation theory consists in a double expansion of the two exponentials in \( Z[J] \): One in the coupling constant \( \lambda \) and one in the number of external sources \( J \). The latter is fixed by the number of external particles in a scattering process, while the former is chosen according to the desired precision of the calculation.

**Choosing the interaction term**  Even restricting possible interactions to be polynomials in the field, we have still an infinite number of choices. Surprisingly, the simple argument of dimensional analysis will be an important guide line in the selection of “good” interaction terms. We first remember that using natural units, \( \hbar = c = 1 \), the dimension of all physical quantities can be expressed by powers of one basic unit which we choose as mass \( m \).

Then we use that the action has dimension zero, \( [S] = m^0 \), and thus the Lagrangian \( [\mathcal{L}] = m^4 \) in four space-time dimensions. From the free Lagrangian, we conclude that the dimension of a bosonic field in four dimension is \( [\phi] = m^1 \). Thus we can order possible self-couplings of a scalar field according to their dimension as
\[
\mathcal{L}_I = c_2 M \phi^3 + c_4 \phi^4 + \frac{c_5}{M} \phi^5 + \ldots,
\]  
where the coupling constants \( c_i \) are dimensionless and we introduced the mass scale \( M \) to ensure \( [\mathcal{L}] = m^4 \). We call \( \phi^n \) a dimension-\( n \) operator. Similar as in the case of the Fermi constant, \( G_F = \sqrt{2} g_F^2 / (8m_W^2) \), the scale \( M \) could be connected to the exchange of a heavy particle. If the scale \( M \) is large compared to the center-of-mass (cms) energy \( \sqrt{s} \) of the process considered, the interaction probability contains typically factors like \( (\sqrt{s}/M)^{n-4} \). Thus for \( \sqrt{s} \ll M \) the interaction is the stronger suppressed the higher the dimension of the corresponding operator is. If on the other hand \( \sqrt{s} \gg M \), then we expect that the effective expansion parameter becomes \( \lambda(\sqrt{s}/M)^{n-4} \gg 1 \) and perturbation theory becomes thus unreliable.
2.6. Green functions for the $\lambda \phi^4$ theory

In a first approach, we neglect therefore operators of dimension five and higher. Simplifying further $L_I$, we want to include only one interaction term. In this case, a $\phi^3$ term would lead to an unstable vacuum. Therefore our choice for the scalar self-interaction is

$$L_I = -\frac{\lambda \phi^4}{4!},$$

where the factor $1/4!$ was added for later convenience. If this choice of interaction is realized in Nature for a specific particle has to be decided by experiment.

2.6. Green functions for the $\lambda \phi^4$ theory

We start now with the perturbative evaluation of Eq. (2.78) for a $\lambda \phi^4/4!$ interaction. From

$$Z[J] = (1 - i\lambda \frac{4!}{\delta J(x)^4} + \ldots) Z_0[J] = Z_0[J] - i\lambda \int d^4x \frac{4! Z_0'[J]}{\delta J(x)^4} + \ldots$$

we see that we will generate a series of the type free Green function plus correction $O(\lambda)$ plus higher order corrections. The calculation is very similar to the calculation of the free four-point function, with the difference that now the four sources are at the same point. We find

$$\left( \frac{\delta}{\delta J(x)} \right)^4 \exp(iW_0[J]) = \left[ 3(i\Delta(0))^2 + 6i\Delta(0) \left( \int d^4y \Delta_F(x-y)J(y) \right) \right]^2 + \left( \int d^4y \Delta_F(x-y)J(y) \right)^4 \exp(iW_0[J]).$$

Next we introduce a graphical representation for the various terms in Eq. (2.82). Each Feynman propagator $\Delta_F(x-y)$ is represented by

$$i\Delta_F(x-y) = x \quad \bullet \quad y,$$

a source term $J(x)$ by

$$i \int d^4x \ J(x) = \quad \bullet$$

and an interaction vertex by

$$-i\lambda \int d^4x = \bullet$$

Each source and vertex has its own coordinates and an integration over all coordinates is implied. In case of a $\phi^4$ interaction, a vertex connects four lines. Using this notation, we can express $Z_1$ as

$$Z_1[J] = \frac{1}{4!} \left( \begin{array}{c} 3 \bigcirc + 6 \bigcirc + \bigcirc \end{array} \right) \exp \left( \frac{1}{2} \quad \bullet \right).$$

A graph consists of lines and dots, i.e. vertices or sources. We distinguish internal and external lines: A line which ends on boths side at a dot with at least two lines attached is called internal; otherwise it is an external line. The three graphs contained in $Z_1[J]$ differ by the number of loops, i.e. by the number of closed lines. A connected graph with $n$ lines and
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$V$ dots has the loop number $l = n - V + 1$. Thus increasing the number of dots reduces the number of loops. A graph with loop number $l = 0$ is called a tree graph.

Note that the first and the second term can be derived from the third one by joining two and one lines, respectively. There are six ways to join one line, and three ways to join two lines. Thus the prefactors of the various terms could be derived by simple symmetry arguments.

Now we can derive disconnected Green functions valid at $\mathcal{O}(\lambda)$ by performing functional derivatives,

$$G^{(n)}(x_1, \ldots, x_n) = \frac{1}{i^n} \delta^n J(x_1) \cdots \delta J(x_n) Z_0[J] \left( 1 + \lambda z_1[J] \right) \bigg|_{J=0}. \quad (2.87)$$

In the graphical notation, differentiating with respect to $J(x)$ amounts to replace the open dot denoting the source $\int d^4y J(y)$ by its position $x$,

$$\frac{1}{i} \frac{\delta}{\delta J(x)} = x. \quad (2.88)$$

**Vacuum diagrams** We call terms in the perturbative evaluation of $Z[J]$ which contain no sources $J$ vacuum diagrams. Setting $J = 0$ eliminates all graphs containing at least one source. Thus these diagrams correspond to loops without external lines and the corresponding Green functions are the “zero-point” Green functions $G^{(0)}$.

Let us assume that the integration measure of the free theory is normalized such that the free “zero-point” function is one, $Z_0[0] = 1$. Interactions induce however a change in this normalization. Setting $J = 0$ in our result for $Z[J]$ at lowest order perturbation theory, Eq. (2.86), gives

$$G^{(0)} \equiv Z[0] = 1 - \frac{i\lambda}{8} \int d^4x (i\Delta F(0))^2. \quad (2.89)$$

Since vacuum diagrams do not contribute to scattering processes, one often prefers to eliminate these diagrams by an appropriate normalization of $Z[J]$. We now show that choosing $N^{-1} = Z[0]$ as normalisation condition eliminates all these graphs.

Calling the normalized functional $\tilde{Z}[J]$ with $\tilde{Z}[J] = Z[J]/Z[0]$, expanding nominator and denominator up to $\mathcal{O}(\lambda)$, we have at lowest order perturbation theory

$$\tilde{Z}[J] = \frac{Z[J]}{Z[0]} = \frac{1 + \lambda z_1[J] + \mathcal{O}(\lambda^2)}{1 + \lambda z_1[0] + \mathcal{O}(\lambda^2)} Z_0[J] = \{1 + \lambda(z_1[J] - z_1[0])\} Z_0[J] + \mathcal{O}(\lambda^2). \quad (2.90)$$

Thus dividing $Z[J]$ by the source-free functional eliminates indeed all vacuum graphs. It becomes obvious that this procedure works at any order perturbation theory, if we look at the generating functional for connected graphs, $W[J]$. As dividing $Z[J]$ by the source-free functional $Z[0]$ corresponds to

$$iW[J] = \ln \tilde{Z}[J] = \ln Z[J] - \ln Z[0], \quad (2.91)$$

it is clear that this procedure eliminates indeed all vacuum graphs. Note that because of $N = \exp \ln(N)$, a normalization different from one is equivalent to adding a constant term to the Lagrangian,

$$\mathcal{L} \to \mathcal{L} + \ln(N)/(VT) = \mathcal{L} - \rho, \quad (2.92)$$

where $VT$ is the four-dimensional integration volume in the action. The normalised generating functional at $\mathcal{O}(\lambda)$ is thus obtained by dropping the two-loop diagram $\propto (i\Delta F(0))^2$ and corresponds to a vacuum with zero vacuum energy density, i.e. a vanishing cosmological constant.
2.6. Green functions for the $\lambda\phi^4$ theory

2-point functions We start by taking one derivative of the normalised generating functional,

$$\frac{1}{i} \frac{\delta}{\delta J(x_1)} \left[ 1 + \frac{1}{4!} \left( 6 \lambda \phi^6 + \lambda \phi^4 \right) \right] \exp \left( \frac{1}{2} \phi^2 \right) =$$

$$= \left[ \frac{1}{4!} \left( 6 \lambda \phi^6 + \lambda \phi^4 \right) \right] \times \exp \left( \frac{1}{2} \phi^2 \right)$$

Every term in this expression contains at least one source $J$, and the one-point function $G^{(1)}(x)$ vanishes therefore. If we proceed to the two-point function $G^{(2)}(x_1, x_2)$, we have to differentiate only those terms with one source,

$$\frac{1}{i} \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} Z[J] =$$

$$= \frac{1}{\delta J(x_2)} \left[ -\lambda \phi^4 \right] \times \exp \left( \frac{1}{2} \phi^2 \right) \left[ -\lambda \phi^4 \right] \times \exp \left( \frac{1}{2} \phi^2 \right)$$

Setting the sources $J$ to zero, the exponential factor becomes one. Converting the graphical formula back into standard notation, we find the 2-point function $G^{(2)}(x_1, x_2)$ as the sum of the free 2-point function $G^{(2)}_0(x_1, x_2)$ and a correction,

$$G^{(2)}(x_1, x_2) = G^{(2)}_0(x_1, x_2) - \frac{i\lambda}{2} \int d^4 x i \Delta_F(x_1 - x) i \Delta_F(x - x) i \Delta_F(x - x_2).$$

This correction is called the self-energy $\Sigma$ of the scalar particle. Note that the pre-factor combines to $-1/2i\lambda$, so there appears an extra factor $1/2$. Such factors are called symmetry factors. They appear because we included a factor $1/4!$ in $Z[J]$ to compensate for the $4!$ permutations of four sources. If a diagram has a certain symmetry, i.e. if it can be rearranged by permutating propagators and/or vertices giving the same expression, the cancellation is only partially. In such cases a symmetry factor appears.

4-point functions The disconnected 4-point function $G(x_1, x_2, x_3, x_4)$ is shown graphically in Fig. 22. The last diagram corresponds to the connected 4-point function $G(x_1, x_2, x_3, x_4)$. Next we want to derive $G(x_1, x_2, x_3, x_4)$ from its generating functional $W[J]$. We insert $Z[J]$
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\[
\begin{align*}
\text{Figure 2.2.: Graphs contributing to the disconnected four-point function } G(x_1, x_2, x_3, x_4). \\
\end{align*}
\]

Into

\[
iW[J] = \ln(Z[J]) = \ln \exp \left( \frac{1}{2} \right) + \ln \left[ 1 + \frac{1}{4!} \left( 6 \quad + \quad \right) \right] + O(\lambda^2)
\]

\[
= \frac{1}{2} + \frac{1}{4!} \left( 6 \quad + \quad \right) + O(\lambda^2) \tag{2.99}
\]

Where we expanded the logarithm, \( \ln(1 + x) \sim x \). Taking four derivatives with respect to \( J \), only the last term survives,

\[
G(x_1, x_2, x_3, x_4) = -i\lambda \int d^4x \Delta_F(x_1 - x) \Delta_F(x_2 - x) \Delta_F(x_3 - x) \Delta_F(x_4 - x). \tag{2.100}
\]

Symmetry factors We illustrate by an example how one can determine the symmetry factor in more complicated cases. Consider the so-called “sunrise diagram”,
2.6. Green functions for the $\lambda\phi^4$ theory

which is a second order diagram, corresponding to the term

$$\frac{1}{2!} \left(\frac{-i\lambda}{4!}\right)^2 \int d^4y_1 d^4y_2 \langle 0 | T[\phi(x_1)\phi(x_2)\phi^4(y_1)\phi^4(y_2)] | 0 \rangle + (y_1 \leftrightarrow y_2)$$

in the perturbative expansion. The exchange graph $y_1 \leftrightarrow y_2$ is identical to the original one, canceling the factor $1/2!$ from the Taylor expansion. We have to count the number of possible ways to combine the fields in the time-ordered product into five propagators. As shorthand notation, we mark a possible combination as $\phi(x_1)\phi(y_1)$.

We have four possibilities to combine $\phi(x_1)$ with $\phi(y_1)$, $\phi(x_1)\phi(y_1)$. Similarly, there are four possibilities for $\phi(x_2)\phi(y_2)$. The remaining six fields can be combined in $3!$ ways into pairs, $\phi(y_1)\phi(y_1)\phi(y_2)\phi(y_2)$. Thus the symmetry factor of this diagram is

$$S = \left(\frac{1}{2!} \times 2\right) \left(\frac{1}{4!}\right)^2 (4 \times 4 \times 3!) = \frac{1}{3!}$$

Feynman rules for the $\lambda\phi^4$ theory in coordinate space We can summarize our results in few simple rules which allows us to write down Green functions directly, without the need to derive them from their generating functional. The rules refer to connected diagrams after integration over internal variables.

1. Draw all topologically different diagrams for the chosen order $\mathcal{O}(\lambda^n)$ and number of external coordinates or particles.
2. To each line connecting the points $x$ and $x'$ we associate a propagator $i\Delta_F(x-x')$.
3. Each vertex has a factor $-i\lambda$ and connects $n$ lines for a $\lambda\phi^n$ interaction.
4. Integrate over all intermediate points.
5. Add a symmetry factor counting the number of possible identical rearrangements of the diagram.
6. The rules above give a $n$-point Green function. A scattering process is described by the transition amplitude $iA$ between a fixed initial and final state which contain real, on-shell particles. Thus the propagators of the external lines, which describe virtual particles, should be replaced by on-shell wave functions—this rule will be derived in chapter 6.2. Moreover, we will show later that we can ignore all loop corrections in external lines.

Feynman rules in momentum space The integration over intermediate points is trivial, because all propagators (and possible wave function on external lines) depend like $\exp(\pm ikx)$ on the position of the interaction point. Hence the space integrations result in 4-momentum conservation at each vertex. Therefore the integration over the momenta of $V-1$ propagator can be trivially performed in a Green function with $V$ vertices, whereas the remaining four-momenta delta function expresses overall momentum conservation. As a result, in a Green function with loop number $l = n - V + 1$ remain $l$ non-trivial four-momenta integrations.

The Feynman rules in momentum space have thus the following changes:

2. To each line we associate a propagator $i\Delta_F(k) = i/(k^2 - m^2 + i\varepsilon)$. 

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3. Fix the external momenta and impose 4-momentum conservation at each vertex.
4. Integrate over all unconstrained momenta with \( \int \frac{d^4k}{(2\pi)^4} \). The number of independent momenta we have to integrate over equals the loop number of the graph.
6. If the diagram should represent the transition amplitude \( \mathcal{A} \) of a scattering process, propagators on the external lines should be replaced by on-shell wave functions. It is advantageous to omit the normalisation factors \( N = \frac{1}{(2\pi)^3(2\omega_k)} \) from the matrix element and to include them later into a flux (for the initial state) and phase space factor (for the final state). In this way, the amplitude \( \mathcal{A} \) is Lorentz invariant and easier to manipulate. Thus for scalar particles, the Feynman rule for external particles is simply to write “1”.

2.7. Loop diagrams

Aim of this section is to illustrate the regularization and renormalization procedure used in quantum field theory calculating three one-loop diagrams of the \( \lambda\phi^4 \) theory. We will have time to digest these examples, before we will come back to the problem of renormalisation in Chapter 8. The basic steps in the evaluation of simple Feynman integrals are summarised in the appendix 2.A.

2.7.1. Self-energy

We consider first the only one-loop diagram contained in \( Z_1[J] \), the 2-point function of a scalar particle at \( \mathcal{O}(\lambda) \),

\[
\mathcal{G}^{(2)}(x_1, x_2) = i\Delta_F(x_1 - x_2) - \frac{\lambda}{2}\Delta_F(0) \int d^4x \Delta_F(x_1 - x)\Delta_F(x - x_2). \tag{2.101}
\]

We start concentrating on the correction term, the self-energy \( \Sigma(x_1 - x_2) \) of the scalar particle, and insert the Fourier representation of the two propagators into the integral,

\[
\Sigma(x_1 - x_2) = -\frac{\lambda}{2}\Delta_F(0) \int d^4x \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \frac{e^{ip(x_1-x)}}{p^2 - m^2 + i\varepsilon} \frac{e^{ip'(x-x_2)}}{p'^2 - m^2 + i\varepsilon}. \tag{2.102}
\]

The \( d^4x \) integration produces \( (2\pi)^4\delta(p - p') \), then one of the momentum integrations can be performed. Together this gives

\[
\Sigma(x_1 - x_2) = -\frac{\lambda}{2}\Delta_F(0) \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x_1-x_2)}}{(p^2 - m^2 + i\varepsilon)^2}. \tag{2.103}
\]

Inserting also for the free Green function its Fourier representation, we arrive at

\[
\mathcal{G}^{(2)}(x_1, x_2) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x_1-x_2)} \left[ \frac{i}{p^2 - m^2 + i\varepsilon} - \frac{\Delta_F(0)}{(p^2 - m^2 + i\varepsilon)^2} \right]. \tag{2.104}
\]

The Green function in momentum space is thus

\[
\mathcal{G}^{(2)}(p) = \frac{i}{p^2 - m^2 + i\varepsilon} \left[ 1 + \frac{\lambda}{2}\Delta_F(0) \right]. \tag{2.105}
\]
Assuming that perturbation theory is justified, the second term in the parenthesis should be small. Thus \(1 + \lambda a = [1 - \lambda a]^{-1} + \mathcal{O}(\lambda^2)\) and
\[
G^{(2)}(p) = \frac{i}{p^2 - m^2 - \frac{i\lambda}{2} \Delta F(0) + i\varepsilon}.
\]

The residue of the free propagator defined the “bare” particle mass at zero order in \(\lambda\). To make this clearer, we write now \(m_0\) instead of \(m\). The physical (or renormalized) mass of the scalar particle is at order \(\lambda\) given by
\[
m_{\text{phys}}^2 = m_0^2 + \delta m^2 = m_0^2 + \frac{i\lambda}{2} \Delta F(0).
\]

Hence interactions shift the “bare” mass \(m_0\) used initially in the classical Lagrangian \(\mathcal{L}\). Theorists tend to call this procedure “renormalization”, although the scalar mass was never properly normalized. It is important to realize that renormalization happens in any interacting theory, independently of the question if \(\delta m^2\) is finite or infinite. A familiar example in a classical context is the Debye screening of an electric charge in a plasma.

As next step, we have to calculate (and to interpret properly)
\[
i\Delta F(0) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}.
\]

Since the mass correction is \(\delta m^2 = i\lambda \Delta F(0)/2\), the Feynman propagator at coincident points \(\Delta F(0)\) has to be purely imaginary. Otherwise the \(\lambda \phi^4\) theory would contain no stable particles.

**Wick rotation** The integrals appearing in loop graphs can be easier integrated, if one performs a Wick rotation from Minkowski to Euclidean space: Rotating the integration contour anti-clockwise to \(-i\infty : +i\infty\) avoids both poles in the complex \(k_0\) plane and is thus admissible. Introducing as new integration variable \(ik_4 = k_0\), it follows
\[
\int_{-\infty}^{\infty} dk_0 \frac{1}{k^2 - m^2 + i\varepsilon} = \int_{-\infty}^{i\infty} dk_0 \frac{1}{k^2 - m^2 + i\varepsilon} = i \int_{-\infty}^{\infty} dk_4 \frac{1}{k^2 - m^2 + i\varepsilon}.
\]

We next combine \(k_E = (k, k_4)\) into a new four-vector. Since
\[
k_E^2 = -(|k|^2 + k_4^2)
\]
we work now (apart from the overall sign) in an Euclidean space. In particular, the denominator never vanishes and we can omit the \(i\varepsilon\). Moreover, the integrand is now spherically symmetric. Thus we have
\[
i\Delta F(0) = \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2}.
\]

As required by our interpretation \(\delta m^2 = i\lambda \Delta F(0)\), the propagator \(\Delta F(0)\) is imaginary. Because the relative sign of the momenta and the mass term indicates, if we work in the Euclidean or Minkowski space, we will omit the index \(E\) in the following. Introducing furthermore spherical coordinates, we see that \(\Delta F(0)\) diverges quadratically for large \(k\),
\[
\lambda i \Delta F(0) \propto \int_0^\Lambda dk \frac{k^3}{k^2 + m^2} \propto \Lambda^2.
\]
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**Dimensional regularization** Using the integral representation

\[
\frac{1}{k^2 + m^2} = \int_0^\infty ds \ e^{-s(k^2 + m^2)}
\]  

(2.113)

and interchanging the integrals, we can reduce the momentum integral to a Gaussian integral. Manipulations like interchanging the order of integrations or a change of integration variables in divergent expressions as Eq. (2.112) are however ambiguous. Before we can proceed, we have to “regularize” therefore the integral, similar as we did introducing a cutoff function into the expression of the zero-point energy.

We will use dimensional regularization, i.e. we will calculate integrals for \(n = 4 - \varepsilon\) dimensions where they are finite. Then we find

\[
i\Delta_F(0) = \int_0^\infty ds \int \frac{d^n k}{(2\pi)^n} e^{-s(k^2 + m^2)} = \frac{1}{(4\pi)^{n/2}} \int_0^\infty ds \ s^{-n/2} e^{-sm^2}.
\]  

(2.114)

The substitution \(x = sm^2\) transforms the integral into one of the standard representations of the gamma function (see the appendix 2.A for some useful formula),

\[
i\Delta_F(0) = \frac{(m^2)^{\frac{n}{2}-1}}{(4\pi)^{n/2}} \Gamma \left(1 - \frac{n}{2}\right).
\]  

(2.115)

This expression diverges for \(n = 2, 4, 6, \ldots\), but is as announced finite for \(n = 4 - \varepsilon\) and small \(\varepsilon\). In the next step, we would like to expand the expression in a Laurent series, separating pole terms in \(\varepsilon\) and a finite reminder.

**Appearance of a dimensionfull scale** As the expression stands, we can not expand the prefactor of the Gamma function for small \(\varepsilon\), because it is dimensionfull. In order to make the factor \(m^{n-2}\) dimensionless, we should supply a new mass scale.

More physically, we can understand the need for an additional dimensionfull scale by the requirement that the interaction \(\int d^n x \mathcal{L}_I\) remains dimensionless if we deviate from \(n = 4\) dimensions. In order to obtain the correct dimension of the interaction term, keeping the dimensions of physical quantities fixed, the coefficient of the \(\phi^4\) term has to aquire the mass dimension \(-n + 4\). In order to achieve this, we introduce a mass scale \(\mu\) (with \(|\mu| = 1\)) as follows,

\[
S_I = \int d^4 x \mathcal{L}_I = -\int d^4 x \frac{\lambda}{4!} \phi^4 \rightarrow -\mu^{4-n} \int d^n x \frac{\lambda}{4!} \phi^4.
\]  

(2.116)

Adding the factor \(\mu^{4-n}\) to our previous result, we obtain

\[
\lambda \mu^{4-n} i\Delta_F(0) = \frac{i\lambda}{(4\pi)^2} m^2 \left(\frac{4\pi \mu^2}{m^2}\right)^{2-n/2} \Gamma(1 - n/2).
\]  

(2.117)

We expand the dimensionless last two factors in this expression around \(n = 4\) using \(\Lambda.45\) for the Gamma function,

\[
\Gamma(1 - n/2) = \Gamma(-1 + \varepsilon/2) = -\frac{2}{\varepsilon} - 1 + \gamma + O(\varepsilon)
\]  

(2.118)

and

\[
a^{-\varepsilon/2} = e^{-(\varepsilon/2) \ln a} = 1 - \frac{\varepsilon}{2} \ln a + O(\varepsilon^2).
\]  

(2.119)
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Note that we require the expansion of the prefactor up to $O(\varepsilon^2)$ because of the pole term in (2.118). Thus the mass correction is given by

$$
\lambda \mu^4 \propto \frac{1}{\varepsilon} - 1 + \gamma + O(\varepsilon) \left[ 1 + \frac{\varepsilon}{2} \ln \left( \frac{4\pi \mu^2}{m^2} \right) + O(\varepsilon) \right].
$$

This expansion allows us to separate the correction into a divergent term $\propto 1/\varepsilon$ and a finite term that depends on the scale $\mu$. We will pursue these topics later in more detail, for the moment we note simply that i) the divergence can be hidden via Eq. (2.107) in $m_{\text{phys}}^2$, and ii) the scale dependence $\mu$ will lead to the “running” of masses and coupling constants.

2.7.2. Vacuum energy density

We can generate out of the self-energy diagram new one-loop graphs by adding or subtracting two external lines. Subtracting two lines generates an one-loop graph without external lines \(^1\), the “zero-point” Green function $G(0)$ at order $\lambda^0$.

One way to calculate this quantity is to evaluate directly $Z_0[0]$ using

$$
det A = \exp \ln det A = \exp tr \ln A
$$

which gives

$$
Z_0[0] = -\frac{1}{2} \exp tr \ln(\Box - m^2).
$$

We postpone the question how such an expression can be evaluated and use instead another approach, recycling our result for the self-energy. Vacuum diagrams are generated by the functional $Z[J]$ setting $J = 0$,

$$
\langle 0 + |0- \rangle = Z[0] = \int D\phi \exp i \int_\Omega d^4 x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right).
$$

We saw in Eq. (2.62) that the zero-point energy is related to the propagator at coincident points. Since we suspect a connection between vacuum diagrams and the zero-point energy, we try to relate $Z[0]$ and $\Delta_F(0)$. Taking a derivative with respect to $m^2$ gives

$$
\frac{\partial}{\partial m^2} \langle 0 + |0- \rangle = -\frac{i}{2} \int d^4 x \langle 0 + |\phi(x)^2|0- \rangle = -\frac{i}{2} \int d^4 x i\Delta_F(0) \langle 0 + |0- \rangle.
$$

The additional factor $\langle 0 + |0-\rangle = N^{-1}$ on the RHS takes into account that we defined the Feynman propagator with respect to a normalised vacuum. Translation invariance implies that $\langle 0 + |0- \rangle$ does not depend on $x$. Thus we obtain

$$
\frac{\partial}{\partial m^2} \ln(0 + |0- \rangle = -\frac{i}{2} \int d^4 x i\Delta_F(0) = -\frac{i}{2} VT i\Delta_F(0)
$$

with $VT$ as the four-dimensional integration volume. Next we use our result (2.115) for the propagator, $i\Delta_F(0) = C(m^2)^{n/2-1}$, to perform the integration over $m^2$,

$$
\ln(0 + |0- \rangle = -i VT \left[ C \frac{m^n}{n} - \rho_0 \right],
$$

\(^1\) Although $G^{(0)}$ is often represented as a closed loop, it has also no internal line; this is in agreement with our general formula $l = n - V + 1$. 

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where we introduced the integration constant \( \rho_0 \). Exponentiating this expression,

\[
\langle 0 + |0-\rangle = \langle 0 + | \exp (-iHT) |0-\rangle = \exp (-i\rho VT)
\]

we see that we should associate \( \rho \) with the energy density of the vacuum (at order \( \mathcal{O}(\lambda^0) \)). Replacing \( C \), we find

\[
\rho = \frac{m^n}{(4\pi)^{n/2} n} \Gamma \left(1 - \frac{n}{2}\right) - \rho_0.
\]

The energy density given by Eq. (2.128) diverges for \( n = 2, 4, 6 \ldots \) as

\[
\Gamma \left(1 - \frac{n}{2}\right) \sim \frac{2}{n-4}.
\]

We can make \( \rho \) finite and equal to the observed value \( \rho_{\Lambda} \), if we choose \( \rho_0 \) as

\[
\rho_0 = \mu^{n-4} \left[ \frac{1}{2} \frac{m^4}{(4\pi)^{n/2}} \frac{1}{n-4} - \rho_{\Lambda} \right].
\]

The prefactor \( \mu^{n-4} \) ensures that the action remains dimensionless also for \( n \neq 4 \) (keeping the dimension of physical quantities constant).

Note that this implies that we should start off with \( \mathcal{L} - \rho_0 \) instead of \( \mathcal{L} \). Even if we dismiss a cosmological constant in the classical Lagrangian, it will appear automatically by quantum corrections. More generally, every possible term that is not forbidden by a symmetry in \( \mathcal{L} \) will show up calculating loop corrections.

We have seen that we can absorb the vacuum energy density \( \rho \) into the normalisation of the path integral,

\[
\int D\phi e^{-\int d^4x \rho} = N \int D\phi \rightarrow \int \tilde{D}\phi.
\]

Therefore, one may wonder if \( \rho \) has a real physical meaning. The crucial difference between the vacuum energy and a simple normalisation constant is that \( \rho \) depends on the parameters (masses, coupling constants) of the considered theory. Since we should define the path integral independent of the Langrangian we integrate, we cannot eliminate the contribution of the vacuum diagrams to the cosmological constant in the gravitational field equations by a simple redefinition of the integration measure.

**Remark:** Equivalence to the zero-point energy:

Performing the \( k^0 \) integral in Eq. (2.108) or using (2.33)

\[
i \Delta_F(0) = \frac{\int d^3k}{(2\pi)^3 i \omega_k} = \frac{\int d^3k}{(2\pi)^3 2 \sqrt{m^2 + k^2}}
\]

and integrating then with respect to \( m^2 \),

\[
i \int dm^2 \Delta_F(0) = \int \frac{d^3k}{(2\pi)^3} \sqrt{m^2 + k^2},
\]

shows that the present expression for the vacuum energy agrees with the one in Eq. (2.63). However, the resulting expression for \( \rho \) differ: While Eq. (2.64) shows that \( \rho \propto \Lambda^4 \) we found \( \rho \propto m^4 \) in the case of dimensional regularization. Thus in this scheme a massless particle as the photon would give a zero contribution to the cosmological constant.

This disturbing difference has to be caused by the regularization scheme. We will discuss later which prediction is more trustable.
2.7. Loop diagrams

2.7.3. Vertex correction

For our last example we add two external lines to the self-energy diagram. This generates

\[ \text{diagrams which describe} \ 2 \rightarrow 2 \text{ scattering at } O(\lambda^2). \]

At tree-level, the same process was given by the four-point function (2.100). The corresponding Feynman amplitude in momentum space is simply \( iA = i\lambda \). Thus we suspect that the loop process leads to the renormalisation of the coupling constant \( \lambda \).

Determining the Feynman amplitude

Instead of calculating the order \( \lambda^2 \) term in the perturbative expansion of the generating functional \( Z[J] \) we use directly the Feynman rules to obtain the Feynman amplitude for this process. According to these rules, the first steps in the calculation of the are to draw all Feynman diagrams, to find the symmetry factor and to associate then the right mathematical expressions to the symbols.

In coordinate space, we have to connect four external points (say \( x_1, \ldots, x_4 \)) with the help of two vertices (say at \( x \) and \( y \)) which combine each four lines. An example is shown here

\[
\begin{array}{c}
\text{(a)} \\
x_2 \quad y \quad x_4 \\
x_1 \quad \text{(b)} \quad x_3
\end{array}
\]

Two other diagrams are obtained connecting \( x_1 \) with \( x_2 \) or \( x_4 \). In order to determine the symmetry factor, we consider the expression for the four-point function corresponding to the graph shown above,

\[
\frac{1}{2!} \left( -i \frac{\lambda}{4!} \right)^2 \int d^4x d^4y (0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(x)\phi^4(y)}|0) + (x \leftrightarrow y),
\]

and count the number of possible contractions: We can connect \( \phi(x_1) \) with each one of the four \( \phi(x) \), and then \( \phi(x_3) \) with one of the three remaining \( \phi(x) \). This gives \( 4 \times 3 \) possibilities. Another \( 4 \times 3 \) possibilities come by the same reasoning from the upper part of the graph. The remaining pairs \( \phi^2(x) \) and \( \phi^2(y) \) can be combined in two possibilities. Finally, the factor \( 1/2! \) from the Taylor expansion is cancelled by the exchange graph. Thus the symmetry factor is

\[
S = \frac{1}{2!} \left( \frac{4 \times 3}{4!} \right)^2 2 = \frac{1}{2}.
\]

Next we associate the right mathematical expressions to the symbols of the graphs: We replace internal propagator by \( i\Delta(k) \), external lines by 1 and vertices by \( -i\lambda \). Imposing four-momentum conservation at the two vertices leaves one free loop momentum, which we call \( p \). The momentum of the other propagator is then fixed to \( p - q \), where \( q^2 = s = (p_1 + p_2)^2 \), \( q^2 = t = (p_1 - p_3)^2 \), and \( q^2 = u = (p_1 - p_4)^2 \) for the three graphs shown in Fig. [2.3]. Thus the Feynman amplitudes in \( n = 4 \) are at order \( O(\lambda^2) \)

\[
iA_i^{(2)} = \frac{1}{2} \lambda^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{|p^2 - m^2 + i\varepsilon|} \frac{1}{|(p - q)^2 - m^2 + i\varepsilon|}.
\]

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The squared cms energy $s$ and the two variables describing the momentum transfer $t$ and $u$ are called Mandelstam variables. For $2 \to 2$ scattering, they are connected by $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$, see problem [2.11]. According to the value of $q^2$ one calls the diagrams the $s$, $t$ and $u$ channel.

Performing again a naive power-counting analysis, we find that the amplitude is logarithmically divergent,

$$A_i^{(2)} \propto \int \frac{d^4 p}{p^4} \propto \ln(\Lambda).$$

(2.136)

If we consider the infinite number of one-loop graphs characterised by $n = V \geq 0$, then we see that adding two external lines increases the number of propagators in the loop by one. As a result, the convergence of the loop integral improves from a quartic divergence (vacuum energy), over a quadratic divergence (self-energy energy) to a logarithmic divergence for the vertex correction. Adding two or more external lines to the vertex correction would therefore produce a finite diagram. At one-loop level, the $\lambda \phi^4$ theory contains thus only a finite number of divergent Feynman graphs. Going beyond one-loop, this interesting behavior persists and as a result a renormalisation of the three parameters contained in the classical Lagrangian, $m^2$, $\lambda$ and $\rho$, is sufficient to eliminate all divergencies.

**Calculating the loop integral** The path to be followed in the evaluation of simple loop integrals as (2.135) can be sketched schematically as follows: Regularise the integral (and add a mass scale if you use DR). Combine then the denominators, and shift the integration variable to eliminate linear terms in the denominator by completing the square. Performing the same shift of variables in the nominator, the linear terms can be dropped as the vanish after integration. Finally, Wick rotate the integrand, and reduce the integral to a known one by a suitable variable substitution. We do the last steps once in the appendix [2.A] where we derive a list of useful Feynman integrals which we simply look up in the future.

We start by rewriting the integral for $n = 4 - \varepsilon$ dimensions as

$$i A_i^{(2)} = \frac{1}{2} \lambda^2 (\mu^2)^{4-n} \int \frac{d^n p}{(2\pi)^n} \frac{1}{D},$$

(2.137)

where we introduced also the short-cut $D$ for the denominator in the integrand. Next we use

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1 - z)]^2}$$

(2.138)
to combine the two denominators, setting \( a = p^2 - m^2 \) and \( b = (p - q)^2 - m^2 \),
\[
D = az + b(1 - z) = p^2 - m^2 - 2pq(1 - z) + q^2(1 - z).
\]
Then we eliminate the term linear in \( p \) substituting \( p^2 = [p - q(1 - z)]^2 \),
\[
D = p^2 - m^2 + q^2z(1 - z).
\]
Since \( d^n p = d^n p' \), we can drop the primes and find
\[
iA^{(2)}_s = \frac{1}{2} \lambda^2(\mu^2)^{4 - n} \int_0^1 dz \int d^n p \frac{1}{(2\pi)^n} \frac{1}{[p^2 - m^2 + q^2z(1 - z)]^2}.
\]
Performing a Wick rotation requires that \( q^2z(1 - z) < m^2 \) for all \( z \in [0 : 1] \), or \( q^2 < 4m^2 \).
The integral is of the type \( I(\omega, 2) \) calculated in the appendix and equals
\[
I(\omega, 2) = i - \frac{1}{(4\pi)^n} \frac{\Gamma(2 - \omega)}{\Gamma(2)} \frac{1}{[m^2 - q^2z(1 - z)]^{2 - \omega}}.
\]
Inserting the result into the Feynman amplitude gives
\[
A^{(2)}_s = \frac{1}{2} \lambda^2(\mu^2)^{4 - n} \frac{\Gamma(2 - n/2)}{(4\pi)^{n/2}} \int_0^1 dz [m^2 - q^2z(1 - z)]^{n/2 - 2}
\]
\[
= \frac{\lambda^2}{32\pi^2}(\mu^2)^{2 - n/2} \frac{\Gamma(2 - n/2)}{(4\pi)^{n/2}} \int_0^1 dz \left[ \frac{m^2 - q^2z(1 - z)}{4\pi\mu^2} \right]^{n/2 - 2}.
\]
In the last step, we made the function \( f \) dimensionless. Now we take the limit \( \varepsilon = 4 - n \to 0 \),
expanding both the Gamma function
\[
\Gamma(2 - n/2) = \Gamma(\varepsilon/2) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)
\]
and
\[
f^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2} \ln f + \mathcal{O}(\varepsilon^2).
\]
From
\[
\frac{\lambda^2}{32\pi^2}(\mu^2)^{\varepsilon} \left( \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \right) \left[ 1 - \frac{\varepsilon}{2} \int_0^1 dz \ln f + \mathcal{O}(\varepsilon^2) \right]
\]
we see that all diagrams give the same divergent part, while we have to replace \( q^2 \) by the value \( \{s, t, u\} \) appropriate for the three diagrams,
\[
A = A^{(1)} + A^{(2)}_s + A^{(2)}_t + A^{(2)}_u + \mathcal{O}(\lambda^3)
\]
\[
= -\lambda\mu^{\varepsilon/2} + \frac{3\lambda^2\mu^\varepsilon}{16\pi^2\varepsilon} - \frac{\lambda^2\mu^\varepsilon}{32\pi^2} \left[ 3\gamma + F(s, m, \mu) + F(t, m, \mu) + F(u, m, \mu) \right].
\]
with
\[
F(q^2, m, \mu) = \int_0^1 dz \ln \left[ \frac{m^2 - q^2z(1 - z)}{4\pi\mu^2} \right].
\]
Note that \( t \) and \( u \) are in the physical region negative and thus the condition \( q^2 < 4m^2 \) is always satisfied for these two diagrams. By contrast, for the \( s \) channel diagram the relation \( q^2 = s > 4m^2 \) holds: In this case, we have to continue analytically the result \( \{s, t, u\} \) into the physical region. We will postpone this task to chapter \( \{s, t, u\} \) and note for the moment only that thereby the argument of the logarithm in \( \{s, t, u\} \) changes sign. Additionally, an imaginary part of the scattering amplitude is generated.
2.7.4. Basic idea of renormalisation

The regularisation of loop integrals has introduced as a new unphysical parameter the renormalisation scale $\mu$. As we perform perturbation theory at order $\lambda^n$, we have to connect the parameters $\{m, \lambda, \rho\}$ of the truncated theory with the physical ones of the full theory. This process is called renormalisation and will eliminate the unphysical parameter $\mu$.

Renormalisation of the coupling  We try to connect the amplitude $iA$ to a physical measurement. We assume that experimentalists measured $\phi \phi \rightarrow \phi \phi$ scattering. It is sufficient that they provide us with a single value, e.g. with the value of the differential cross section $d\sigma/d\Omega$ at zero-momentum transfer close to threshold $s = 4m^2$. Then

$$F(4m^2, m, \mu) = C\lambda^2 \ln \left[ \frac{4m^2}{\mu^2} \right] + \text{const},$$

while for $s \gg m^2$ we find

$$F(s, m, \mu) = C\lambda^2 \ln \left[ \frac{s}{\mu^2} \right] + \text{const}.$$  

Subtracting the infinite parts (and the constant term), we obtain for $s, t, u \gg m^2$,

$$A = -\lambda - C\lambda^2 \left[ \ln \left( \frac{s}{\mu^2} \right) + \ln \left( \frac{t}{\mu^2} \right) + \ln \left( \frac{u}{\mu^2} \right) \right] + O(\lambda^3) \equiv -\lambda - C\lambda^2 L(s/\mu^2),$$

where we introduced also the sloppy notation $L$ for the three log terms in the square bracket. This expression for $A$ is finite but still arbitrary since it contains $\mu$.

We use now the experimental measurement at the scale $s = 4m^2$ to connect via

$$A = -\lambda - C\lambda^2 L(4m^2/\mu^2)$$

the measured value $\lambda_0$ of the coupling to our calculation,

$$-\lambda_0 = -\lambda - C\lambda^2 L(4m^2/\mu^2) + O(\lambda^3).$$

Now we solve for $\lambda$,

$$-\lambda = -\lambda_0 + C\lambda^2 L(4m^2/\mu^2) + O(\lambda^3)$$

$$= -\lambda_0 + C\lambda_0^2 L(4m^2/\mu^2) + O(\lambda_0^3).$$

In the second line, we could replace $\lambda^2$ by $\lambda_0^2$, because their difference is of $O(\lambda^3)$. Next we insert $\lambda$ back into the matrix element $A$ for general $s$ and replace then again $\lambda^2$ by $\lambda_0^2$,

$$A = -\lambda - C\lambda^2 L(s/\mu^2) + O(\lambda^3)$$

$$= -\lambda_0 + C\lambda_0^2 L(4m^2/\mu^2) - C\lambda_0^2 L(s/\mu^2) + O(\lambda_0^3)$$

$$= -\lambda_0 - C\lambda_0^2 L(s/4m^2) + O(\lambda_0^3).$$

Combining the log’s, the scale $\mu$ has canceled and we find

$$A = -\lambda_0 - \frac{\lambda_0^2}{32\pi^2} \left[ \ln \left( \frac{s}{4m^2} \right) + \ln \left( \frac{t}{4m^2} \right) + \ln \left( \frac{u}{4m^2} \right) \right] + O(\lambda_0^3).$$

Thus the matrix element is finite and depends only on the measured value $\lambda_0$ of the coupling constant and the kinematical variables $s$ and $t$. 

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Running coupling  We look now from a somewhat different point of view at the problem of the apparent $\mu$ dependence of physical observables. Assume that we have subtracted the infinite parts (and the constant term) of the amplitude $A$, obtaining Eq. (2.151). We now demand that the scattering amplitude $A$ as a physical observable is independent of the arbitrary scale $\mu$, $dA/d\mu = 0$. If we did a perturbative calculation up to $O(\lambda^n)$, the condition $dA/d\mu = 0$ can hold only up to terms $O(\lambda^{n+1})$. The explicit $\mu$ dependence of the amplitude, $\partial A/\partial \mu \neq 0$, can be only cancelled by a corresponding change of the parameters $m$ and $\lambda$ contained in the classical Lagrangian, converting them into “running” parameters $m(\mu)$ and $\lambda(\mu)$. Then the condition $dA/d\mu = 0$ becomes

$$\left( \frac{\partial}{\partial \mu} + \frac{\partial}{\partial \mu} \frac{m^2}{\partial m^2} + \frac{\partial}{\partial \mu} \frac{\partial}{\partial \lambda} \right) A(s,t,m(\mu),\lambda(\mu),\mu) = 0. \quad (2.157)$$

The only explicit $\mu$ dependence of $A$ is contained in the $F(q^2,m,\mu)$ functions, giving in the limit $\varepsilon \to 0$

$$\frac{\partial}{\partial \mu} A(s,t,m(\mu),\lambda(\mu),\mu) = -3 \frac{\lambda^2}{32\pi^2} \frac{\partial}{\partial \mu} F(q^2,m,\mu) = \frac{3\lambda^2}{16\pi^2\mu}. \quad (2.158)$$

Since the change of $m(\mu)$ and $\lambda(\mu)$ is given by loop diagrams, it includes at least an additional factor $\lambda$. Therefore the action of the derivatives $\partial m^2$ and $\partial \lambda$ on the 1-loop contribution $A^{(2)}$ leads to a term of $O(\lambda^3)$ which can be neglected. Thus the only remaining term will be given by $\partial \mu$ acting on the tree-level term $A^{(1)}$. Note that this holds also at higher orders and ensures that $\partial \mu \lambda$ at $O(\lambda^{n+1})$ is determined by the parameters calculated at $O(\lambda^n)$. Combining the two contributions we find

$$\mu \frac{\partial \lambda}{\partial \mu} = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3). \quad (2.159)$$

Thus the scattering amplitude $A$ is independent of the scale $\mu$, if we transform the coupling constant $\lambda$ into a scale dependent “running” coupling $\lambda(\mu)$ whose evolution is given by Eq. (2.159). If we truncate the perturbation series at a finite order, the cancellation of the scale dependence is however incomplete.

Separating variables in Eq. (2.159), we find

$$\lambda(\mu) = \frac{\lambda_0}{1 - 3\lambda_0/16\pi^2 \ln(\mu/\mu_0)} \quad (2.160)$$

with $\lambda_0 \equiv \lambda(\mu_0)$ as initial condition. Thus the running coupling $\lambda(\mu)$ increases logarithmically for increasing $\mu$.

Comparing (8.86) to our result for the scattering amplitude (2.156),

$$A = -\lambda_0 \left( 1 + \frac{\lambda_0}{32\pi^2} \left[ \ln(s/4m^2) + \ln(t/4m^2) + \ln(u/4m^2) \right] \right) + O(\lambda_0^3), \quad (2.161)$$

we see that we can rewrite the amplitude using a symmetric point $q^2 = s = t = u$ and $\mu_0^2 = 4m^2$ as

$$A = -\lambda_0 \left( 1 + \frac{3\lambda_0}{32\pi^2} \ln(q^2/\mu_0^2) \right) = -\lambda(\mu). \quad (2.162)$$

This shows that the $q^2$ dependence of the amplitude $A$ in the limit $q^2 \gg m^2$ is determined completely by the $\mu^2$ dependence of the running coupling $\lambda(\mu)$.
2.A. Appendix: Evaluation of Feynman integrals

Combination of propagators The standard strategy in the evolution of loop integrals is the combination of the \( n \) propagator denominators into a single propagator-like denominator of higher power. One uses either Schwinger’s proper-time representation

\[
\frac{i}{p^2 - m^2 + i\varepsilon} = \int_0^\infty ds \ e^{-is(p^2 - m^2 + i\varepsilon)}
\]  

(2.163)

or the Feynman parameter integral

\[
\frac{1}{x_1 \cdots x_n} = \Gamma(n) \int_0^1 dx_1 \cdots \int_0^1 dx_n \ \delta(1 - \sum_i x_i) \ [x_1 \cdots x_n]^{-n}
\]

\[
= \int_{\text{triangle}} dx_1 \cdots dx_n \ [x_1 \cdots x_n]^{-n} .
\]  

(2.164)

In order to derive this formula for \( n = 2 \), consider

\[
\frac{1}{b - a} \int_a^b \frac{dx}{x^2} = \frac{1}{b - a} \left( \frac{1}{a} - \frac{1}{b} \right) = \frac{1}{ab}
\]  

(2.165)

for \( a, b \in \mathbb{C} \). Setting \( x = az + b(1 - z) \) and changing the integration variable, we obtain

\[
\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1 - z)]^2} .
\]  

(2.166)

The cases \( n > 2 \) can be derived by induction.

Evaluation of Feynman integrals We want to calculate integrals of the type

\[
I(\omega, \alpha) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{|k^2 - M^2 + i\varepsilon|^\alpha}
\]  

(2.167)

defined in Minkowski space. Performing a Wick rotation to Euclidean space and introducing spherical coordinates results in

\[
I(\omega, \alpha) = i(-1)^\alpha \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{|k^2 + M^2 + i\varepsilon|^\alpha} = i(-1)^\alpha \frac{1}{(2\pi)^{2\omega}} \Omega_{2\omega} \int_0^\infty dk \frac{k^{2\omega - 1}}{|k^2 + M^2|^\alpha},
\]  

(2.168)

where we denoted the volume \( \text{vol}(S^{2\omega - 1}) \) of a unit sphere in \( 2\omega \) dimensions\(^2\) by \( \Omega_{2\omega} \). You are asked in problem 2.10 to show that \( \Omega_{2\omega} = 2\pi^{\omega}/\Gamma(\omega) \) and thus \( \Omega_4 = 2\pi^2 \). Substituting \( k = m\sqrt{x} \) and using the integral representation \( \Gamma(x) \) for Euler’s beta function allows us to express the \( k \) integral as a product of Gamma functions,

\[
\int_0^\infty dk \frac{k^{2\omega - 1}}{|k^2 + M^2|^\alpha} = \frac{1}{2} M^{2\omega - 2\alpha} \int_0^\infty dx \frac{x^{\omega - 1}}{[1 + x]^\alpha} = \frac{1}{2} M^{2\omega - 2\alpha} B(\omega, \alpha - \omega) = \frac{1}{2} M^{2\omega - 2\alpha} \frac{\Gamma(\omega)\Gamma(\alpha - \omega)}{\Gamma(\alpha)} .
\]  

(2.169)

\(^2\)A \( n - 1 \)-dimensional sphere \( S^{n-1}(R) \) encloses the \( n \)-dimensional volume \( x_1^2 + \ldots + x_n^2 \leq R^2 \), while its own \( n - 1 \)-dimensional volume is given by \( x_1^2 + \ldots + x_n^2 = R^2 \). The volume of a unit 1-sphere is a length, \( \text{vol}(S^0) = 2 \), of a unit 2-sphere an area, \( \text{vol}(S^2) = 4\pi \), and of a unit 3-sphere a volume, \( \text{vol}(S^3) = 2\pi^2 \). If we say that the volume of a sphere is \( 4\pi R^2 / 3 \), we mean in fact the volume of the 3-ball \( B^3(R) \), \( x_1^2 + x_2^2 + x_3^2 \leq R^2 \) which is enclosed by the 2-sphere \( S^2(R) \).
Combining this result with \( \Omega_{2\omega} = 2\pi^\omega / \Gamma(\omega) \) we obtain

\[
I(\omega, \alpha) = \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\omega^2 - M^2 + i\epsilon} = i \frac{(-1)^\alpha}{(4\pi)^\omega} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} \frac{1}{[M^2 + i\epsilon]^\omega - \alpha} \cdot \tag{2.170}
\]

We can generate additional formulas by adding first a dependence on an external momentum \( p^\mu \), shifting the integration variable \( k \rightarrow k + p \),

\[
\int \frac{d^2 \omega k}{(2\pi)^2} \frac{1}{\omega^2 + 2pk - M^2 + i\epsilon} = i \frac{(-1)^\alpha}{(4\pi)^\omega} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} \frac{1}{[M^2 + p^2 + i\epsilon]^\omega - \alpha} \cdot \tag{2.171}
\]

Taking then derivatives with respect to the external momentum \( p^\mu \) results in

\[
I_\mu(\omega, \alpha) = \int \frac{d^2 \omega k}{(2\pi)^2} \frac{k_\mu}{\omega^2 - M^2 + i\epsilon} = -p_\mu \omega(\omega, \alpha) \quad \tag{2.172}
\]

and

\[
I_{\mu\nu}(\omega, \alpha) = \int \frac{d^2 \omega k}{(2\pi)^2} \frac{k_\mu k_\nu}{\omega^2 - M^2 + i\epsilon} = \tag{2.173}
\]

\[
= i \frac{(-\omega)}{(2\pi)^2} \frac{\omega}{\epsilon} \frac{\omega(1 - \omega)}{\Gamma(\alpha)} \frac{(M^2 - p^2)}{[M^2 - p^2 + i\epsilon]^\alpha - \omega} \cdot \tag{2.174}
\]

Contracting both sides with \( k_\mu k_\nu \) and using \( \eta^{\mu\nu} \eta_{\mu\nu} = 2\omega \) gives

\[
I_2(\omega, \alpha) = \int \frac{d^2 \omega k}{(2\pi)^2} \frac{k^2}{\omega^2 - M^2 + i\epsilon} = i \frac{(-\omega)}{(2\pi)^2} \frac{\omega}{\epsilon} \frac{\omega(1 - \omega)}{\Gamma(\alpha)} \frac{(M^2 - p^2 - \omega M^2)}{[M^2 - p^2 + i\epsilon]^\alpha - \omega} \cdot \tag{2.175}
\]

Special cases often needed are

\[
I(\omega, 2) = \frac{i}{(4\pi)^\omega} \frac{\Gamma(2 - \omega)}{[M^2 - p^2 + i\epsilon]^\omega - 2}, \quad \tag{2.176}
\]

\[
I_2(\omega, 2) = -\frac{i}{(4\pi)^\omega} \omega \Gamma(1 - \omega) \frac{\omega(1 - \omega)}{[M^2 - p^2 + i\epsilon]^\omega - 1}, \quad \tag{2.177}
\]

and

\[
I(2, 3) = \frac{i}{32\pi^2} \frac{1}{M^2 - p_1^2 + i\epsilon}. \quad \tag{2.178}
\]

**Summary**

The Feynman propagator obtained by the \( m^2 - i\epsilon \) prescription is the unique Green function which can be analytically continued to an Euclidean Green function. It propagates particles (with positive frequencies) forward in time, while anti-particles (with negative frequencies) propagate backward in time. The exchange of time-like quanta with zero energy between two static sources leads to the Yukawa potential. The corresponding force mediated by a scalar field is attractive. The three loop diagrams we calculated in the \( \lambda \phi^4 \) theory were infinite and had to be regularised. Renormalising the three parameters contained in the classical Lagrangian of the \( \lambda \phi^4 \) theory, \( \rho_0, m^2 \), and \( \lambda \), eliminated the divergencies and converted them into “running” quantities.
2. Scalar fields

Further reading

The quantisation of fields using both canonical quantisation and the path integral approach is discussed extensively in [GR96]. For a treatment of the $\lambda\phi^4$ theory in $n = 6$ dimensions—a theory which resembles a bit more QED than the $\lambda\phi^4$ theory we discussed—see [Sre07].

Problems

2.1 Complex scalar field.

Derive the Lagrangian and the Hamiltonian for a complex scalar field, considering $f = (\phi_1 + i\phi_2)/\sqrt{2}$ and $\phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$ as the dynamical variables. Find then the conserved current of this complex field, proceeding similar as in the case of the Schrödinger equation.

2.2 Maxwell Lagrangian.

Derive the Lagrangian for the photon field $A_\mu$ from the source-free Maxwell equation $\partial_\nu F^{\nu\mu} = 0$ following the steps from (2.13) to (2.14) in the scalar case. What is the meaning of the unused set of Maxwell equations?

2.3 Lorentz invariant integration measure.

Argue that $d^3k/(2\omega_k)$ is a Lorentz invariant integration measure by showing that

$$\int d^4k \delta(k^2 - m^2)\vartheta(k^0)f(k^0, k) = \int \frac{d^3k}{2\omega_k} f(\omega_k, k)$$

for any function $f$.

2.4 Yukawa potential.

Show that the Yukawa potential $V(r) = e^{-mr}/(4\pi r)$ is the Fourier transform of $(k^2 + m^2)^{-1}$, cf. Eq. (2.32).

2.5 Vacuum energy.

Rederive (2.63) expressing $\phi$ by annihilation and creation operators. Show that rewriting all creation operators on the left of the annihilation operators results in $\rho_1 = 0$. (This prescription is called “normal ordering”.)

2.6 $\zeta$ function regularisation.

a.) The function $f(t) = t/(e^t - 1)$ is the generating function for the Bernoulli numbers $B_n$, i.e.

$$f(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$  

Calculate the first Bernoulli numbers up to $B_2$.

b.) The Riemann $\zeta$ function can be defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s > 1$ and then analytically continued into the complex $s$ plane. The Bernoulli numbers are connected to the Riemann $\zeta$ function with negative odd argument as $\zeta(1-2n) = -\frac{B_{2n}}{2n}$.

Show using

$$\frac{1}{a} = \frac{1}{a e^a - 1} = \frac{1}{a} \sum_{n=0}^{\infty} \frac{B_n a^n}{n!}$$

that you can split the sum into a divergent and a finite part. The divergent term will cancel in the difference of the vacuum energy with and without plates, and the remaining finite term is determined by $B_2/2$.

2.7 Casimir effect.

Repeat the calculation of the Casimir effect for a scalar field in 1+1 dimensions using dimensional regularization.

2.8 Feynman propagator.

Discuss the behavior of the scalar Green function $\Delta_F(0, r)$ for large $r = |\mathbf{x}|$.

2.9 Green functions.

Show that the connected and the unconnected $n$-point Green functions are identical for $n = 2$, while they differ in general for $n \geq 3$.

2.10 Volume of $n$ dimensional sphere.

a.) Calculate the volume of the unit sphere $S^{n-1}$ defined by $x_1^2 + \ldots + x_n^2 = 1$ in $\mathbb{R}^n$.

b.) Generalize the result to arbitrary (not necessarily integer) dimensions and show that it agrees with the familiar results for $n = 1, 2$ and 3.

2.11 Mandelstam variables.

Show that for $2 \to 2$ scattering $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$ is valid. Express $t$ as function of the scattering angle between $p_1$ and $p_3$, derive
the lower and upper limits of $t$ and the relation between $d\sigma/d\Omega$ and $d\sigma/dt$.

**2.12 Renormalisation invariance of the propagator.**

Derive analogously to Eq. (2.157ff) an equation $dm^2(\mu)/d\mu = f(m^2)$ requiring that the propagator (2.106) is independent of the scale $\mu$.

**2.13 Renormalisation with an Euclidean momentum cutoff.**

Calculate the self-energy (mass correction) and vertex correction using an Euclidean momentum cutoff $\Lambda$. Derive the RGE equations $dm^2(\Lambda)/d\Lambda = f(m^2)$, $d\lambda(\Lambda)/d\Lambda = f(\lambda)$ and $d\rho(\Lambda)/d\Lambda = f(\rho)$ and compare them to the ones derived using DR.
3. Global symmetries and Noether’s theorem

Emmy Noether showed 1917 that any global continuous symmetry of a classical system described by a Lagrangian $L$ leads to a locally conserved current. We can divide such symmetries into two classes: Symmetries of the space-time manifold and internal symmetries of a group of fields. Prominent examples are the Poincaré symmetry of Minkowski space and global symmetries that lead to the conservation laws of electric charge or baryon number, respectively.

A locally conserved vector current leads classically always to global charge conservation, while this does not hold for tensors of higher rank: In the latter case, global charge conservation requires additionally the existence of a Killing vector field which in turn indicates a corresponding symmetry of space-time. The non-existence of such a field in a general space-time is the reason why e.g. the energy of a photon is not conserved in an expanding Universe.

3.1. Internal symmetries

We have used up to now mainly space-time symmetry of Minkowski space, namely the requirement of Lorentz invariance, to deduce possible terms in the action. If we allow for more than one field, e.g. several scalar fields, the new possibility of internal symmetries arise. For instance, we can look at a theory of two massive scalar fields,

$$L = \frac{1}{2} \left( \partial_\mu \phi_1 \right)^2 - \frac{1}{2} m_1^2 \phi_1^2 - \frac{1}{4} \lambda_1 \phi_1^4 + \frac{1}{2} \left( \partial_\mu \phi_2 \right)^2 - \frac{1}{2} m_2^2 \phi_2^2 - \frac{1}{4} \lambda_2 \phi_2^4 - \frac{1}{2} \lambda_3 \phi_1 \phi_2^2. \quad (3.1)$$

As it stands, the theory contains five arbitrary parameters, two masses $m_i$ and three coupling constants $\lambda_i$. For arbitrary values of these parameters, no new additional symmetry results. In nature, we find however often a set of particles with nearly the same mass and (partly) the same couplings. One of the first examples was suggested by Heisenberg after the discovery of the neutron, which has a mass very close to the one of the proton, $m_n \sim m_p$. With respect to strong interactions, it is useful to view the proton and neutron as two different “isospin” states of the nucleon, similar as an electron has two spin states. An example of an exact symmetry are particles and their anti-particles, as e.g. the charged pions $\pi^\pm$ which can be combined into one complex scalar field.

If we set in our case $m_1 = m_2$ and $\lambda_2 = \lambda_1$, the Lagrangian becomes invariant under the exchange $\phi_1 \leftrightarrow \phi_2$. Adding the further condition that $\lambda_3 = \lambda_1 = \lambda_2$, we arrive at

$$L = \frac{1}{2} \left[ \left( \partial_\mu \phi_1 \right)^2 + \left( \partial_\mu \phi_2 \right)^2 \right] - \frac{1}{2} m^2 \left( \phi_1^2 + \phi_2^2 \right) - \frac{\lambda}{4} \left( \phi_1^2 + \phi_2^2 \right)^2. \quad (3.2)$$

Now any orthogonal transformation $O \in O(2)$ in the two-dimensional field space $\{\phi_1, \phi_2\}$ leads to the same Langrangian $L$. In particular, the Lagrangian is invariant under a rotation $R(\alpha) \in SO(2)$ which mixes $\{\phi_1, \phi_2\}$ as

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (3.3)$$
The fields transform as a vector $\phi = \{\phi_1, \phi_2\}$ and a rotation leaves the length of the vector $\phi$ invariant. Generalising this to $n$ scalar fields, we can write down immediately a theory that is invariant under transformations $\phi_a \rightarrow R_{ab}\phi_b$ where $R$ is an element of $SO(n)$,

$$L = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} (\phi^2)^2. \quad (3.4)$$

Note that the Lagrangian is only invariant under global, i.e., space-time independent rotations.

An interaction vertex connects two particles of type $a$ and two particles of type $b$, so it is of the type $-i\lambda(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$. Since all particles have the same mass, all $R_{ab}\phi_b$ are eigenstates of the free Hamiltonian. Thus the propagator $D_{ab}(x - x')$ is diagonal, $D_{ab}(x - x') \propto \delta_{ab}$.

### 3.2. Noether’s theorem

From our experience in classical and quantum mechanics, we expect that global continuous symmetries lead also in field theory to conservation laws for the generators of the symmetry. In order to derive such a conservation law, we assume that our collection of fields $\phi_a$ has a continuous symmetry group. Thus we can consider an infinitesimal change $\delta \phi_a$ of the fields that keeps $L(\phi_a, \partial_\mu \phi_a)$ invariant,

$$0 = \delta L = \frac{\delta L}{\delta \phi_a} \delta \phi_a + \frac{\delta L}{\delta \partial_\mu \phi_a} \delta \partial_\mu \phi_a. \quad (3.5)$$

Now we exchange $\delta \partial_\mu = \partial_\mu \delta$ in the second term and use then the Lagrange equations, $\delta L / \delta \phi_a = \partial_\mu (\delta L / \delta \partial_\mu \phi_a)$, in the first one. Then we can combine the two terms using the product rule,

$$0 = \delta L = \partial_\mu \left( \frac{\delta L}{\delta \partial_\mu \phi_a} \right) \delta \phi_a + \frac{\delta L}{\delta \partial_\mu \phi_a} \partial_\mu \delta \phi_a = \partial_\mu \left( \frac{\delta L}{\delta \partial_\mu \phi_a} \delta \phi_a \right). \quad (3.6)$$

Hence the invariance of $L$ under the change $\delta \phi_a$ implies the existence of a conserved current, $\partial_\mu j^\mu = 0$, with

$$j^\mu = \frac{\delta L}{\delta \partial_\mu \phi_a} \delta \phi_a. \quad (3.7)$$

If the transformation $\delta \phi_a$ leads to change in $L$ that is a total four-divergence, $\delta L = \partial_\mu K^\mu$, and boundary terms can be dropped, then the equations of motion remain invariant. The conserved current $j^\mu$ is changed to

$$j^\mu = \frac{\delta L}{\delta \partial_\mu \phi_a} \delta \phi_a - K^\mu. \quad (3.8)$$

In Minkowski space, we can convert this differential form of a conservation law into a global one, obtaining a globally conserved charge

$$Q = \int_V d^3 x \, j^0. \quad (3.9)$$

Often (but not always) this charge has a profound physical meaning.
Internal symmetries As an example, we can use our $n$ scalar fields invariant under the group $\text{SO}(n)$. We need the infinitesimal generators $T_i$ of rotations,

$$\phi'_a = R_{ab} \phi_b \approx (1 + \alpha_i T_i + \mathcal{O}(\alpha_i^2))_{ab} \phi_b. \quad (3.10)$$

$\text{SO}(n)$ has an anti-symmetric Lie algebra with $n(n-1)/2$ generators. Thus a theory invariant under $\text{SO}(n)$ has $n(n-1)/2$ conserved currents. The special case $n = 2$ has as an important application the $\pi^\pm$ system. We combine the two real fields $\phi_1$ and $\phi_2$ into the complex field

$$\phi = (\phi_1 + i\phi_2)/\sqrt{2},$$

then the Lagrangian becomes

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2. \quad (3.11)$$

Under the combined phase transformations $\phi \to e^{i\vartheta} \phi$ and $\phi^\dagger \to e^{-i\vartheta} \phi^\dagger$, the Lagrangian $\mathcal{L}$ is clearly invariant. With $\delta \phi = i\vartheta \phi$, $\delta \phi^\dagger = -i\vartheta \phi^\dagger$, the conserved current follows as

$$j^\mu = iq \left[ \phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger) \phi \right], \quad (3.12)$$

where we absorbed the parameter $\vartheta$ in the definition of the charge $q$. The conserved charge $Q = \int d^3x j^0$ can take any value in $\mathbb{R}$. Therefore we cannot interpret $j^0$ as the probability density to observe a $\phi$ particle. Instead, we should associate $Q$ with a conserved additive quantum number as the electric charge. In the similar case of the proton and antiproton, we could view the conserved charge also as nucleon number, or in modern language, baryon number.

Space-time symmetries of Minkowski space The Poincaré group as symmetry group of Minkowski space has ten generators. If the Lagrangian does not depend explicitly on space-time coordinates, i.e. $\mathcal{L} = \mathcal{L}(\phi_a, \partial_\mu \phi_a)$, ten conservation laws for the fields $\phi_a$ follow. We consider first the behaviour of the fields $\phi_a$ and the Lagrangian under an infinitesimal translation $x^\mu \to x^\mu + \varepsilon^\mu$. As in the case of internal symmetries, we consider only global transformations and thus $\varepsilon$ does not depend on $x$. From

$$\phi_a(x) \to \phi_a(x + \varepsilon) \approx \phi_a(x) + \varepsilon^\mu \partial_\mu \phi_a(x) \quad (3.13)$$

we find the change

$$\delta \phi_a(x) = \varepsilon^\mu \partial_\mu \phi_a(x) = \partial_\mu (\varepsilon^\mu \phi_a(x)).$$

If the Lagrange density $\mathcal{L}$ contains no explicit space-time dependence, it will change simply as $\mathcal{L}(x) \to \mathcal{L}(x + \varepsilon)$ or

$$\delta \mathcal{L}(x) = \varepsilon^\mu \partial_\mu \mathcal{L}(x) = \partial_\mu (\varepsilon^\mu \mathcal{L}(x)). \quad (3.14)$$

Thus $K^\mu = \varepsilon^\mu \mathcal{L}(x)$ and inserting both in the Noether current gives

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} [\varepsilon^\nu \partial_\nu \phi_a] \varepsilon^\mu \mathcal{L} = \varepsilon_\nu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \frac{\partial \phi_a}{\partial x_\nu} - \eta^\mu{}^\nu \mathcal{L} \right] = \varepsilon_\nu T^{\mu\nu} \quad (3.15)$$

1 Although the Langrangian is invariant under the larger group $\text{O}(n)$, we consider only the subgroup $\text{SO}(n)$ which is continuously connected to the identity. The additional discrete transformations contained in $\text{O}(n)$ can be used to classify solutions of the Lagrangian, but do not lead to additional conservation laws.
with $T^\mu{}^\nu$ as (energy-momentum) stress tensor and the four-momentum $p^\nu$ as four conserved Noether charges,

$$p^\nu = \int d^3x \, T^{0\nu}. \quad (3.16)$$

The conserved energy-momentum stress tensor that is defined by Eq. (3.15) is called canonical. The definition (3.15) does not guaranty that $T^\mu{}^\nu$ is symmetric. A symmetric stress tensor, $T^\mu{}^\nu = T^\nu{}^\mu$, is however the condition for the conservation of total angular momentum, as will show in the next paragraph\footnote{Another reason to require a symmetric stress tensor $T^\mu{}^\nu$ is that it serves as source term for the symmetric gravitational field.}. Since the Lagrange density is only determined up to a four-divergence, we can symmetrize always $T^\mu{}^\nu$ adding a four-dimensional divergence.

**Example:** The general expression (3.15) for the canonical stress tensor becomes for a free complex scalar field

$$T^\mu{}^\nu = 2 \partial^\mu \phi^* \partial^\nu \phi - \eta^\mu{}^\nu \mathcal{L}. \quad (3.17)$$

Thus the canonical stress tensor of a scalar field is already symmetric. Its $00$ component agrees with twice the result for the energy-density $\rho$ and the Hamilton density $H$ of a single real scalar field,

$$T^{00} = 2 |\partial_t \phi|^2 - |\nabla \phi|^2 + m^2 |\phi|^2 =\rho = H. \quad (3.18)$$

Changing from the continuum normalisation to a box of size $V = L^3$ amounts to replace $(2\pi)^3$ by $L^3$. Thus the normalisation constant $N^{-2} = (2\pi)^3 2\omega$ becomes for finite volume $N^{-2} = 2\omega V$. Thence the energy-density $T^{00} = \omega/V$ agrees with the expectation for one particle with energy $\omega$ per volume $V$.

The remaining components of $T^\mu{}^\nu$ are fixed by its tensor structure,

$$T^\mu{}^\nu = 2 N^2 (\partial^\mu k^\nu - \partial^\nu k^\mu) \frac{\omega V}{(3.19)}.$$

Since the stress tensor $T^\mu{}^\nu$ is symmetric, we can find a frame in which $T^\mu{}^\nu$ is diagonal with $T \propto \text{diag}(\omega, v_x p_x, v_y p_y, v_z p_z)/V$. Note that the spatial part of the stress tensor corresponds to the pressure tensor, cf. Eq. (9.54). Thus a scalar field can be described by the stress tensor of an ideal fluid, $T^\mu{}^\nu = \text{diag}(\rho, P_x, P_y, P_z)$, with energy density $\rho$ and pressure $P$.

**Angular momentum** If the tensor $T^\mu{}^\nu$ is symmetric, we can construct six more conserved quantities. Define

$$M^{\mu\nu\lambda} = x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu}, \quad (3.20)$$

then $M^{\mu\nu\lambda}$ is conserved with respect to the index $\mu$,

$$\partial_\mu M^{\mu\nu\lambda} = \delta^\nu_\mu T^{\mu\lambda} - \delta^\lambda_\mu T^{\mu\nu} = T^{\nu\lambda} - T^{\lambda\nu} = 0, \quad (3.21)$$

provided that $T^{\nu\lambda} = T^{\lambda\nu}$. In this case,

$$J^{\mu\nu} = \int d^3x \, M^{0\mu\nu} = \int d^3x \left[ x^\mu T^{0\nu} - x^\nu T^{0\mu} \right], \quad (3.22)$$
is a globally conserved tensor. The antisymmetry of $J^{\mu\nu}$ implies that there exist six conserved charges. The three charges

$$J^{ij} = \int d^3x \left[ x^i T^{0j} - x^j T^{0i} \right], \quad (3.23)$$

correspond to the conservation of total angular momentum, since $T^{0j}$ is the momentum density. The remaining three charges $J^{0i}$ express the fact that the center-of-mass moves with constant velocity.

While $J^{\mu\nu}$ transforms as expected for a tensor under Lorentz transformations, it is not invariant under translations $x^\mu \to x^\mu + \epsilon^\mu$. Instead, the angular momentum changes as

$$J^{\mu\nu} \to J^{\mu\nu} + \epsilon^\mu p^\nu - \epsilon^\nu p^\mu. \quad (3.24)$$

Clearly, this is a consequence of the definition of the orbital angular momentum with respect to the center of rotation. We want therefore to split the total angular momentum $J^{\mu\nu}$ into the orbital angular momentum $L^{\mu\nu}$ and an intrinsic part connected to a non-zero spin of the field. The latter we require to be invariant under translations. We set

$$S_\alpha = \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} u^\delta, \quad (3.25)$$

where $u^\alpha$ is the four-velocity of the center-of-mass system (cms). Because of the antisymmetry in $\beta\gamma$ of $\varepsilon_{\alpha\beta\gamma\delta}$, the change in (3.24) induced by a translation drops out in $S_\alpha$. In the cms, $u^\alpha = (1, 0, 0, 0)$ and thus $S_0 = 0$ and $S_\alpha u^\alpha = 0$. The other components are $S_1 = J^{23}$, $S_2 = J^{31}$, and $S_3 = J^{12}$. Thus the vector $S_\alpha$ describes as desired the intrinsic angular momentum of a field. It is called the Pauli-Lubanski spin vector.

### 3.3. Symmetries of a general space-time

In the case of a Riemannian space-time manifold $(\mathcal{M}, g)$, we say the space-time possess a symmetry if it looks the same as one moves from a point $P$ along a vector field $\xi$ to a different point $\tilde{P}$. More precisely, we mean with "looking the same" that the metric tensor transported along $\xi$ remains the same.

Such symmetries may be obvious, if one uses coordinates adapted to these symmetries: For instance, the metric may be independent from one or several coordinates. Let us assume that the metric is e.g. independent from the time coordinate $x^0$. Then $x^0$ is a cyclic coordinate, $\partial L/\partial x^0 = 0$, of the Lagrangian $L = d\tau/d\sigma$ of a free test particle moving in $\mathcal{M}$. With $L = d\tau/d\sigma$, the resulting conserved quantity $\partial L/\partial x^0 = \text{const.}$ can be written as

$$\frac{\partial L}{\partial x^0} = g_{0\beta} \frac{dx^\beta}{d\tau} = g_{0\beta} \frac{dx^\beta}{d\sigma} = \xi \cdot u \quad (3.26)$$

with $\xi = e_0$ and $u$ as the four-velocity. Hence the quantity $\xi \cdot u = p^0/m$ is conserved along the solutions $x^\alpha(\sigma)$ of the Lagrange equation of a free particle on $\mathcal{M}$, i.e. along geodesics: The motion of all test particles in the corresponding space-time conserve energy. The vector field $\xi$ that points in the direction in which the metric does not change is called Killing vector field.
Since we allow arbitrary coordinate systems, space-time symmetries are in general not evident by a simple inspection of the metric tensor. We say the metric is invariant moving along the Killing vector field $\xi$, when the resulting change $\delta g^{\mu\nu}$ of the metric is zero,

$$\delta \xi g^{\mu\nu} = \tilde{g}^{\mu\nu}(\tilde{x}) - g^{\mu\nu}(\tilde{x}) = 0. \quad (3.27)$$

Here we make an active coordinate transformation, moving from $x$ to $\tilde{x}$ which we assume for simplicity to be separated by an infinitesimal distance. Then we can work in the approximation

$$\tilde{x}^i = x^i + \varepsilon \xi^i(x^k) + \mathcal{O}(\varepsilon^2), \quad \varepsilon \ll 1, \quad (3.28)$$

and neglect all terms quadratic in $\varepsilon$. Equation (3.27) expresses the condition that the metric tensor after being transported along $\xi$ from $x$ to $\tilde{x}$ has the same functional dependence on the coordinates as the original metric tensor.

We start recalling the transformation law for a rank two tensor as the metric under an arbitrary coordinate transformation,

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} g^{\alpha\beta}(x). \quad (3.29)$$

Applied to the transport along $\xi$ defined in (3.28), we obtain

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x) = (\delta^\mu_\alpha + \varepsilon \xi^\mu_\alpha)(\delta^\nu_\beta + \varepsilon \xi^\nu_\beta)g^{\alpha\beta}(x) = g^{\mu\nu}(x) + \varepsilon(\xi^{\mu,\nu} + \xi^{\nu,\mu}) + \mathcal{O}(\varepsilon^2). \quad (3.30)$$

In order to be able to compare the new $\tilde{g}^{\mu\nu}(\tilde{x})$ with $g^{\mu\nu}(x)$, we have to express $\tilde{g}^{\mu\nu}(\tilde{x})$ as function of $x$. A Taylor expansion gives

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \tilde{g}^{\mu\nu}(x + \varepsilon \xi) = \tilde{g}^{\mu\nu}(x) + \varepsilon \xi^\alpha \partial_\alpha \tilde{g}^{\mu\nu}(x) + \mathcal{O}(\varepsilon^2). \quad (3.31)$$

Setting equal Eqs. (3.30) and (3.31), we obtain

$$g^{\mu\nu}(x) + \varepsilon(\xi^{\mu,\nu} + \xi^{\nu,\mu}) = \tilde{g}^{\mu\nu}(x) + \varepsilon \xi^\alpha \partial_\alpha \tilde{g}^{\mu\nu}(x). \quad (3.32)$$

Thus the metric is kept invariant, if the condition

$$\xi^{\mu,\nu} + \xi^{\nu,\mu} - \xi^k \partial_k g^{\mu\nu} = 0 \quad (3.33)$$

or

$$g^{\alpha\beta} \partial_\alpha \xi^\nu + g^{\nu\alpha} \partial_\alpha \xi^\mu - \xi^\alpha \partial_\alpha g^{\mu\nu} = 0 \quad (3.34)$$

is satisfied. Expressing partial derivatives as covariant ones, the terms containing connection coefficients cancel and we obtain the Killing equations

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \quad (3.35)$$

The vectors $\xi$ are called Killing vectors of the metric. Moving along a Killing vector field, the metric is kept invariant.
3. Global symmetries and Noether’s theorem

We now check that Eq. (3.35) leads indeed to a conservation law, as required by our initial definition of a Killing vector field. We multiply the equation for geodesic motion,

\[
\frac{Du^\mu}{d\tau} = 0,
\]

by the Killing vector \( \xi_\mu \) and use Leibniz’s product rule together with the definition of the absolut derivative (1.97),

\[
\xi_\mu \frac{Du^\mu}{d\tau} = \frac{d}{d\tau} (\xi_\mu u^\nu) - \nabla_\mu \xi_\nu u^\mu u^\nu = 0.
\]

(3.37)

The second term vanishes for a Killing vector field \( \xi_\mu \), because the Killing equation implies the antisymmetry of \( \nabla_\mu \xi_\nu \). Hence the quantity \( \xi_\mu u^\mu \) is indeed conserved along any geodesics.

**Example:** Find all ten Killing vector fields of Minkowski space and specify the corresponding symmetries and conserved quantities.

The Killing equation \( \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \) simplifies in Minkowski space to

\[
\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0.
\]

(3.38)

Taking one more derivative and using the symmetry of partial derivatives, we arrive at

\[
\partial_\rho \partial_\mu \xi_\nu + \partial_\rho \partial_\nu \xi_\mu = 2 \partial_\mu \partial_\rho \xi_\nu = 0.
\]

(3.39)

Integrating this equation twice, we find

\[
\xi_\mu = \omega_\mu ^ {\nu} x^\nu + a_\mu.
\]

(3.40)

The matrix \( \omega^{\mu\nu} \) has to be antisymmetric in order to satisfy Eq. (3.38). Thus the Killing vector fields are determined by ten integration constants. They agree with the infinitesimal generators of Lorentz transformations, cf. appendix B.3.

The four parameters \( a_\mu \) generate translations, \( x^\mu \rightarrow x^\mu + a^\mu \), described by four Killing vector fields which can be chosen as the Cartesian basis vectors of Minkowski space,

\[
T_0 = \partial_t, \quad T_1 = \partial_x, \quad T_2 = \partial_y, \quad T_3 = \partial_z.
\]

For a particle with momentum \( p^\mu = mu^\mu \) moving along \( x^\mu(\lambda) \), the existence of a Killing vector \( T^\mu \) implies

\[
\frac{d}{d\lambda} (T^\mu \cdot u) = \frac{d}{m d\lambda} (T^\mu \cdot p) = 0,
\]

i.e. the conservation of the four-momentum component \( p_\mu \).

Consider next the \( ij \) (=spatial) components of the Killing equation. Three additional Killing vectors are

\[
J_1 = y \partial_z - z \partial_y , \quad \text{cyclic permutations}.
\]

(3.41)

The existence of Killing vectors \( J_i \) implies that \( J_1 \cdot p \) is conserved along a geodesics of particle. But

\[
J_1 \cdot p = yp_z - zp_y = L_x
\]

and thus the angular momentum around the origin of the coordinate system is conserved.

The other three components satisfying the 0\( \alpha \) component of the Killing equations \( (\omega^0_{\alpha} = \omega_{\alpha 0}) \),

\[
K_1 = t \partial_z + z \partial_t , \quad \text{cyclic permutations}.
\]

(3.42)

The conserved quantity \( tp_z - zE = \text{const.} \) now depends on time and is therefore not as popular as the previous ones. Its conservation implies that the center of mass of a system of particles moves with constant velocity \( v_\alpha = p_\alpha / E \).
Global conservation laws

An immediate consequence of Eq. (1.129) is a covariant form of Gauß’ theorem for vector fields. In particular, we can conclude from local current conservation, \( \nabla_{\mu} j^{\mu} = 0 \), the existence of a globally conserved charge. If the conserved current \( j^{\mu} \) vanishes at infinity, then we obtain also in a general space-time

\[
\int_{\Omega} d^{4}x \sqrt{|g|} \nabla_j j^j = \int_{\Omega} d^{4}x \partial_{\mu}(\sqrt{|g|} j^{\mu}) = 0. \tag{3.43}
\]

For a non-zero current, the volume integral over the charge density \( j^{0} \) remains constant,

\[
\int_{\Omega} d^{4}x \sqrt{|g|} \nabla_{\mu} j^{\mu} = \int_{V(t_2)} d^{3}x \sqrt{|g|} j^{0} - \int_{V(t_1)} d^{3}x \sqrt{|g|} j^{0} = 0. \tag{3.44}
\]

Thus the conservation of Noether charges of internal symmetries as the electric charge, baryon number, etc., is not affected by an expanding universe.

Next we consider the energy-momentum stress tensor as example for a locally conserved symmetric tensors of rank two. Now, the second term in Eq. (1.132) prevents us to convert the local conservation law into a global one. If the space-time admits however a Killing field \( \xi \), then we can form the vector field \( P^\mu = T^{\mu\nu} \xi_{\nu} \) with

\[
\nabla_{\mu} P^{\mu} = \nabla_{\mu} (T^{\mu\nu} \xi_{\nu}) = \xi_{\nu} \nabla_{\mu} T^{\mu\nu} + T^{\mu\nu} \nabla_{\mu} \xi_{\nu} = 0. \tag{3.45}
\]

Here, the first term vanishes since \( T^{\mu\nu} \) is conserved and the second because \( T^{\mu\nu} \) is symmetric, while \( \nabla_{\mu} \xi_{\nu} \) is antisymmetric. Therefore the vector field \( P^\mu = T^{\mu\nu} \xi_{\nu} \) is also conserved, \( \nabla_{\mu} P^{\mu} = 0 \), and we obtain thus the conservation of the component of the energy-momentum vector in direction \( \xi \).

In summary, global energy conservation requires the existence of a time-like Killing vector field. Moving along such a Killing field, the metric would be invariant. Since we expect in an expanding universe a time-dependence of the metric, a time-like Killing vector field does not exist and the energy contained in a “comoving” volume changes with time.

3.4. Outlook: Classical versus quantum symmetries

Anomalies

Our discussion of symmetries has been purely classical. A classical symmetry holds also in quantum theory, if the path integral (i.e. functionals like \( Z[J] \)) remains invariant under the symmetry transformation \( \delta \phi_{a} \). This requires however not only the invariance of \( \mathcal{L} \) but also of the path integral measure \( D\phi_{a} \). If the latter is not invariant, then the classical symmetry is broken by quantum effects and one speaks of an “anomaly”. The two most important example are

- the chiral anomaly: Weak interactions of left and right chiral fermions differ. The integration measure in the path integral for fermions is not invariant under electroweak gauge transformations, thus a consistent quantum theory seems impossible. Only way out is that the anomalous terms of all matter fields cancels. This happens if the electric charge of all fermions vanishes, \( \sum_{f} Q_{f} = 0 \), or in one generation

\[
\sum_{f} Q_{f} = Q_{\nu} + Q_{t} + 3Q_{U} + 3Q_{D} = 0.
\]
First, we note (without knowing why) that the condition is indeed fulfilled for the particle content of the standard model. Second, that the faith in this relation lead to the prediction of the top quark, twenty years before its discovery.

• the breaking of conformal invariance in string theory. In this case, the anomalous term vanishes for a definite number of space-time dimensions, \( D = 10 \) or 26.

Ward identities The generating functional for a single, real scalar field given in Eq. (2.73) can be generalized in a straightforward way to a complex scalar field. We treat \( \phi \) and \( \phi^* \) as the two independent degrees of freedom, and add therefore also two independent sources \( J \) and \( J^* \). Coupling them as \( \mathcal{L}_s \equiv J \phi^* + J^* \phi \) to the fields keeps the Lagrangian real. We denote the total Lagrangian as

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{cl}} + \mathcal{L}_s ,
\]

with \( \mathcal{L}_{\text{cl}} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \) as the Lagrangian used to derive to the classical equation of motions. Thus the generating functional is

\[
Z[J, J^*] = \int D\phi D\phi^* \exp \{ \int d^4x \ (\mathcal{L}_{\text{cl}} + J \phi^* + J^* \phi) \} .
\]

We assume now that the classical Langrangian \( \mathcal{L}_{\text{cl}} \) is invariant under an internal global symmetry. To be concrete, we consider again the U(1) symmetry with \( \delta \phi = -i\varepsilon \phi, \delta \phi^* = i\varepsilon \phi^* \). If we keep the external sources \( J \) and \( J^* \) constant, the source term \( \mathcal{L}_s \) breaks the symmetry. This seems to imply that the generating functional \( Z[J, J^*] \) and thus also the Green functions and physical observables are not invariant under the U(1) symmetry of the classical Lagrangian. On the other hand, the fields \( \phi \) and \( \phi^* \) are integrated out in the generating functional \( Z[J, J^*] \) which therefore can not depend on a change of the fields generated by a symmetry of \( \mathcal{L}' \). As a result, identities between different Green functions can be derived.

We sketch now the idea how these so-called Ward identities are derived. For \( \delta \phi = i\varepsilon \phi, \delta \phi^\dagger = -i\varepsilon \phi^\dagger \), the change of the action is

\[
\delta \int d^4x \mathcal{L}_{\text{eff}} = \delta \int d^4x \mathcal{L}_s = \int d^4x i\varepsilon (J \phi^* - J^* \phi) \]  \hspace{1cm} (3.48)

The resulting change of the generating functional is thus

\[
\delta Z[J, J^*] = \int D\phi D\phi^* \int d^4x i\varepsilon (J \phi^* + J^* \phi) \exp \{ \int d^4x' \mathcal{L}_{\text{eff}} \} .
\]

The variation \( \delta Z[J, J^*] \) has to vanish. Moreover, we can as usually substitute fields by functional derivatives of sources. Then the part from Eq. (8.153) can be moved outside the functional integral, leading to the equation

\[
0 = \int d^4x \left[ J \frac{\delta}{\delta J} - J^* \frac{\delta}{\delta J^*} \right] \{ iW \} .
\]

Taking additional functional derivatives we can generate identities between connected Green functions.
3.4. Outlook: Classical versus quantum symmetries

Summary of chapter

Noether’s theorem shows that continuous global symmetries lead to conservation laws. Since any local symmetry $O(x)$ contains as a special case global transformations $O$, local symmetries lead to the same conservation laws as the corresponding global symmetries. In the cases of the conservation of electric and colour charge, the global symmetry is a consequence of an underlying local gauge symmetry which we will study later in chapter 7 in detail. In most other cases however, as e.g. the conservation of baryon or lepton number, no underlying local symmetry exist and one speaks therefore of accidental symmetries.

Further reading

Scale invariance which is briefly introduced in problem 3.5 is discussed in more detail in [Pok87]. For the analogue of the classical conserved currents in quantum theory see [DGH94].

Problems

3.1 Lagrangian for $N$ scalar fields.
The most general expression for the Lagrange density $\mathcal{L}$ of $N$ scalar fields $\phi_i$, which is Lorentz invariant and at most quadratic in the fields is

$$\mathcal{L} = \frac{1}{2} A_{ij} \partial^{\mu} \phi_i \partial_{\mu} \phi_j - \frac{1}{2} B_{ij} \phi_i \phi_j - C.$$

Argue why the coefficient matrices $A, B, C$ have to be real. Show that $\mathcal{L}$ can be recast into “canonical form”

$$\mathcal{L} = \frac{1}{2} \partial^{\mu} \phi_i \partial_{\mu} \phi_i - \frac{1}{2} b_{ij} \phi_i \phi_j - C$$

by linear field redefinitions.

3.2 Stress tensor for point particles.
Find the stress tensor for an ensemble of $N$ non-relativistic a.) point particles; ii) On scales $L$ such that $\Delta N/L^3 \gg 1$, we can describe the phase space density by a smooth function $f(x, p)$. b.) In the approximation $n \gg 1$, where the distribution can be approximated. Discuss the physical meaning of the different elements of $T^{\mu\nu}$.

3.3 Stress tensor for an ideal fluid.
a.) Find first the stress tensor of dust, i.e. of pressure-less matter.
b.) Generalise this result to an ideal fluid. (Hint: The state of an ideal fluid is completely determined by its energy density $\rho$ and its pressure $P$; in the rest-frame if the fluid, $P_{ij} = P \delta_{ij}$.)

3.4 Stress tensor for the electromagnetic field.
Determine the canonical energy-momentum stress tensor $T^{\mu\nu}$ of the free Maxwell field. Symmetrize $T^{\mu\nu}$ and determine the energy-density $\rho = T^{00}$. Find the trace of $T^{\mu\nu}$.

3.5 Scale invariance.
Consider the effect of a scale transformation $x \to e^\alpha x$ on a scalar field with Lagrange density

$$\mathcal{L} = \frac{1}{2} \partial^{\mu} \phi \partial_{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \phi^4$$

assuming that it acts linearly on the fields,

$$\phi_a(x) \to e^D \phi_a(e^\alpha x).$$

Here $\alpha$ is a number and $D$ a matrix in field space. We write down first the infinitesimal version of the scale transformation, with the aim to show that $\mathcal{L}$ can be made invariant for a specific value of $D$, if $m = 0$.

i) Find the corresponding conserved current $s^a$. ii) Show that the current $s^a$ can be written as $s^a = x^\nu T_{\mu\nu}$, where $T_{\mu\nu}$ is an “improved” energy-momentum tensor. Hint: Proceed similar as in the case of the angular momentum tensor. Derive
4. Spin-1 and spin-2 fields

The electric as well as the gravitational potential follow a $1/r$ law. According to our discussion of the Yukawa potential, we expect therefore that the two forces are mediated by massless particles. Since they exist as macroscopic fields, these particles should follow Bose-Einstein statistics. Such particles have integer spin and are described by tensor fields.

The quantisation of fields with higher spin becomes increasingly difficult—for the interacting spin-2 case no satisfactory solution has been found up to present. Our aim in this chapter is therefore much more restricted than in the scalar case: we derive the (linear) field equations describing the propagation of electromagnetic and gravitational perturbations, their solutions and Feynman propagators.

4.1. Tensor fields

Massive fields can be boosted to their rest-frame, $k^\mu = (m, 0)$. Wigner showed that the transformation properties of a massive field under Poincaré transformations are completely specified knowing its transformation properties under rotations in its rest-frame. As we will see, this implies that tensor fields of rank $n$ correspond to particles with spin $s = n$.

Lorentz invariance requires that each component of a free field $\psi_a$, where $a$ represents a collection of tensor and spinor indices, satisfies the Klein-Gordon equation. Additionally, we have to impose constraints which eliminate undesired components,

$$ (\Box + m^2) \psi_a(x) = 0, \quad \text{and} \quad f_i(\psi_a(x)) = 0. $$

For instance, massive fields with spin $s$ have $2s + 1$ components, i.e. a massive spin-1 field has three and a massive spin-2 field has five polarisation states. On the other hand, a vector field $A^\mu$ has four components, and a symmetric 2-rank tensor field $h^{\mu\nu}$ has ten components. Thus there exist already in the case of massive spin-1 and spin-2 fields too many degrees of freedom that have to be projected out. The reason for this mismatch is that in general a tensor of rank $n$ is reducible, i.e. it contains components of rank $< n$. For instance, the trace $h^{\mu}_\mu$ of a second rank tensor transforms clearly as a scalar. Therefore we should choose the constraints for massive fields with spin $s$ such that all components with spin $< s$ are eliminated.

**Example:** An object which contains invariant subgroups with respect to a symmetry operation is called reducible. Two examples for reducible tensors are:

a) The reducible subgroups of an arbitrary tensor of rank two with respect to general coordinate transformations.

We can split any tensor of rank two in its symmetric and anti-symmetric part,

$$ T^{\mu\nu} = \frac{1}{2} (T^{\mu\nu} + T^{\nu\mu}) + \frac{1}{2} (T^{\mu\nu} - T^{\nu\mu}) = S^{\mu\nu} + A^{\mu\nu}. $$

As the (anti-)symmetrisation is linear, it is clearly invariant under the change $\tilde{T}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \frac{\partial \tilde{x}^\sigma}{\partial x^\nu} T^{\rho\sigma}(x)$. Moreover, the trace of a tensor is an invariant. Thus we can break up $S^{\mu\nu}$ into its trace $S = S^{\mu}_\mu$ and
4.1. Tensor fields

its traceless part \( S_{\mu\nu} - S^\delta_{\mu\nu} / d \) in \( d \) dimensions.

b) The reducible subgroups of a symmetric tensor \( h^{\mu\nu} \) of rank two with respect to spatial rotations. We split \( h^{\mu\nu} \) into a scalar \( h^{00} \), a vector \( h^{0i} \) and a reducible tensor \( h^{ij} \),

\[
h^{\mu\nu} = \begin{pmatrix} h^{00} & h^{0i} \\ h^{i0} & h^{ij} \end{pmatrix}.
\]

Then we decompose \( h^{ij} \) again into its trace \( h^{ii} \) and its traceless part \( h^{ij} - h^i_j / (d - 1) \). The latter has \( 6 - 1 = 5 \) degrees of freedom in \( d = 4 \), as required for a massive spin-2 field.

This problem is more severe for massless fields, where only two physical degrees of freedom exist. In this case, even the irreducible tensors contain too many degrees of freedom. They can be consistently eliminated only, if some redundancy of the field variables exists which in turn leads to a symmetry of the fields. In this chapter, we discuss the consequences of this redundancy called gauge symmetry on the level of the wave equations and their solutions for the photon and the graviton.

**Tensor structure of the propagator** We can gain some insight into the general tensor structure of the Feynman propagator with spin \( s > 0 \) using the definition of the 2-point Green function as the time-ordered vacuum expectation values of fields. In general, we can express an arbitrary solutions of a free spin \( s = 0, 1 \) and 2 field by its Fourier components as

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ a(k)e^{-i(\omega_k t - kx)} + \text{h.c.} \right],
\]

\[
A^\mu(x) = \sum_r \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ a_r(k)\epsilon^\mu_r e^{-i(\omega_k t - kx)} + \text{h.c.} \right],
\]

\[
h^{\mu\nu}(x) = \sum_r \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ a_r(k)\epsilon^{\mu\nu}_r e^{-i(\omega_k t - kx)} + \text{h.c.} \right],
\]

where the momentum is on-shell, \( k^\mu = (\omega_k, k) \) and \( r \) labels the spin or polarisation states. The constraints \( f_i(\psi(x)) = 0 \) are now conditions on the polarisation vector and tensor, respectively, which depend on \( k \).

Proceeding as in the scalar case, (2.54), we expect that e.g. the propagator for a vector field is

\[
iD_F^{\mu\nu}(x) = \langle 0 | T \{ A^\mu(x) A^{\nu*}(0) \} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4 2k} \left[ \epsilon^\mu_\tau(k)e^{-i(\omega_k t - kx)}\partial(x^0) + \epsilon^{\mu*}_\tau(k)e^{i(\omega_k t - kx)}\partial(-x^0) \right]
\]

\[
= \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{P}^{\mu\nu}(k)}{k^2 - m^2 + i\varepsilon}.
\]

The expression (4.6) is in line with the interpretation of the propagator as the probability for the creation of a particle at \( x \) with any momentum \( k \) and polarisation \( r \), its propagation to \( x' \) followed by its annihilation. In the last step, we assumed that the tensor \( \mathcal{P}^{\mu\nu}(k) \) which corresponds to a sum over the polarisation states \( \epsilon^\mu_\tau(k) \) of the particle introduces no additional singularities.

\[1\]See the appendix B.4 for a brief discussion.
4. Spin-1 and spin-2 fields

In the following, we will assume that the propagators for \( s > 0 \) can be derived from the scalar one, \( \Delta_F(k) \), knowing the polarisation states as

\[
D^{\mu\nu}_F(k) = \sum_r \varepsilon^\mu_r(k) \varepsilon^{\nu\ast}_r(k) \Delta_F(k),
\]

(4.8)

\[
D^{\mu\nu\rho\sigma}_F(k) = \sum_r \varepsilon^\mu_r(k) \varepsilon^{\nu\rho\ast}_r(k) \Delta_F(k).
\]

(4.9)

Later on we will come back to this topic deriving the propagators directly from the Lagrange density for spin-1 and spin-2 fields.

4.2. Vector fields

**Proca and Maxwell equations**  A massive vector field \( A^\mu \) has four components in \( d = 4 \) space-time dimensions, while it has only \( 2s + 1 = 3 \) independent spin components. Correspondingly, a 4-vector \( A^\mu \) transforms under a rotation as \((A^0, \mathbf{A})\), i.e. it contains a scalar and a three-vector. Therefore we have to add to the four Klein-Gordon equations for \( A^\mu \) one constraint which eliminates \( A^0 \): The only linear, Lorentz invariant possibility is

\[
(\Box + m^2) A^\mu(x) = 0 \quad \text{and} \quad \partial_\mu A^\mu = 0.
\]

(4.10)

In momentum space, this translates into \((k^2 - m^2)A^\mu(k) = 0 \) and \( k_\mu A^\mu(k) = 0 \). In the rest frame of the particle, \( k^\mu = (m, \mathbf{0}) \), and the constraint becomes \( A^0 = 0 \). Hence a field satisfying \((4.10)\) has clearly only three components as required for a massive \( s = 1 \) field. We can choose the three polarisation vectors which label the three degrees of freedom in the rest frame e.g. as the Cartesian unit vectors, \( \varepsilon_i \propto \mathbf{e}_i \).

The two equations can be combined into one equation called Proca equation,

\[
(\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu) A_\nu = m^2 A^\mu = 0.
\]

(4.11)

To show the equivalence of this equation with \((4.10)\), act with \( \partial_\mu \) on it,

\[
(\partial^\nu \Box - \Box \partial^\nu) A_\nu + m^2 \partial_\mu A^\mu = m^2 \partial_\mu A^\mu = 0.
\]

(4.12)

Hence, a solution of the Proca equation fulfils automatically the constraint \( \partial_\mu A^\mu = 0 \) for \( m^2 > 0 \). On the other hand, we can neglect the second term in \((4.11)\) for \( \partial_\nu A^\nu = 0 \) and obtain the Klein-Gordon equation.

We now go over to the case of a massless spin-1 field which is described by the Maxwell equations. In classical electromagnetism, the field-strength tensor \( F_{\mu\nu} \) is an observable quantity, while the potential \( A_\mu \) is merely a convenient auxiliary quantity. From the definition

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

(4.13)

it is clear that \( F_{\mu\nu} \) is invariant under the transformations

\[
A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x).
\]

(4.14)

Thus \( A'_\mu(x) \) is for any \( \Lambda(x) \) physically equivalent to \( A_\mu(x) \), leading to the same field-strength tensor and thus e.g. to the same Lorentz force on a particle. The transformations \((4.14)\) are
called gauge transformations. Note that the mass term \( m^2 A^\mu \) in the Proca equation breaks gauge invariance.

If we insert into the Maxwell equation the definition of the potential,

\[
\partial_\mu F^{\mu \nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial_\mu \partial^\mu A^\mu = j^\nu ,
\]

we see that this expression equals the \( m = 0 \) limit of the Proca equation. Gauge invariance allows us to choose a potential \( A^\mu \) such that \( \partial_\mu A^\mu = 0 \). Such a choice is called fixing the gauge, and the particular case \( \partial_\mu A^\mu = 0 \) is denoted as Lorenz gauge. In the Lorenz gauge, the wave equation simplifies to

\[
\Box A^\mu = j^\mu .
\]

Additionally, we can add to the potential \( A^\mu \) any function \( \chi \) satisfying \( \Box \chi = 0 \). We can use this freedom to set \( A^0 = 0 \). Inserting then a plane-wave \( A^\mu \propto \varepsilon^\mu \epsilon kx \) into the free wave equation, \( \Box A^\nu = 0 \), we find that \( k \) is a null-vector and that \( \varepsilon^\mu \varepsilon^\nu = -\epsilon \cdot k = 0 \). Thus the photon propagates with the speed of light, is transversely polarised and has two polarisation states as expected for a massless particle.

Closely connected to the gauge invariance of electrodynamics is the fact that its source, the electromagnetic current, is conserved. The antisymmetry of \( F^{\mu \nu} \), which is the basis for the symmetry (4.14), leads also to \( \partial_\mu \partial_\nu F^{\mu \nu} = 0 \). Thus the Maxwell equation \( \partial_\mu F^{\mu \nu} = j^\nu \) implies the conservation of the electromagnetic current \( j^\mu \),

\[
\partial_\mu \partial_\nu F^{\mu \nu} = \partial_\nu j^\nu = 0 .
\]

4.2. Vector fields

The propagator \( D_{\mu \nu} \) for a massive spin-1 field is determined by

\[
\left[ \eta^{\mu \nu} (\Box + m^2) - \partial^\mu \partial^\nu \right] D_{\nu \lambda}(x) = \delta_\mu^\nu \delta(x) .
\]

Inserting the Fourier transformation of the propagator and of the delta function gives

\[
\left[ (-k^2 + m^2) \eta^{\mu \nu} + k^\mu k^\nu \right] D_{\nu \lambda}(k) = \delta_\mu^\lambda .
\]

We will apply the tensor method to solve this equation: In this approach, we use first all tensors available in the problem to construct the required tensor of rank 2. In the case at hand, we have at our disposal only the momentum \( k_\mu \) of the particle—which we can combine to \( k_\mu k_\nu \)—and the metric tensor \( \eta_{\mu \nu} \). Thus the tensor structure of \( D_{\mu \nu}(k) \) has to be of the form

\[
D_{\mu \nu}(k) = A \eta_{\mu \nu} + B k_\mu k_\nu
\]

with two unknown scalar functions \( A(k^2) \) and \( B(k^2) \). Inserting this ansatz into (4.19) and multiplying out, we obtain

\[
\left[ (-k^2 + m^2) \eta^{\mu \nu} + k^\mu k^\nu \right] \left[ A \eta_{\nu \lambda} + B k_\nu k_\lambda \right] = \delta_\mu^\lambda ,
\]

\[
-A k^2 \delta_\mu^\lambda + A m^2 \delta_\mu^\lambda + A k^\mu k_\lambda + B m^2 k^\mu k_\lambda = \delta_\mu^\lambda ;
\]

\[
-A (k^2 - m^2) \delta_\mu^\lambda + (A + B m^2) k^\mu k_\lambda = \delta_\mu^\lambda .
\]

In the last step, we regrouped the LHS into the two tensor structures \( \delta_\mu^\lambda \) and \( k^\mu k_\lambda \). A comparison of their coefficients gives then \( A = -1/(k^2 - m^2) \) and

\[
B = \frac{A}{m^2} = \frac{1}{m^2(k^2 - m^2)} .
\]
Thus the massive spin-1 propagator follows as
\[
D_F^{\mu\nu}(k) = \frac{-\eta^{\mu\nu} + k^\mu k^\nu/m^2}{k^2 - m^2 + i\varepsilon}.
\]  
(4.22)

Next we check our claim that the propagator \( D_F^{ab}(k) \) of spin \( s > 0 \) fields can be obtained as sum over their polarisation states \( \varepsilon^{(r)}_a \) times the scalar propagator \( \Delta(k) \). As the theory is Lorentz invariant, we can choose the frame most convenient for this comparison which is the restframe of the massive particle. Then \( k^\mu = (m, 0) \) and the three polarisation vectors can be chosen as the Cartesian basis vectors,

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \sum_r \varepsilon^{(r)}_\mu \varepsilon^{r}_{\nu}.
\]
(4.23)

Thus both methods agree and allow us to derive the Feynman propagator. In the latter approach, working from the RHS to the LHS of (4.23), we derive first the expression valid for the Feynman propagator in a specific frame. Then we have to rewrite the expression in an invariant way using the relevant tensors, here \( \eta^{\mu\nu} \) and \( k^\mu k^\nu \).

**Propagator for massless spin-1 fields** As we have seen, we can set \( m = 0 \) in the Proca equation and obtain the Maxwell equation. The corresponding limit in (4.22) leads however to an ill-defined result. As we know that the number of degrees of freedom differs between the massive and the massless case, this is not too surprising. If we try next the limit \( m \to 0 \) in (4.21), then we find

\[
-Ak^2\delta_\mu^\lambda + Ak^\mu k_\lambda = \delta_\mu^\lambda.
\]
(4.24)

The last equation has for arbitrary \( k \) with \( A = -1/k^2 \) and \( A = 0 \) no solution. Moreover, the function \( B \) is undetermined. We can understand this physically, since for a massless field current conservation holds. But \( \partial_\mu J^\mu(x) = 0 \) implies \( k_\mu J^\mu(k) = 0 \) and thus the \( k^\mu k^\nu \) term does not influence physical quantities: In physical measurable quantities, as e.g. \( W[J] \), the propagator is always matched between conserved currents, and the longitudinal part \( k^\mu k^\nu \) drops out.

We now try to construct the photon propagator from its sum over polarisation states. First we consider a linearly polarised photon with polarisation vectors \( \varepsilon^{(r)}_\mu \) lying in the plane perpendicular to its momentum vector \( k \). If we perform a Lorentz boost on \( \varepsilon^{(1)}_\mu \), we will find

\[
\tilde{\varepsilon}^{(1)}_\mu = \Lambda_\mu^\nu \varepsilon^{(1)}_\nu = a_1 \varepsilon^{(1)}_\mu + a_2 \varepsilon^{(2)}_\mu + a_3 k_\mu,
\]
(4.25)

where the coefficients \( a_i \) depend on the direction \( \beta \) of the boost. Thus, in general the polarisation vector will not be anymore perpendicular to \( k \). Similarly, if we perform a gauge transformation

\[
A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)
\]
(4.26)

with

\[
\Lambda(x) = i\lambda \exp(-ikx) + \text{h.c.},
\]
(4.27)

then

\[
A'_\mu(x) = \varepsilon'_\mu \exp(-ikx) + \text{h.c.}
\]
(4.28)
with \( \varepsilon'_\mu = \varepsilon_\mu + \lambda k_\mu \). Since \( \lambda \) is arbitrary, only the components of \( \varepsilon^\mu \) transverse to \( k \) can have physical significance. Using \( \varepsilon_\mu k^\mu = 0 \), we can eliminate both the time-like and the longitudinal component choosing \( \varepsilon_0 = -\varepsilon_3 \).

We will use now current conservation, \( k_\mu J^\mu(k) = 0 \), to derive a convenient expression for the propagator of a massless vector particle. The two polarisation vectors of a photon should satisfy the normalisation \( \varepsilon^\mu (a) \varepsilon^\mu (b) = \delta^{ab} \). For a linearly polarised photon propagating in \( z \) direction, \( k^\mu = (\omega, 0, 0, \omega) \), the polarisation vectors are \( \varepsilon^{(1)}_\mu = \delta_1^\mu \) and \( \varepsilon^{(2)}_\mu = \delta_2^\mu \). If we perform the sum over the two polarisation states, we find

\[
\sum_r \varepsilon^{(r)*}_{\mu} \varepsilon^{(r)}_{\nu} = \text{diag} \{0, 1, 1, 0\}. \tag{4.29}
\]

If we try to rewrite this expression in an invariant way using \( \eta_{\mu\nu} \) and \( k_\mu k_\nu / k^2 \), we fail: We cannot cancel at the same time \( \eta_{00} = +1 \) and \( \eta_{33} = -1 \) by \( k_\mu k_\nu / k^2 \). We introduce therefore additionally the momentum vector \( \tilde{k}^\mu = (\omega, 0, 0, -\omega) \) obtained by a spatial reflection from \( k^\mu \). This allows us to write the polarisation sum as an invariant tensor expression,

\[
\sum_r \varepsilon^{(r)*}_{\mu} \varepsilon^{(r)}_{\nu} = -\eta_{\mu\nu} + \frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{kk} \equiv -\lambda_{\mu\nu}. \tag{4.30}
\]

Current conservation, \( k_\mu J^\mu(k) = 0 \), implies that the second term in the polarisation sum does not contribute to physical observables. For the same reason, we can add an arbitrary term \( \xi k_\mu k_\nu \). We use this freedom to eliminate the \( \tilde{k} \) dependence and to set

\[
J^\mu \left( \sum_r \varepsilon^{(r)*}_{\mu} \varepsilon^{(r)}_{\nu} \right) J^\nu = J^\mu \left( -\eta_{\mu\nu} + (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) J^\nu. \tag{4.31}
\]

Now we can read off the photon propagator in a form suitable for loop calculations as

\[
D_F^{\mu\nu}(k) = -\eta^{\mu\nu} + (1 - \xi) k^\mu k^\nu / k^2. \tag{4.32}
\]

A specific choice of the parameter \( \xi \) called gauge fixing parameter corresponds to the choice of a gauge in Eq. \((4.13)\). In particular, the Feynman gauge \( \xi = 1 \), which leads to a form of the propagator often most convenient in calculations, correspond to the Lorenz gauge in Eq. \((4.15)\). In this gauge, \( \sum_{r=0}^3 \varepsilon^{(r)*}_{\mu} \varepsilon^{(r)}_{\nu} = \text{diag} \{-1, 1, 1, 1\} \) : the propagator contains nonphysical degrees of freedom, time-like and longitudinal photons, which contributions cancel in physical observables. Similarly, for all other values \( \xi \) the propagator is explicitly Lorentz invariant but contains unphysical degrees of freedom. We will see later that it is a general feature of gauge theories as electromagnetism that we have to choose between a covariant gauge which introduces unphysical degrees of freedom and a gauge which contains only the transverse degree of freedom but selects a specific frame.

**Asymptotic behaviour** The behaviour of a propagator for large (Euclidean) momenta \( k \) determines the degree of divergence of loop integrals. The massless spin-1 propagator decreases like \( D_F^{\mu\nu}(k) \propto 1/k^2 \) for large \( k \), while the massive propagator behaves as \( D_F^{\mu\nu}(k) \propto \text{const} \). This implies that the divergences in loop diagrams are more severe for massive vector particles than for massless ones. In particular, inserting additional massive propagators into a loop graph does not improve its convergence and thus a theory with massive spin-1 particles contains an infinite number of divergent diagrams at each loop order.
4. Spin-1 and spin-2 fields

**Repulsive Coulomb potential by vector exchange** We consider again two static point charges as external sources, but using now a vector current \( J^\mu = J^\mu_1(x_1) + J^\mu_2(x_2) \). Since \( J^\mu = (\rho, j) \), only the zero component, \( J^0 = \delta_0^0 \delta(x - x_1) \), contributes for a static source to \( W[J] \). Moreover, we can neglect the longitudinal part \( k^\mu k^\nu /m^2 \) of the propagator. This is justified, since the concept of a potential energy makes only sense in the non-relativistic limit, i.e. for \( V \ll m \) or equivalently \( r \gg 1/m \). Hence

\[
W_{12}[J] = -\frac{1}{2} \int d^4x d^4x' \int \frac{d^4k}{(2\pi)^4} J^\mu_1(x) \frac{-\eta_{\mu\nu} e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon} J^\nu_2(x') \quad (4.33)
\]

Comparing with our earlier result for scalar exchange in Eq. (2.42), it becomes clear without further calculation that spin-1 exchange between equal charges is repulsive. In the limit \( m \to 0 \), we obtain the Coulomb potential with the correct sign for electromagnetic interactions.

**Lorentz transformations on \( A^{(r)}_\mu(x) \)** Before we move on, let us examine how the quantity \( A^{(r)}_\mu(x) \) describing the two physical states of a photon transform under a Lorentz transformation. We expect a solution of (4.15) to transform as a four-vector, but in determining \( A^{(r)}_\mu(x) \) we have also to fix a gauge in order to eliminate the two unphysical degree of freedoms. Performing a Lorentz transformation changes the gauge, and therefore the transformation law of a photon state is

\[
\tilde{A}^\mu = A^\mu + \partial^\mu \Lambda, \quad (4.35)
\]

where \( \Lambda \) is such that in both frames the same gauge can be used. You are asked to examine the Lorentz transformation properties of \( A^{(r)}_\mu(x) \) in more detail in problem 4.3.

### 4.3. Gravity

**Wave equation** From Newton’s law we know that gravity is fundamentally attractive and of long-range. Thus the gravitational force has to be mediated by a massless particle which can not be a spin \( s = 1 \) particle.

Analog to the electric field \( \mathbf{E} = -\nabla \phi \) we can introduce a classical gravitational field \( \mathbf{g} \) as the gradient of the gravitational potential, \( \mathbf{g} = -\nabla \phi \). We obtain then \( \nabla \cdot \mathbf{g}(x) = -4\pi G \rho(x) \) and as Poisson equation

\[
\Delta \phi(x) = 4\pi G \rho(x), \quad (4.36)
\]

where \( \rho \) is the mass density, \( \rho = \text{dm}/d^3x \).

Special relativity gives us two hints how we should transfer this equation into a relativistic framework: First, the Laplace operator \( \Delta \) on the LHS is the \( c \to \infty \) limit of minus the d’Alembert operator \( \Box \). Second, the RHS is the \( v/c \to 0 \) limit of something incorporating not only the mass density but all types of energy densities. To proceed, consider first how the mass density \( \rho \) transforms under a Lorentz transformation: An observer moving with the
4.3. Gravity

speed $\beta$ relative to the rest frame of the matter distribution $\rho$ measures the energy density $\rho' = \gamma dm/(\gamma^{-1} dV) = \gamma^2 \rho$, where $\gamma = \sqrt{1 - \beta^2}$. This is the transformation law of the 00 component of a tensor of rank 2, alas the energy-momentum stress tensor $T^\mu_\nu$.

Thus the field equations for a purely scalar theory of gravity would be

$$\Box \phi = -4\pi GT^\mu_\mu. \quad (4.37)$$

Such a theory predicts no coupling between photons and gravitation, since $T^\mu_\mu = 0$ for the electromagnetic field, and is in contradiction to the observed gravitational lensing of light. A purely vector theory for gravity fails too, since it predicts not attraction but repulsion of two masses. Hence we are forced to consider a symmetric spin-2 field $\bar{h}^\mu_\nu$ as mediator of the gravitational force; its source is the energy-momentum stress tensor

$$\Box \bar{h}^\mu_\nu = -2\kappa T^\mu_\nu. \quad (4.38)$$

Let us consider as a warm-up first the massive case: A symmetric, massive spin-2 field has ten independent component, but only $2s + 1 = 5$ physical spin degrees of freedoms. Thus we have to impose five constraints additional to the source-free equation

$$(\Box + m^2)\bar{h}^\mu_\nu = 0. \quad (4.39)$$

Proceeding as in the $s = 1$ case, (4.10), we use as constraint $\partial_\mu \bar{h}^\mu_\nu = 0$ which provides now four constraints. These constraints we can use to set $\bar{h}^0_\mu = 0$. We obtain the missing fifth constraint subtracting the trace $\bar{h}^{ii}$ which transforms as a scalar from $\bar{h}^{ij}$.

We move now to the massless case considering a plane wave $h^\mu_\nu = \epsilon^\mu_\nu \exp(ikx)$. In analogy to the photon case, we expect that also the graviton has only two, transverse degrees of freedom. If we choose the plane wave propagating in the $z$ direction, $k = ke_z$, then we expect that the polarisation tensor can be expressed as

$$\epsilon^\mu_\nu = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \epsilon_{11} & \epsilon_{12} & 0 \\
0 & \epsilon_{12} & -\epsilon_{11} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \quad (4.40)$$

Here we used that the polarisation tensor has to be symmetric and traceless. The choice (4.40) is called the transverse traceless (TT) gauge.

**Metric perturbations as a tensor field** In the case of the photon, we could reduce the degrees of freedom from four to two, because of the redundancy implied by the gauge symmetry of electromagnetism. Moreover, the gauge symmetry lead to the conservation of the electromagnetic current. The two obvious questions to address next are which symmetry and which conservation law are connected to gravitation.

The second part of the questions is the simpler one, since we know already that in flat space $\partial_\mu T^\mu_\nu = 0$ holds. Thus for gravity energy-momentum conservation will play the role of current conservation, implying that a gravitational wave is transverse, $k_\mu T^\mu_\nu(k) = 0$. In order to answer the first part of the question, we have to consider the properties of $\bar{h}^\mu_\nu$.

A unique characteristic of gravitation compared to all other known interactions is that its “charge”, the gravitational mass, is equal to the inertial mass. The equivalence principle implies that test-particles with arbitrary properties move along the same world-line, if they
4. Spin-1 and spin-2 fields

are released at the same initial point and move only under the influence of the gravitational force. This universality motivated Einstein to describe the effect of gravity by the curvature of space-time.

We associate therefore the symmetric tensor field \( \bar{h}_{\mu\nu} \) with small perturbations around the Minkowski metric \( \eta_{\mu\nu} \),

\[
g_{\mu\nu} = \eta_{\mu\nu} + \lambda h_{\mu\nu} , \quad \lambda h_{\mu\nu} \ll 1 . \tag{4.41}
\]

(Here we dropped the bar, anticipating that \( \bar{h}_{\mu\nu} \) may be a function of \( h_{\mu\nu} \); in particular, the metric tensor we use to describe space-time is dimensionless, while the bosonic field \( h_{\mu\nu} \) should have mass dimension one. This change of dimension implies that the coupling strength \( \lambda \) is dimensionfull, \( \lambda \propto G_{N}^{-1/2} \), as suggested by \( [G_{N}] = m^{-2} \).)

We choose a Cartesian coordinate system \( x \) and ask ourselves which transformations are compatible with the splitting (4.41) of the metric. If we consider global (i.e. space-time independent) Lorentz transformations \( \Lambda_{\alpha}^{\beta} \), then \( x_{\alpha}' = \Lambda_{\alpha}^{\beta} x_{\beta} \).

The splitting (4.41) is however clearly not invariant under general coordinate transformations, as they allow e.g. the rescaling \( g_{\mu\nu} \rightarrow \Omega g_{\mu\nu} \).

It is more fruitful to view this equation not as a coordinate but as a gauge transformation analogous to (4.14): Both \( h'_{\mu\nu} \) and \( h_{\mu\nu} \) describe the same physical situation, since the (linearized) Einstein equations do not fix uniquely \( h_{\mu\nu} \) for a given source. In momentum space, (4.44) becomes a condition on the polarisation tensor,

\[
\varepsilon_{\mu\nu}' = \varepsilon_{\mu\nu} - \xi_{\mu} k_{\nu} - \xi_{\nu} k_{\mu} . \tag{4.45}
\]
Graviton propagator We follow now the same approach as in the derivation of the photon propagator. For a graviton propagating in $z$ direction, $k^\mu = (\omega, 0, 0, \omega)$, we choose as the two polarisation states $\varepsilon^{(1)}_{\mu\nu}$ setting $\varepsilon_{11} = 1/\sqrt{2}$ and $\varepsilon_{12} = 0$ and $\varepsilon^{(2)}_{\mu\nu}$ setting $\varepsilon_{11} = 0$ and $\varepsilon_{12} = 1/\sqrt{2}$, respectively. They satisfy the normalisation $\varepsilon^{a}_{\mu\nu}\varepsilon^{b}_{\mu\nu} = \delta^{ab}$. Thus the graviton propagator in the Feynman gauge is given by

$$T^{\mu\nu} = T^{\mu\nu}_{\text{11}} - \frac{1}{2} \eta^{\mu\nu} \eta_{\rho\sigma} + \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} + \frac{1}{2} \eta_{\mu\sigma} \eta_{\nu\rho}.$$

We continue to proceed in the same way as for the photon. Energy-momentum conservation, $k_\mu T^{\mu\nu} = 0$, implies that $k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu$ term in $\lambda_{\mu\nu}$ does not contribute to physical observables. We can therefore drop again all terms proportional to the graviton momentum $k_\mu$.

$$T^{\mu\nu} \left( \sum_r \varepsilon^{(r)}_{\mu\nu} \varepsilon^{(r)}_{\rho\sigma} \right) T^{\rho\sigma} = T^{\mu\nu} \left( -\frac{1}{2} \eta^{\mu\nu} \eta_{\rho\sigma} + \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} + \frac{1}{2} \eta_{\mu\sigma} \eta_{\nu\rho} \right) T^{\rho\sigma}.$$

Thus the graviton propagator in the Feynman gauge is given by

$$D_F^{\mu\nu\rho\sigma}(k) = \frac{1}{k^2 + i\varepsilon} \left( -\eta^{\mu\nu} \eta^{\rho\sigma} + \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} \right).$$

Other $R_\xi$ like gauges are obtained by the replacement $\eta^{\mu\nu} \rightarrow \eta^{\mu\nu} - (1 - \xi) k^\mu k^{\nu}/k^2$. In the case of gravity, the Feynman gauge $\xi = 1$ is most often called harmonic gauge, but also the names Hilbert, Loren(t)z, de Donder and confusingly many others are in use.
4. Spin-1 and spin-2 fields

Attractive potential by spin-2 exchange We consider now the potential energy created by two point masses as external sources interacting via a tensor current \( T_{\mu\nu} = T_{\mu\nu}^1(x_1) + T_{\mu\nu}^2(x_2) \). Specialising to static sources, only the zero-zero component, \( T_{\mu\nu}^{00} = \delta_0^\mu \delta_0^\nu \delta(x - x_i) \), contributes to \( W[J] \). Hence

\[
W_{12}[J] = -\frac{1}{2} \int d^4x \int d^4x' \int \frac{d^4k}{(2\pi)^4} T_{00}^1(x) D_{F00;00}(k) e^{-i\mathbf{k}(\mathbf{x} - \mathbf{x'})} T_{00}^2(x').
\] (4.50)

Looking at the nominator of the graviton propagator, we find \(-1 + 1 + 1 = 1 > 0\). Thus spin-2 exchange is attractive, as required for the force mediating gravity. In problem 4.4 you are asked to determine the coupling \( \lambda \) in \( \mathcal{L} = \lambda h_{\mu\nu} T_{\mu\nu} \) comparing (4.50) to Newton’s gravitational potential.

Helicity We determine now how a metric perturbation \( h_{\mu\nu} \) transforms under a rotation with the angle \( \alpha \). We choose the wave propagating in \( z \) direction, \( \mathbf{k} = k \mathbf{e}_z \), the TT gauge, and the rotation in the \( xy \) plane. Then the general Lorentz transformation \( \Lambda_{\nu}^{\mu} \) becomes

\[
\Lambda_{\nu}^{\mu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha & 0 \\
0 & -\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (4.51)

Since \( \mathbf{k} = k \mathbf{e}_z \) and thus \( \Lambda_{\nu}^{\mu} k_\nu = k_\mu \), the rotation affects only the polarisation tensor. We rewrite \( \varepsilon'_{\mu\nu} = \Lambda_{\rho}^{\mu} \Lambda_{\sigma}^{\nu} \varepsilon_{\rho\sigma} \) in matrix notation,

\[
\varepsilon' = \Lambda \varepsilon \Lambda^T.
\] (4.52)

It is sufficient in TT gauge to perform the calculation for the \( xy \) sub-matrices. The result after introducing circular polarisation states \( \varepsilon_{\pm} = \varepsilon_{11} \pm i\varepsilon_{12} \) is

\[
\varepsilon'_{\pm} = \exp(\mp 2i\alpha) \varepsilon_{\pm}.
\] (4.53)

The same calculation for a circularly polarised photon gives \( \varepsilon'_{\pm} = \exp(\mp i\alpha) \varepsilon'_{\pm} \).

Any plane wave \( \psi \) which is transformed into \( \psi' = e^{-i\mathbf{k}(\mathbf{x} - \mathbf{x'})} \psi \) by a rotation of an angle \( \alpha \) around its propagation axis is said to have helicity \( h \). Thus if we say that a photon has spin 1 and a graviton has spin 2, we mean more precisely that electromagnetic and gravitational plane waves have helicity one and two, respectively. Doing the same calculation in an arbitrary gauge, one finds that the remaining, unphysical degrees of freedom transform as helicity one and zero (problem 4.6).

4.4. Source of gravity

The dynamical energy-momentum tensor If we compare the wave equation for a photon and a graviton, then there is an important difference: The former is in the classical limit exact. The photon carries no charge and does not contribute to its source term. As a result, the wave equation is linear. In contrast, a gravitational wave carrying energy and momentum acts as its own source. The LHS of (4.38) should be therefore the limit of a more complicated equation, which we write symbolically as \( G_{\mu\nu} = -2\kappa T_{\mu\nu} \). In the absence of matter, \( G_{\mu\nu} = 0 \),
4.4. Source of gravity

and the LHS reduces to the yet unknow (exact) field equations for the gravitational field $g^{\mu\nu}$. These field equations should be given as the variation of an appropriate action of gravity, called the Einstein-Hilbert action $S_{\text{EH}}$, with respect to the metric tensor $g^{\mu\nu}$.

Even without knowing the action of gravity, we can derive an important conclusion. The yet unknown $G_{\mu\nu}$ on the LHS should be given as the variation of the action $S_{\text{EH}}$ of gravity with respect to the metric tensor $g^{\mu\nu}$. If the total action is the sum of $S_{\text{EH}}$ and the action $2\kappa S_m$ including all relevant matter fields, then the variation of the matter action $S_m$ should give the energy-momentum tensor as the source of the gravitational field,

$$\frac{2}{\sqrt{|g|}} \delta S_m = T_{\mu\nu}. \tag{4.54}$$

Since the presence of gravity implies a curved space-time, the replacements $\{\partial_\mu, \eta_{\mu\nu}, d^4x/\sqrt{|g|}\} \to \{\nabla_\mu, g_{\mu\nu}, d^4x/\sqrt{|g|}\}$ have to performed in $S_m$ before the variation is performed.

The tensor $T_{\mu\nu}$ defined by this equation is called dynamical energy-momentum stress tensor. In order to show that this rather bold definition makes sense, we have to prove that $\nabla_\mu (\delta S_m/\delta g^{\mu\nu}) = 0$, and we have to convince ourselves that this definition reproduces the standard results we know already.

**Conservation of the stress tensor** We start by proving that the dynamical energy-momentum tensor defined by Eq. (19.1) is locally conserved. We consider the change of the matter action under a variation of the metric,

$$\delta S_m = \frac{1}{2} \int_{\Omega} d^4x \sqrt{|g|} T_{\mu\nu} \delta g^{\mu\nu} = -\frac{1}{2} \int_{\Omega} d^4x \sqrt{|g|} T^{\mu\nu} \delta g_{\mu\nu}. \tag{4.55}$$

We allow infinitesimal but otherwise arbitrary coordinate transformations,

$$\bar{x}^\mu = x^\mu + \xi^\mu(x). \tag{4.56}$$

For the resulting change in the metric $\delta g_{\mu\nu}$ we can use Eqs. (3.30) and (3.31),

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{4.57}$$

We use that $T^{\mu\nu}$ is symmetric and that general covariance guarantees that $\delta S_m = 0$ for a coordinate transformation,

$$\delta S_m = -\int_{\Omega} d^4x \sqrt{|g|} T^{\mu\nu} \nabla_\mu \xi_\nu = 0. \tag{4.58}$$

Next we apply the product rule,

$$\delta S_m = -\int_{\Omega} d^4x \sqrt{|g|} (\nabla_\mu T^{\mu\nu}) \xi_\nu + \int_{\Omega} d^4x \sqrt{|g|} \nabla_\mu (T^{\mu\nu} \xi_\nu) = 0. \tag{4.59}$$

The second term is a four-divergence and thus a boundary term that we can neglect. The remaining first term vanishes for arbitrary $\xi_\mu$ only, if the energy-momentum tensor is conserved,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.60}$$

Hence the local conservation of energy-momentum is a consequence of the general covariance of the gravitational field equations, in the same way as current conservation follows from gauge invariance in electromagnetism. You should convince yourself that the dynamical energy-momentum stress tensor evaluated for the examples of the Klein-Gordon and the Maxwell field agrees with the symmetrized canonical stress tensor, cf. problem 4.10.
4. Appendix: Large extra dimensions and massive gravity

Large extra dimensions As mentioned in chapter 3, quantum corrections break the conformal invariance of string theory except we live in a world with \( n = 10 \) or 26 space-time dimensions. There are two obvious answers to this result: First, one may conclude that string theory is disproven by reality or, second, one may adjust reality. Consistency of the second approach with experimental data could be achieved, if the \( n - 4 \) dimensions are compactified with a sufficiently small radius \( R \), such that they are not visible in experiments sensible to wavelengths \( \lambda \gg R \).

Let us check what happens to a scalar particle with mass \( m \), if we add a fifth, compact dimension \( y \). The Klein-Gordon equation for a scalar field \( \phi(x^\mu, y) \) becomes

\[
(\Box_5 + m^2) \phi(x^\mu, y) = 0
\]

with the five-dimensional d’Alembert operator \( \Box_5 = \Box - \partial_y^2 \). The equation can be separated, \( \phi(x^\mu, y) = \phi(x^\mu) f(y) \), and since the fifth dimension is compact, the spectrum of \( f \) is discrete. Assuming periodic boundary conditions, \( f(x) = f(x + R) \), gives

\[
\phi(x^\mu, y) = \phi(x^\mu) \cos(\frac{n\pi y}{R}).
\]

The energy eigenvalues of these solutions are \( \omega_{k,n}^2 = k^2 + m^2 + (n\pi/R)^2 \). From a four-dimensional point of view, the term \((n\pi/R)^2\) appears as a mass term, \( m_n^2 = m^2 + (n\pi/R)^2 \).

Since we usually consider states with different masses as different particles, we see the five-dimensional particle as a tower of particles with mass \( m_n \) but otherwise identical quantum numbers. Such theories are called Kaluza-Klein theories, and the tower of particles Kaluza-Klein particles. If \( R \ll \lambda \), where \( \lambda \) is the length-scale experimentally probed, only the \( n = 0 \) particle is visible and physics appears to be four-dimensional.

Since string theory includes gravity, one often assumes that the radius \( R \) of the extra-dimensions is determined by the Planck length, \( R = 1/M_{Pl} = (8\pi G_N)^{1/2} \approx 10^{-34} \) cm. In this case it is difficult to imagine any observational consequences of the additional dimensions. Antoniadis asked in 1990 what happens if some of the extra dimensions are large,

\[
R_{1,\ldots,\delta} \gg R_{\delta + 1,\ldots,6} = 1/M_{Pl}.
\]

Since the \( 1/r^2 \) behaviour of the gravitational force is not tested below \( d \sim \) mm scales, one can imagine that large extra dimensions exists that are only visible to gravity: Relating the \( d = 4 \) and \( d > 4 \) Newton’s law \( F \sim \frac{m_1 m_2}{r^{d+2}} \) at the intermediate scale \( r = R \), we can derive the “true” value of the Planck scale in this model: Matching of Newton’s law in 4 and \( 4 + \delta \) dimensions at \( r = R \) gives

\[
F(r = R) = G_N \frac{m_1 m_2}{R^2} = \frac{1}{M_D^{2+\delta}} \frac{m_1 m_2}{R^{2+\delta}}.
\]

This equation relates the size \( R \) of the large extra dimensions to the true fundamental scale \( M_D \) of gravity in this model,

\[
G_N^{-1} = 8\pi M_{Pl}^2 = R^\delta M_D^{\delta + 2},
\]

while Newton’s constant \( G_N \) becomes just an auxiliary quantity useful to describe physics at \( r \gtrsim R \). (You may compare this to the case of weak interactions where Fermi’s constant

96
$G_F \propto g^2/m_W^2$ is determined by the weak coupling constant $g$ and the mass $m_W$ of the $W$-boson).

Next we ask, if $M_D \sim \text{TeV}$ is possible, i.e. if one may test such theories at accelerators as LHC? Inserting the measured value of $G_N$ and plus $M_D = 1$ TeV in Eq. (4.64) we find the required value for the size $R$ of the large extra dimension as

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>1</th>
<th>2</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>R/cm</td>
<td>$10^{13}$</td>
<td>0.1</td>
<td>$10^{-12}$</td>
</tr>
</tbody>
</table>

Thus the case $\delta = 1$ is excluded by the agreement of the dynamics of the solar system with 4-dimensional Newtonian physics. The cases $\delta \geq 2$ are possible, because Newton’s law is experimentally tested only for scales $r > \sim 1 \text{ mm}$.

**Massive gravity** Theories with extra dimensions contain often from our 4-dimensional point of view a Kaluza-Klein tower of massive gravitons. Such modified theories of gravity have found large interest since one may hope to find an alternative explanation for the accelerated expansion of the Universe.

A striking difference between the spin-2 and the spin-0 and 1 cases is that the limit $m \to 0$ of the massive spin-2 propagator and thus of the potential energy $V_{12}$ is not smooth: In problem 4.5, you are asked to derive the massive spin-2 propagator. As result, you should find

$$D_F^{\mu\nu;\rho\sigma}(k) = \frac{1}{2} \left( -\frac{2}{3} G^{\mu\nu} G^{\rho\sigma} + G^{\mu\rho} G^{\nu\sigma} + G^{\mu\sigma} G^{\nu\rho} \right) k^2 - m^2 + i\varepsilon,$$

where

$$G^{\mu\nu}(k) = -\eta^{\mu\nu} + k^\mu k^\nu/m^2$$

is the polarisation tensor for a massive spin-1 particle. Thus the nominator in the massive spin-2 propagator is as in the massless case a linear combination of the tensor products of two spin-1 polarisation tensors. However, the coefficients of the first term differ and thus the $m \to 0$ limit of the massive propagator does not agree with the massless case. In particular, the difference cannot be compensated by a rescaling of the coupling constant $\tilde{G}_N$, because it is not an overall factor: Imagine for instance that we determine the value of $\tilde{G}_N$ by calculating the potential energy of two non-relativistic sources like the Sun and the Earth. This requires in massive gravity—for an arbitrarily small graviton mass—a gravitational coupling constant $\tilde{G}_N$ a factor $3/4$ smaller than in the massless case. Having fixed $\tilde{G}_N$, we can predict the deflection of light by the Sun. Since the first term in the propagator couples the traces $T_\mu^\mu$ of two sources, it does not contribute to the deflection of light. As a result of the reduced coupling strength, the deflection angle of light by the Sun decreases by the same factor and any non-zero graviton mass would be in conflict with observations.

When this result was first derived in 1970, its authors explained this discontinuity by the different number of degrees of freedom in the two theories: Even if the Compton wavelength of a massive graviton is larger than the observable size of the Universe, and thus the Yukawa factor $\exp(-mr)$ indistinguishable from one, the additional spin states of a massive graviton may change physics. Two years later, Vainshtein realised that perturbation theory may break down in massive gravity and thus a calculation using one-graviton exchange is not reliable. More precisely, a theory of massive gravity contains an additional length scale $R_V = (GM/m^4)^{1/5}$ and for distances $r \ll R_V$ the theory has to be solved exactly.
4. Spin-1 and spin-2 fields

Summary of chapter

Tensor fields satisfy second-order differential equations and have mass dimension one. A bosonic field with spin \( s \) has a polarisation tensor of rank \( 2s \) and thus their propagators are even in the momentum \( k \). For a massive bosonic field, the polarisation tensors contain tensor products of \( k_\mu k_\nu \) and therefore its propagator scales as \( D_{\mu_1 \cdots \mu_s \nu_1 \cdots \nu_s}(k) \propto k^{2s-2} \). This implies that the divergences of loop diagrams aggravate for higher spin fields.

Massless spin one and two fields have only two, transverse degrees of freedom. A Lorentz invariant description for such fields is only possible, if the remaining number of non-physical degrees of freedom is redundant. This redundancy implies that fields connected by a gauge transformation are equivalent and describe the same physical system. In the case of photons, the gauge symmetry implies that they couple to a conserved current, in the case of gravitons that they couple to a conserved energy-momentum conservation tensor.

Further reading

Problems

4.1 Polarisation of massive spin-1 particle.
Determine the polarisation vectors \( \varepsilon_\mu^r(k) \) of a massive spin-1 particle for arbitrary \( k^\mu \).

4.2 Feynman propagator as sum over solutions.
Follow the steps from (2.29) to (2.33) in the scalar case for a massive spin-1 propagator.

4.3 Constrained systems ♥.
Calculate the Hamiltonian density of the photon field. Apply canonical quantisation and derive the Lorentz transformation property of the photon field operator.

4.4 Gravitational coupling.
Determine the coupling \( \lambda \) in \( \mathcal{L} = \lambda \eta_{\mu \nu} T_{\mu \nu} \) comparing (4.50) to Newton’s gravitational potential.

4.5 Massive spin-2 propagator.
Derive the propagator (4.65) of a massive spin-2 particle: Use the tensor method to write down the most general combination of tensors built from \( \eta_{\mu \nu} \) and \( k_\mu \) (or from \( G_{\mu \nu} \) and \( k_\mu \)), compatible with the symmetries and constraints on the polarisation tensor.

4.6 Helicity.
Show that the unphysical degrees of freedom of an electromagnetic wave transform as helicity 0, and of a gravitational wave as helicity 0 and 1.

4.7 Helicity.
Show that \( \hat{h} \) in \( \psi' = e^{-i\hat{h}a} \psi \) for a plane wave is the eigenvalue of the helicity operator \( J \cdot p / |p| \).

4.8 Scalar QED ♣.
4.8.1 Introduce in the Lagrangian (3.11) of a complex scalar field the interaction with photons via the minimal substitution, \( \partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \), and read off the interaction vertices.
4.8.2 Calculate the matrix element for “scalar Compton scattering”, \( \phi \gamma \to \phi \gamma \) and show that it is gauge invariant. [Try to simplify the matrix element choosing a “good” frame and polarisation vectors.]

4.9 The \( \phi \phi h_{\mu \nu} \) vertex.
Expand the action of a scalar particle coupled to gravity,
\[
S = \int d^4x \sqrt{|g|} \left( g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - \rho \lambda \right)
\]
with \( g_{\mu \nu} = \eta_{\mu \nu} + \lambda h_{\mu \nu} \) to first order in \( \lambda \). Find the Feynman rule for the \( \phi \phi h_{\mu \nu} \) vertex.

4.10 Dynamical stress tensor.
Show that the formula (19.1) for the stress tensor can be simplified to
\[ T_{\mu\nu} = 2\partial L / \partial g^{\mu\nu} - g_{\mu\nu} L. \]

Derive the dynamical energy-momentum stress tensor \( T_{\alpha\beta} \) of a real scalar field \( \phi \) and of the photon field \( A^\mu \).
5. Fermions and the Dirac equation

Up to now we have discussed fields which transform as tensors under Lorentz transformations and have even spin or helicity. Since we can boost massive particles into their rest-frame, we can use our knowledge of non-relativistic quantum mechanics to anticipate that additional representations of the rotation group SO(3) and thus also of the Lorentz group SO(1,3) exist which correspond to particles with half-integer spin. Such particles are described by anti-commuting variables which is the fundamental reason for the Pauli principle, or the stability of matter.

5.1. Spinor representation of the Lorentz group

In order to introduce spinors we have to find the corresponding representation of the Lorentz group. As always it is simpler to work at “linear order,” which is in this case the Lie algebra.

The Lie algebra of the Poincaré group\(^1\) contains ten generators, the three generators \(J\) of rotations, the three generators \(K\) of Lorentz boosts and the four generators for translations. The Killing vector fields \(V\) of Minkowski space generate these symmetries, and therefore the generators \(T\) at each point are given by the Killing vector fields. Thus we can use (3.41) and (3.42) to calculate their commutation relations as (cf. problem 5.1)

\[
\begin{align*}
[J_i, J_j] &= i\varepsilon_{ijk}J_k, \\
[J_i, K_j] &= i\varepsilon_{ijk}K_k, \\
[K_i, K_j] &= -i\varepsilon_{ijk}J_k.
\end{align*}
\]

(5.1a, 5.1b, 5.1c)

Here we followed physicist’s convention and identified \(iT = V\), so that the generators are hermitian. Moreover, we restrict our attention to the Lorentz group which is sufficient to derive the concept of a Weyl spinor. Note that the algebra of the boost generators \(K\) is not closed. Thus in contrast to rotations, boosts do not form a subgroup of the Lorentz group. The fact that the commutator of two (not parallel) boosts involves a rotation,

\[
e^{iK_x\delta\psi}e^{iK_y\delta\eta}e^{-iK_x\delta\psi}e^{-iK_y\delta\eta} = 1 - [K_x, K_y]\delta\psi\delta\eta + \ldots = 1 + iJ_z\delta\psi\delta\eta + \ldots
\]

(5.2)

is the origin of the Thomas precession.

The commutation relations above are satisfied by \(\pm i\sigma/2\), suggesting that we can rewrite the Lorentz group as a product of two SU(2) factors. We try to decouple the two sets of generators \(J\) and \(K\) by introducing two non-hermitian ladder operators

\[
J^\pm = \frac{1}{2}(J \pm iK).
\]

(5.3)

\(^1\)See Appendix B.3 for a brief review of the Lorentz group.
5.1. Spinor representation of the Lorentz group

Their commutations relations are

\[ [J^+, J^+_j] = i \epsilon_{ijk} J^+_k, \]  
\[ [J^-, J^-_j] = i \epsilon_{ijk} J^-_k, \]  
\[ [J^+_j, J^-_j] = 0 \quad i, j = 1, 2, 3. \]  

Thus \( J^- \) and \( J^+ \) commute with each other and generate each a SU(2) group. The Lorentz group is therefore \( \sim SU(2) \otimes SU(2) \), and states transforming in a well-defined way are labeled by a pair of two angular momenta, \((j^-, j^+)\), corresponding to the eigenvalues of \( J^-_z \) and \( J^+_z \), respectively. From our knowledge of the angular momentum algebra in nonrelativistic quantum mechanics, we conclude that the dimension of the representation \((j^-, j^+)\) is \((2j^- + 1)(2j^+ + 1)\). Because of \( J^+ = J^- + J^+ \), the representation \((j^-, j^+)\) contains all possible spins \( j \) in integer steps from \(|j^- - j^+|\) and \( j^- + j^+ \).

The representation \((0, 0)\) has dimension one, transforms trivially, \( J = K = 0 \), and corresponds therefore to the scalar representation. The smallest non-trivial representations are \( J^+ = 0 \), i.e. \((j^-, 0)\) with \( J^{(1/2)} = i K^{(1/2)} \), and \( J^- = 0 \), i.e. \((0, j^+)\) with \( J^{(1/2)} = -i K^{(1/2)} \). Both representations have spin 1/2 and dimension two. We define therefore two types of two-component spinors,

\[ \phi_L : \quad (1/2, 0), \quad J^{(1/2)} = \sigma/2, \quad K^{(1/2)} = +i \sigma/2, \]  
\[ \phi_R : \quad (0, 1/2), \quad J^{(1/2)} = \sigma/2, \quad K^{(1/2)} = -i \sigma/2, \]  

which we call left-chiral and right-chiral Weyl spinors. These Weyl spinors form the fundamental representation of the Lorentz group: All higher spin states can obtained as tensor products involving these two spinors. Their transformation properties under a finite Lorentz transformation with parameters \( \alpha \) and \( \eta \) follow by exponentiating their generators as \( \exp(-i J \alpha + i K \eta) \) (compare \[\underline{\text{4.3}}\] for our choice of signs),

\[ \phi_L \rightarrow \phi'_L = \exp \left[ -i \sigma \alpha \frac{\sigma \eta}{2} \right] \phi_L \equiv S_L \phi_L, \]  
\[ \phi_R \rightarrow \phi'_R = \exp \left[ -i \sigma \alpha \frac{\sigma \eta}{2} \right] \phi_R \equiv S_R \phi_R. \]  

We defined in section \[\underline{\text{4.3}}\] the helicity \( h \) of a particle as the phase \( e^{-ih\alpha} \) gained by a plane wave which is rotated the angle \( \alpha \) around its propagation axis. Thus both Weyl spinors have helicity \( h = 1/2 \). While the two types of Weyl spinors transform the same way under rotations their behavior under Lorentz boosts is opposite.

We ask now if can convert a a left- into a right-chiral spinor and vice versa. Thus we should find a spinor \( \tilde{\phi}_L \) constructed out of \( \phi_L \) which transforms as \( S_R \tilde{\phi}_L \). Changing \( S_L \) into \( S_R \) requires reversing the relative sign between the rotation and the boost term, which we achieve by complex conjugating \( \phi_L \),

\[ \tilde{\phi}_L' = \left[ 1 + \frac{i \sigma^+ \alpha}{2} - \frac{\sigma^+ \eta}{2} + \ldots \right] \tilde{\phi}_L. \]

\(^2\) More precisely, they have the same Lie algebra and are thus locally isomorphic but differ globally.
Because of $\sigma_1^L = \sigma_1, \sigma_2^L = -\sigma_2, \sigma_3^L = \sigma_3,$ and $\sigma_1^R = -\sigma_2^R, \sigma_2^R = -\sigma_3^R, \sigma_3^R = -\sigma_1^R,$ we obtain the desired transformation property multiplying $\phi_L^*$ with $\sigma_2,$

$$
\sigma_2 \phi_L^* = \sigma_2 \left[ 1 + \frac{i(\sigma_1^L,-\sigma_2^L,\sigma_3^L)\alpha}{2} - \frac{(\sigma_1^L,-\sigma_2^L,\sigma_3^L)\eta}{2} \right] \phi_L^* \tag{5.10}
$$

Thus the two representations are inequivalent, i.e. they are not connected by a unitary transformation, and therefore $\phi_L$ and $\phi_R$ describe different physics. Obviously, we can add to $\sigma_2 \phi_L^*$ an arbitrary phase $e^{i\theta}$ without changing the transformation properties. We define

$$
\phi_R^c \equiv -i \sigma_2 \phi_L^*, \tag{5.12}
$$

which ensures that $(\phi_R^c)^* = \phi_L.$ When we discuss later the coupling of a fermion to an external field, we will see that $\phi_R^c$ is the charge conjugated spinor of $\phi_L.$

The transformations $S_L$ and $S_R$ that belong to the orthochronous Lorentz group do not mix the $L$ and $R$ Weyl spinors. Consider however the effect of parity, $P\mathbf{x} = -\mathbf{x},$ on the generators $K$ and $J.$ The velocity changes sign, $\mathbf{v} \rightarrow -\mathbf{v},$ i.e. transforms as a polar vector, while the angular momentum $J$ as axial vector remains invariant. Thus parity interchanges $(1/2,0)$ and $(0,1/2)$ and hence $\phi_L$ and $\phi_R,$ as one would expect from a left- and right-chiral object. If parity is a symmetry of the theory examined, one can therefore not consider separately the two spinors $\phi_L$ and $\phi_R.$ Instead, it proves useful to combine them into a four-spinor (Dirac or bi-spinor)

$$
\psi = \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix}. \tag{5.13}
$$

For the construction of a Lagrangian we need for the mass term scalars and for the kinetic energy vectors built out of the Weyl fields; they should be real to provide a real Lagrangian. In contrast to the real Lorentz transformation $N_{\nu}^\mu$ acting on tensor fields, the matrices $S_{L/R}$ are however complex and thus the Weyl fields are complex too. This suggests together with the fact that a measurement device should be the same after a rotation by $2\pi$ that observables are bilinear quantities in the fermion fields, such that they transform tensorial and their eigenvalues are real.

Out of the two Weyl spinors, we can form four different products $\phi_{L/R}^\dagger \phi_{L/R}$ leading to the combinations $S_{L/S}^L S_{R/S}, S_{L/S}^R S_{L/S}^L, S_{R/S}^L S_{R/S}^R.$ The rotation $i\sigma_3/2$ cancels in all four products, since it enters with the same sign in $S_L$ and $S_R,$ and the Pauli matrices are hermitian, $\sigma^\dagger = \sigma.$ By contrast, the cancelation of the boost $\sigma\eta/2$ requires a combination of a $L$ and $R$ field,

$$
\phi_{L/R}^\dagger \phi_{L/R} = \phi_{L/R}^\dagger \left[ 1 + i \frac{\sigma_3}{2} \right] \phi_{L/R} = \phi_{L/R}^\dagger \phi_{L/R}, \tag{5.14}
$$

and similarly for $\phi_{L/R}^\dagger \phi_{L/R}.$ Thus $\phi_{L/R}^\dagger \phi_{L/R}$ and $\phi_{L/R}^\dagger \phi_{L/R}$ transform as Lorentz scalars, but not $\phi_{L/R}^\dagger \phi_{L/R}$ and $\phi_{L/R}^\dagger \phi_{L/R}.$ So what are the transformation properties of the latter two products? Performing an infinitesimal boost along the $z$ axis, we find

$$
\phi_{L/R}^\dagger \phi_{L/R} = \phi_{L/R}^\dagger \left[ 1 + \frac{\sigma_3 \eta}{2} \right] \phi_{L/R} = \phi_{L/R}^\dagger \phi_{L/R} + \eta \phi_{L/R} \sigma_3 \phi_{L/R}. \tag{5.15}
$$

This looks like an infinitesimal Lorentz transformation of the time-like component $j^0 = \phi_{L/R}^\dagger \phi_{L/R}$ of a four-vector $j^\mu,$ if we can associate the spatial part $\mathbf{j}$ with $\phi_{L/R}^\dagger \sigma \phi_{L/R}.$ Checking how $\mathbf{j}$
5.2. Dirac equation

We can obtain the spinor \( \phi_{L/R}(p) \) describing a particle with momentum \( p \) by boosting the one describing a particle at rest, \( \phi_{L/R}(0) \),

\[
\phi_R(p) = \exp \left[ \frac{\sigma \eta}{2} \right] \phi_R(0) = \exp \left[ \frac{\eta \sigma n}{2} \right] \phi_R(0) = \left[ \cosh(\eta/2) + \sigma n \sinh(\eta/2) \right] \phi_R(0). \tag{5.17}
\]

If we replace the boost parameter \( \eta \) by the Lorentz factor \( \gamma = \cosh \eta \) and use the identities \( \cosh(\eta/2) = \sqrt{(\cosh \eta + 1)/2} \) and \( \sinh(\eta/2) = \sqrt{(\cosh \eta - 1)/2} \), we can express the spinor as

\[
\phi_R(p) = \left[ \left( \frac{\gamma + 1}{2} \right)^{1/2} + \sigma \hat{p} \left( \frac{\gamma - 1}{2} \right)^{1/2} \right] \phi_R(0). \tag{5.18}
\]

Here \( \hat{p} = p/|p| \) is the unit vector in direction of \( p \). Inserting \( \gamma = E/m \) and combining the two terms in the angular bracket, we arrive at

\[
\phi_R(p) = \frac{E + m + \sigma p}{\sqrt{2m(E + m)}} \phi_R(0). \tag{5.19}
\]

Similarly, we find

\[
\phi_L(p) = \frac{E + m - \sigma p}{\sqrt{2m(E + m)}} \phi_L(0). \tag{5.20}
\]

Thus the \( L \) and \( R \) spinors differ only by the sign of the helicity operator \( \sigma p \) which measures the projection of the spin \( \sigma \) on the momentum \( p \) of the particle. For a particle at rest, this difference disappears and we set therefore \( \phi_L(0) = \phi_R(0) \). This allows us to eliminate the zero momentum spinors, giving

\[
\phi_L^R(p) = \frac{E + \sigma p}{m} \phi_R^L(p). \tag{5.21}
\]

In matrix form, these two equations correspond to

\[
\begin{pmatrix}
-m & E - \sigma p \\
E + \sigma p & -m
\end{pmatrix}
\begin{pmatrix}
\phi_L(p) \\
\phi_R(p)
\end{pmatrix} = 0.	ag{5.22}
\]

We introduced the \( 4 \times 4 \) matrices

\[
\gamma^\mu = \begin{pmatrix}
0 & \sigma^\mu \\
\sigma^\mu & 0
\end{pmatrix}. \tag{5.23}
\]

\(^3\)Recall the relations \( E = m \cosh \eta \) and \( p = m \sinh \eta \) connecting \( E, p, \) and the rapidity \( \eta \).

\(^4\)Some authors [Ryd96] use \( \gamma^I \rightarrow -\gamma^I \) and exchange \( \phi_L \) and \( \phi_R \) in the four-spinor.
Then we arrive with $\gamma^\mu p_\mu = \gamma^0 E - \gamma p$ at the compact expression

$$(\gamma^\mu p_\mu - m) \psi(p) = 0.$$  \hspace{1cm} (5.24)

Identifying $p_\mu = i\partial_\mu$ we obtain the Dirac equation.

The representation used for the Dirac spinor and the gamma matrices is called chiral or Weyl representation. Others can be obtained by a unitary transformation, $U\tilde{\gamma}^\mu U^\dagger = \gamma^\mu$ and $U\tilde{\psi} = \psi$.

**Dirac’s way towards the Dirac equation**  The Klein-Gordon equation was historically the first wave equation derived in relativistic quantum mechanics. Applied to the hydrogen atom, it failed to reproduce the correct energy spectrum. Dirac tried to derive as an alternative an equation linear in the derivatives $\partial_\mu$. Since Lorentz invariance requires that $\partial_\mu$ has to be contracted with another four-vector, a first order equation has the form

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$  \hspace{1cm} (5.25)

Main task for Dirac was to uncover the nature of the quantities $\gamma^\mu$ in this equation. They cannot be normal numbers, since then they would form a four-vector, specify one direction in space-time and thus break Lorentz invariance.

We will follow now Dirac’s approach, which will lead us to a definition of the gamma matrices and their properties which is independent of the considered representation. We start multiplying the Dirac equation with $-(i\gamma^\mu \partial_\mu + m)$ and compare the result to the Klein-Gordon equation,

$$-(i\gamma^\mu \partial_\mu + m)(i\gamma^\nu \partial_\nu - m)\psi = (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = (\square + m^2)\psi = 0.$$ \hspace{1cm} (5.26)

Using the symmetry of partial derivatives, we can rewrite

$$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu.$$ \hspace{1cm} (5.27)

Remembering next the definition of the d’Alembert operator, $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$, we see that the $\gamma^\mu$ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$ \hspace{1cm} (5.28)

These anti-commutation relations define a Clifford algebra, implying

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1 \quad \text{and} \quad \gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$$ \hspace{1cm} (5.29)

for $\mu \neq \nu$. The last condition shows again that the $\gamma^\mu$ cannot be normal numbers. Note that $\gamma^0$ is hermetian, while the $\gamma^i$ are anti-hermetian,

$$\gamma^0 = \gamma^{0\dagger} \quad \text{and} \quad \gamma^i = -\gamma^{i\dagger}.$$ \hspace{1cm} (5.30)

One can readily check that the $\gamma^\mu$ matrices in the Weyl representation satisfy these relations.

The definition (5.28) implies that we can apply in the usual way the metric tensor to raise or to lower the indices of the gamma matrices, $\gamma_\mu = \eta_{\mu\nu} \gamma^\nu$. Thus we can write $\gamma^\nu \partial_\nu = \gamma_\nu \partial^\nu$. Since the contraction of the gamma matrices $\gamma^\mu$ with a four-vector $A^\mu$ will appear frequently, we introduce the so-called Feynman slash,

$$A \equiv A_\mu \gamma^\mu.$$ \hspace{1cm} (5.31)
as useful shortcut. This notation also stresses that the gamma matrices \( \gamma^\mu \) allow us to map a four-vector \( A^\mu \) on an element \( \mathcal{A} \) of the Clifford algebra which then can be applied on a spinor \( \psi \). Although we suppress the spinor indices, you should keep in mind that the matrices \((\gamma_{ab})^\mu\) carry both tensor and spinor indices. Thus the gamma matrices transform not only as a vector under Lorentz transformations, but additionally the two indices \( a \) and \( b \) should transform according to the spinor representation of the Lorentz group.

**Clifford algebra and bilinear quantities** We now determine the minimal matrix representation for the Clifford algebra defined by Eq. (5.28). Thus we should find the number of independent products that we can form out of the four gamma matrices. Five obvious elements are the unit matrix \( 1 = (\gamma^0)^2 \) and the four gamma matrices \( \gamma^\mu \) themselves. Because of \((\gamma^\mu)^2 = \pm 1\), the remaining products should consist of \( \gamma^\mu \) matrices with different indices. Thus the only product of four \( \gamma^\mu \) matrices that we have to consider is \( \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). This combination will appear from now on very often and deserves therefore a special name. Including the imaginary unit to make it hermetian, we define

\[
\gamma^5 \equiv \gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3. \tag{5.32}
\]

Because the four gamma-matrices in \( \gamma^5 \) anti-commute, we can rewrite its definition introducing the completely anti-symmetric tensor \( \varepsilon_{\alpha\beta\gamma\delta} \) in four dimensions as

\[
\gamma^5 = \frac{i}{24} \varepsilon_{\alpha\beta\gamma\delta} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta. \tag{5.33}
\]

This suggests that bilinear quantities containing one \( \gamma^5 \) matrix transform as pseudo-tensors, i.e. change sign under a parity transformation \( \mathbf{x} \rightarrow -\mathbf{x} \). Two important properties of the \( \gamma^5 \) matrix are \((\gamma^5)^2 = 1\) and \(\{\gamma^\mu, \gamma^5\} = 0\).

Next we consider products of three \( \gamma^\mu \) matrices. For instance,

\[
\gamma^1 \gamma^2 \gamma^3 = \gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i \gamma^0 \gamma^5. \tag{5.34}
\]

Hence these products are equivalent to \( \gamma^\mu \gamma^\nu \), giving us four more basis elements.

Finally, we are left with products of two \( \gamma^\mu \) matrices. We start from the defining equation (5.28),

\[
\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu + 2\eta^\mu\nu, \tag{5.35}
\]

divide by two, add \( \gamma^\mu \gamma^\nu / 2 \) and introduce the commutator of two gamma matrices,

\[
\gamma^\mu \gamma^\nu = \frac{1}{2} \gamma^\mu \gamma^\nu - \frac{1}{2} \gamma^\nu \gamma^\mu + \eta^\mu\nu = \frac{1}{2} [\gamma^\mu, \gamma^\nu] + \eta^\mu\nu. \tag{5.36}
\]

Adding again for later convenience an imaginary unit, we define the anti-symmetric tensor \( \sigma^{\mu\nu} \) as

\[
\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu]. \tag{5.37}
\]

The six matrices \( \sigma^{\mu\nu} \) are the remaining basis elements of the four-dimensional Clifford algebra. All together, the basis has dimension 16,

\[
\Gamma = \{1, \gamma^5, \gamma^\mu, \gamma^5 \gamma^\mu, \sigma^{\mu\nu}\}, \tag{5.38}
\]
5. Fermions and the Dirac equation

as the $4 \times 4$ matrices. Hence an arbitrary $4 \times 4$ matrix can be decomposed into a linear combination of the 16 basis elements of the Clifford algebra. Moreover, the smallest matrix representation of the Clifford algebra are $4 \times 4$ matrices. Some useful properties of gamma matrices are collected in the Appendix A.3.

Knowing the dimension of the $\gamma$ matrices, we can count the number of degrees of freedom represented by a Dirac spinor $\psi$. As the $\gamma$ matrices and the Lorentz transformation acting on spinors are complex, the field $\psi$ is complex too and has thus four complex degrees of freedom. We know already that the Dirac equation describes spin $1/2$ particles, which come with $2s + 1 = 2$ spin degrees of freedom for a particle plus 2 for its anti-particle. Thus in this case the number of physical states matches the four components of the fields $\psi$. Note also the difference to the case of a complex scalar or vector field: There we introduced two complex fields $\phi^\pm = (\phi_1 \pm i\phi_2)/\sqrt{2}$, which are connected by $(\phi^\pm)^* = \phi^\mp$. The real fields $\phi_1$ and $\phi_2$ are not mixed by Lorentz transformations and thus we count two real degrees of freedom.

We come now to the construction of bilinear quantities out of the Dirac spinors. It is tempting to try e.g. as scalar $\psi^\dagger \psi$ and as vector current $j^\mu = \psi^\dagger \gamma^\mu \psi$. Using the chiral representation, we find however

$$\psi^\dagger \psi = \begin{pmatrix} \phi_L^\dagger, & \phi_R^\dagger \end{pmatrix} \begin{pmatrix} \phi_L \phi_R \phi_L + \phi_R \phi_R \end{pmatrix},$$

while we learnt that

$$\phi_L^\dagger \phi_R + \phi_R^\dagger \phi_L = \psi^\dagger \gamma^0 \psi$$

is Lorentz invariant. Similarly, $\psi^\dagger \gamma^\mu \psi$ is neither Lorentz invariant nor real. The latter problem arises, because $\gamma^0$ is hermitian and the $\gamma^i$ are anti-hermitian. We can express the (anti-) hermitian property of the gamma matrices succinctly as

$$\gamma^\mu^\dagger = \gamma^0 \gamma^\mu \gamma^0 = \begin{cases} (\gamma^0)^2 \gamma^0 = \gamma^0, \\ -\gamma^i (\gamma^0)^2 = -\gamma^i. \end{cases}$$

Both problems are solved, if bilinear quantities are constructed as

$$\psi^\dagger \gamma^0 \Gamma \psi \equiv \bar{\psi} \Gamma \psi,$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ is called the adjoint spinor to $\psi$ and $\Gamma$ is any of the 16 basis elements given in Eq. (5.38). In this way, the complex conjugated of a bilinear becomes

$$(\bar{\psi} \Gamma \psi')^* = (\bar{\psi} \Gamma \psi')^\dagger = \psi'\dagger \Gamma^\dagger \gamma^0 \psi = \psi'\dagger \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 \psi = \bar{\psi} \Gamma \psi$$

with

$$\Gamma = \gamma^0 \Gamma^\dagger \gamma^0.$$

The analogue $\psi^\dagger \psi$ to the probability density $\psi^* \psi$ of the Schrödinger equation is thus the zero-component of a four-current, $\psi^\dagger \psi = \psi^\dagger \gamma^0 \gamma^0 \psi = \bar{\psi} \gamma^0 \psi = j^0$, as one should expect in a relativistic theory.

**Solutions** We search for plane wave solutions $u e^{-ipx}$ and $v e^{ipx}$ of the Dirac equation with $m > 0$ and $E = p^0 = |\mathbf{p}| > 0$. The algebra is simplified, if we construct the solutions first in the rest frame of the particle. Then $\mathbf{p} = m\gamma^0$, and thus the use of the Dirac representation,

$$\gamma^0 = 1 \otimes \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \gamma^i = \sigma^i \otimes i\tau_2 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

(5.43)
5.2. Dirac equation

\[ \gamma^5 = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]  

(5.44)

where \( \gamma^0 \) is diagonal is most convenient. Here \( \sigma_i \) and \( \tau_i \) are the Pauli matrices, \( \otimes \) denotes the tensor product, 0 and 1 are \( 2 \times 2 \) matrices. The Dirac equation becomes

\[ (\not{p} - m)u = m(\gamma^0 - 1)u = 0 \]  

(5.45a)

\[ (\not{p} + m)v = m(\gamma^0 + 1)v = 0. \]  

(5.45b)

Since \((\gamma^0 \pm 1)^2 = 2(1 \pm \gamma^0)\), we see that \((1 \pm \gamma^0)/2\) are projection operators \( P_{\pm} \), satisfying

\[ P_{\pm}^2 = P_{\pm}, \quad P_{\pm}P_{\mp} = 0, \quad \text{and} \quad P_+ + P_- = 1. \]

Inserting \( \gamma^0 \) into (5.44), the four solutions follow as

\[ u(m, +) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u(m, -) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v(m, -) = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v(m, +) = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]  

(5.46)

The additional \( s = \pm 1 \) label should be the quantum number of a suitable operator labelling the two spin states of a Dirac particle. Note the opposite order of the spin label in the \( v \) spinor compared to \( u \). We will see later that this choice is required by the structure of the relativistic spin operator \( s^\mu \). As an intuitive argument, we add that this labelling corresponds to our interpretation of antiparticles as particles moving backwards in time: The spinor \( v \) describes two states with negative energy, negative 3-momentum \( p \) and negative spin \( s \) relative to \( u \).

The solutions are orthogonal,

\[ \bar{u}(p, s)u(p, s') = N^2 \delta_{s, s'} \quad \text{and} \quad \bar{v}(p, s)v(p, s') = -N^2 \delta_{s, s'}, \]  

(5.47)

but not normalised to one. Note also the minus sign introduced in \( \bar{v}v \) by the corresponding minus in the \((3,4)\) corner of \( \gamma^0 \). Since we know that \( \bar{\psi} \psi \) is the zero component of a four-vector, the normalisation of the corresponding spinor products is

\[ u^\dagger(p, s)u(p, s') = N^2 \frac{E_p}{m} \delta_{s, s'} \quad \text{and} \quad v^\dagger(p, s)v(p, s') = -N^2 \frac{E_p}{m} \delta_{s, s'}. \]  

(5.48)

Summing over spins, we obtain in the rest frame

\[ \sum_s u_a(m, s)\bar{u}_b(m, s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{ab} N^2 = \frac{1}{2}(\gamma^0 + 1)_{ab} N^2, \]  

(5.49)

\[ \sum_s v_a(m, s)\bar{v}_b(m, s) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}_{ab} N^2 = \frac{1}{2}(\gamma^0 - 1)_{ab} N^2. \]  

(5.50)

We saw that \( \gamma^0 \pm 1 \) corresponds in an arbitrary frame to \((\not{p} \mp m)/m\). Thus in general these relations become

\[ \Lambda_+ \equiv \sum_s u_a(p, s)\bar{u}_b(p, s) = N^2 \left( \frac{\not{p} + m}{2m} \right)_{ab}, \]  

(5.51)

\[ \Lambda_- \equiv -\sum_s v_a(p, s)\bar{v}_b(p, s) = N^2 \left( -\frac{\not{p} + m}{2m} \right)_{ab}, \]  

(5.52)
where we defined $\Lambda_{\pm}$ as the projection operator on states with positive and negative energy, respectively.

The two most common normalisation conventions for the Dirac spinors are $\mathcal{N} = \sqrt{2m}$ and $\mathcal{N} = 1$. We will use the former, $\mathcal{N} = \sqrt{2m}$, which has two advantages: First, spurious singularities in the limit $m \to 0$ disappear. Second, the normalisation of fermion states and thus also the phase space volume becomes identical to the one of bosons.

The solutions of the Dirac equation for an arbitrary frame can be simplest obtained remembering $(\slashed{p} - m)(\slashed{p} + m) = p^2 - m^2 = 0$, i.e.

$$u(p, \pm) = \frac{\slashed{p} + m}{\sqrt{2m(m + E)}} u(0, \pm) \quad \text{and} \quad v(p, \pm) = \frac{-\slashed{p} + m}{\sqrt{2m(m + E)}} v(0, \pm). \quad (5.53)$$

Here, the normalisation was fixed using (5.47).

**Spin** We have seen that the Dirac equation describes particle with spin $s = 1/2$. Thus the $\pm$ degeneracy of the $u$ and $v$ spinors should correspond to the different spin states of a $s = 1/2$ particle. We introduce the spin operator

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad (5.54)$$

as an obvious generalisation of the non-relativistic spin matrices. This operator has the eigenvalues $\Sigma_{\pm} u(m, \pm) = \pm u(m, \pm)$ and $\Sigma_{\pm} v(m, \pm) = \pm v(m, \pm)$ and can therefore be used to classify the spin states of a Dirac particle in the rest frame, where $[H_D, \Sigma_z] \propto [\gamma^0, \Sigma_z] = 0$. Note however that $[H_D, \Sigma_z] \neq 0$ for $p^2 \neq m^2$, and thus the eigenvalue of $\Sigma_z$ is not conserved for a moving particle. This comes not as a surprise, because the total angular momentum $L + s$ and not only the spin $s$ should be conserved.

We are looking now for the relativistic generalisation of the three-dimensional spin operator $\Sigma$. It should be a product of gamma matrices which contains in the rest frame of the particle $\sigma$ in the diagonal. We note first that $\gamma^5 \sigma^i = (0 \ 0 \ 0 \ 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has the required structure. Then we define the spin vector $s^\mu$ with the properties $s^2 = -1$, $s^\mu = (0, s)_{\mu = p = m}$ and thus $s \cdot p = 0$. Since

$$\gamma^5 \not{s}_{\mu = m = p} = \gamma^5 s^i \gamma^i = \begin{pmatrix} \sigma s & 0 \\ 0 & -\sigma s \end{pmatrix}, \quad (5.55)$$

we see that $\gamma^5 \not{s}$ measures in the rest frame the projection of the spin along the chosen axis $s$. Moreover, $\gamma^5 \not{s}$ commutes with the Dirac Hamiltonian, $[\gamma^5 \not{s}, \not{p}] = 0$, and has, because of $(\gamma^5 \not{s})^2 = 1$, as eigenvalues $\pm 1$. If we apply $\gamma^5 \not{s}$ on the spinor $v$ in the rest frame,

$$\gamma^5 \not{s} v(m, s) = \begin{pmatrix} \sigma s & 0 \\ 0 & -\sigma s \end{pmatrix} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ -\sigma s \chi \end{pmatrix} = s_1 \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \quad (5.56)$$

we see that $\chi_1$ has the eigenvalue $s_1 = -1$, while $\chi_2$ has the eigenvalue $s_2 = +1$. This explains the “wrong” order of the two spin states of $v(p, s)$ in (5.46). Finally, we can define a projection operator on a definite spin state by

$$\Lambda_s = \frac{1}{2} (1 + \gamma^5 \not{s}). \quad (5.57)$$

Thus we can obtain from an arbitrary Dirac spinor $\psi$ a state with definite sign of the energy and spin by applying the two projection operators $\Lambda_{\pm}$ and $\Lambda_s$. 

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5.2. Dirac equation

**Hamiltonian form** The Dirac equation can be transformed into Hamiltonian form by multiplying with $\gamma^0$,

$$i\partial_t \psi = H_D \psi = (-i\gamma^0\gamma^i \partial_i + \gamma^0 m)\psi.$$  \hfill (5.58)

Looking back at the (anti-) hermiticity properties \[^{[5.30]}\] of the $\gamma^\mu$ matrices, we see that they correspond to the one required to make the Dirac Hamiltonian hermitian. By tradition, one re-writes $H_D$ often with $\beta = \gamma^0$ and $\alpha^i = \gamma^0\gamma^i$ as

$$i\partial_t \psi = H_D \psi = (\alpha \cdot p + \beta m)\psi.$$  \hfill (5.59)

Considering the semi-classical limit, one sees that the matrix $\alpha$ has the meaning of a velocity operator, see problem ??.

**Lagrange density** For a scalar field, we could rewrite after a partial integration the Lagrange density as $\mathcal{L} = -\phi^4(\Box + m^2)\phi$. More generally, the definition of the propagator as two-point function requires that the Lagrange density can be expressed as $\mathcal{L} \propto \bar{\Psi} D \Psi$ for a field $\Psi$ satisfying $D \Psi = 0$ with the differential operator $D$. This suggests to try for the Dirac field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$  \hfill (5.60)

with $\psi$ and $\bar{\psi}$ as independent variables. The Lagrange equation for $\bar{\psi}$ gives trivially the Dirac equation,

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\psi}} - \frac{\delta \mathcal{L}}{\delta \bar{\psi}} = (i\gamma^\mu \partial_\mu - m)\psi = 0,$$  \hfill (5.61)

while the Lagrange equation for $\psi$ gives the adjoint equation for $\bar{\psi}$.

Using the gamma matrices and the Dirac spinor in the chiral representation it is straightforward to express the Dirac Lagrangian \[^{[5.60]}\] by Weyl fields,

$$\mathcal{L} = i\phi_R^\dagger \sigma^\mu \partial_\mu \phi_R + i\phi_L^\dagger \bar{\sigma}^\mu \partial_\mu \phi_L - m(\phi_L^\dagger \phi_R + \phi_R^\dagger \phi_L).$$  \hfill (5.62)

This implies that the Dirac equation is invariant under Lorentz transformations, because we have already checked that all ingredients of \[^{[5.62]}\] are invariant. Note also that out of the two possible combinations of the two Lorentz scalars we found, only the one invariant under parity, $P\phi_L = \phi_R$, entered the mass term. Moreover, $P(\sigma^\mu \partial_\mu) = \bar{\sigma}^\mu \partial_\mu$, and thus the combination of the kinetic energies of $\phi_L$ and $\phi_R$ is also invariant under parity.

Keeping in mind that $[\mathcal{L}] = m^4$, we see that the dimension of a fermion field in four dimension is $[\psi] = m^{3/2}$. Thus we can order possible couplings of a fermion to spin-1 particles according to their dimension as

$$\mathcal{L}_I = c_1 A_\mu \bar{\psi} \gamma^\mu \psi + c_2 A_\mu \bar{\psi} \gamma^5 \gamma^\mu \psi + \frac{c_3}{M} F_{\mu\nu} \bar{\psi} \sigma^{\mu\nu} \psi + \frac{c_4}{M} F_{\mu\nu} \bar{\psi} \gamma^5 \sigma^{\mu\nu} \psi + \ldots,$$  \hfill (5.63)

where the coupling constants $c_i$ are dimensionless and we introduced the mass scale $M$.

**Feynman propagator** The Green functions of the Dirac equation are defined by

$$(i\partial_t - m)S(x,x') = \delta(x-x').$$  \hfill (5.64)

Translation invariance implies $S(x,x') = S(x-x')$ and thus it is again convenient to perform a Fourier transformation. Then the Fourier components $S(p)$ have to obey

$$(\phi - m)S(p) = 1.$$  \hfill (5.65)
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After multiplication with \( \dot{p} + m \), we can solve for the propagator in momentum space,

\[
iS_F(p) = \frac{\dot{p} + m}{p^2 - m^2 + i\varepsilon} = \frac{i}{\dot{p} - m + i\varepsilon},
\]

where the last step is only meant as a symbolical short-cut. Here, we chose again with the \(+i\varepsilon\) prescription the causal or St"uckelberg-Feynman propagator for the electron and, more generally, for spin 1/2 particles. Note also the connection to the scalar propagator \( \Delta_F \).

Example: Feynman propagator as sum over solutions:

We follow the steps from (2.29) to (2.33) in the scalar case, finding now

\[
S_F(x) = i \int \frac{d^3p}{(2\pi)^3} \frac{(\dot{p} + m)e^{-ip \cdot x}}{2\pi (p_0 - E_p + i\varepsilon)(p_0 + E_p - i\varepsilon)}.
\]

Now we change as in the bosonic case the integration variable as \( p \to -p \) in the second term,

\[
iS_F(x) = \int d^3p \left[ (\dot{p} + m)e^{-iE_p t - px} \dot{\vartheta}(t) + (-\dot{p} + m)e^{iE_p t + px} \dot{\vartheta}(-t) \right].
\]

Using finally (5.69), we arrive at

\[
iS_F(x) = \int \frac{d^3p}{2E_p(2\pi)^3} \sum_s \left[ u(p, s)\bar{\vartheta}(p, s)e^{-iE_p t - px} \dot{\vartheta}(x^0) - v(p, s)\bar{\vartheta}(p, s)e^{iE_p t + px} \dot{\vartheta}(-x^0) \right].
\]

We see that our choice for the normalisation of the Dirac spinors means that fermionic states are normalised as bosonic states, \( \langle p|p' \rangle = 2E_p(2\pi)^3\delta(p - p') \). The minus sign between the positive energy solution propagating forward in time and the negative energy solution propagating backward in time is a direct consequence of our \(-i\varepsilon\) prescription. This sign implies that fermionic fields anti-commute, and explains the Pauli exclusion principle and thus the stability of matter.

Hence this sign is clearly one of the more important ones. Let us have therefore a look back to understand what causes ultimately the minus sign in the fermion propagator. To simplify the discussion, we neglect the inessential mass term. In the positive frequency term \( (p^0, 0 - p^0)e^{ipx} \), we pick up a minus from the residuum, \( p^0 \to -E_p \), and a minus from the variable change, \( p \to -p \), resulting in \( \dot{p} \to -\dot{p} \), which gives combined

\[
S_F(x) \propto \dot{p} e^{ipx} \dot{\vartheta}(t) - \dot{p} e^{ipx} \dot{\vartheta}(-t).
\]

Thus the relative minus sign has its origin in the fact that the numerator of the fermion propagator is odd in the momentum, while a bosonic propagator is even. In turn, the fermion propagator is linear in the momentum, because the fermion wave equation is a first-order equation and thus the mass dimension of the fermion field 3/2.
Axial and vector U(1) symmetries  Out of the 16 bilinear forms, two transform as vectors under proper Lorentz transformations, \( j^L = \bar{\psi} \gamma^\mu \psi \) and \( j^R = \bar{\psi} \gamma^5 \gamma^\mu \psi \). We now want to check if these two currents are conserved. In the first case, we expect from non-relativistic quantum mechanics that the global phase transformation,

\[
\psi(x) \to \psi'(x) = e^{i \delta} \psi(x) \quad \text{and} \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i \delta} \bar{\psi}(x),
\]

leads to the conserved current \( j^L = \bar{\psi} \gamma^\mu \psi \). Inspection of the Lagrange density shows immediately that \( \delta \mathcal{L} = 0 \).

In the second case, the underlying symmetry is using \( \{ \gamma^5, \gamma^\mu \} = 0 \),

\[
\psi(x) \to e^{i \delta \gamma^5} \psi(x) \quad \text{and} \quad \bar{\psi}(x) \to \bar{\psi}'(x) = (e^{i \delta \gamma^5} \psi(x))^\dagger \gamma^0 = \bar{\psi}(x) e^{i \delta \gamma^5}.
\]

The resulting (infinitesimal) change is

\[
\delta \mathcal{L} = 2 m i \bar{\psi} \gamma^5 \psi.
\]

Thus the axial or chiral symmetry \( U_A(1) \) is broken by the mass term, leading to the non-conservation of the current \( j^R \).

Chirality  To understand this better we re-express the Dirac Lagrangian using eigenfunctions of \( \gamma^5 \). We can split any solution \( \psi \) of the Dirac equation into

\[
\psi_L = \frac{1}{2}(1 - \gamma^5) \psi \equiv P_L \psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1 + \gamma^5) \psi \equiv P_R \psi.
\]

\( P_L \) and \( P_R \) are projection operators satisfying \( P_L + P_R = 1 \), \( P_L^2 = P_L \) and \( P_R P_L = P_L P_R = 0 \). Since \( \gamma^5 \psi_L = -\psi_L \) and \( \gamma^5 \psi_R = \psi_R \), \( \psi_{L,R} \) are eigenfunctions of \( \gamma^5 \) with eigenvalue \( \pm 1 \). Expressing the mass term through these fields as

\[
\bar{\psi} \gamma^0 \psi = \bar{\psi} \left( P_L^2 + P_R^2 \right) \psi = \psi_L^\dagger \left( P_R \gamma^0 P_L + P_L \gamma^0 P_R \right) \psi_L + \bar{\psi}_R \psi_R,
\]

and similarly for the kinetic term,

\[
\bar{\psi} \partial_\mu \psi = \bar{\psi} \left( P_L^2 + P_R^2 \right) \partial_\mu \psi = \psi_L^\dagger \left( P_R \gamma^\mu P_L + P_L \gamma^\mu P_R \right) \partial_\mu \psi_L + \bar{\psi}_R \partial_\mu \psi_R,
\]

the Dirac Lagrange density becomes

\[
\mathcal{L} = i \bar{\psi}_L \partial_\mu \psi_L + i \bar{\psi}_R \partial_\mu \psi_R - m \left( \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \right).
\]

Comparing this expression to (5.62), we see that we can identify the Dirac fields \( \psi_{L,R} \) in the chiral representation with the Weyl fields \( \phi_{L,R} \) as follows,

\[
\psi_L = \begin{pmatrix} \phi_L \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = \begin{pmatrix} 0 \\ \phi_R \end{pmatrix}.
\]

Thus the projection operators (5.75) allows us to define the left- and right-chiral components of a Dirac field in an arbitrary representation.

The two kinetic terms which are invariant under chiral transformations connect left- to left-chiral and right- to right-chiral fields, while the mass term mixes left- and right-chiral fields. Such a mass term is called Dirac mass. The distinction between left- and right-chiral fields is Lorentz invariant: In terms of Weyl spinors, we saw that the Lorentz transformations \( S_L \) and \( S_R \) do not mix \( \phi_L \) and \( \phi_R \)—which qualified them to form the irreducible representation of the Lorentz group. In terms of Dirac spinors, the relation \( [\gamma^5, \sigma^{\mu\nu}] = 0 \) guarantees that left and right chiral fields transform separately under a Lorentz transformation, \( \psi'_{L/R} = S(\Lambda) \psi_{L/R} \). However, the mass term of a massive Dirac particle will mix left- and right-chiral fields as they evolve in time.
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**Helicity** The helicity operator $s \cdot p / |p|$ measures the projection of the spin $\Sigma / 2$ on the momentum $p$ of a particle,

$$\frac{\Sigma \cdot p}{2|p|} \psi = h \psi .$$

(5.80)

The helicity operator and the Dirac Hamiltonian commute, $[H_D, \Sigma \cdot p] = 0$, because there is no orbital angular momentum in the direction of $p$. Therefore common eigenfunctions of $H_D$ and $h$ called helicity states can be constructed. Positive helicity particles are called right-handed, negative helicity particles are called left-handed. For a massive particle, helicity is a frame-dependent quantity: If we choose e.g. a frame with $\beta \| p$ and $\beta > p$, then the particles moves in the opposite direction and $h$ changes sign. Since we cannot “overtake” a massless particle, helicity becomes in this case a Lorentz invariant quantity.

Helicity and chirality eigenstates can be seen as complimentary states. The former one is a conserved, frame-dependent quantum number, while the latter is frame-independent, but not conserved. Thus helicity states are e.g. useful to describe scattering processes where the detector measures spin in a definite frame. If on the other hand the interactions of a fermion are spin-dependent, then one should choose chiral fields, since the Lagrangian should be Lorentz invariant.

**Lorentz transformations** Our derivation of the Weyl spinors as the fundamental representation of the Lorentz group provided automatically their transformation properties under a finite Lorentz transformation. Using the Weyl representation, the transformation law for a Dirac spinor follows as

$$\psi(x) \rightarrow \psi(x') = S(\Lambda) \psi(x) = \left( \begin{array}{c} \phi_L(x') \\ \phi_R(x') \end{array} \right) = \left( \begin{array}{cc} S_L & 0 \\ 0 & S_R \end{array} \right) \left( \begin{array}{c} \phi_L(x) \\ \phi_R(x) \end{array} \right) .$$

(5.81)

We want to express the transformation matrix $S(\Lambda)$ by gamma matrices, such that it is representation independent. We set

$$S(\Lambda) = \exp \left( -i X_{\mu \nu} \omega^{\mu \nu} / 4 \right) ,$$

(5.82)

where the antisymmetric matrix $\omega^{\mu \nu}$ parametrises the Lorentz transformation and the six generators $(X_{ab})_{\mu \nu}$ have to be determined. Then we split $\omega^{\mu \nu}$ into boosts and rotations,

$$\frac{1}{2} \omega_{\mu \nu} X^{\mu \nu} = \omega_{0i} X^{0i} + \omega_{12} X^{12} + \omega_{13} X^{13} + \omega_{23} X^{23} .$$

(5.83)

Parameterising the rotation by $\alpha^i = \varepsilon^{ijk} \omega^j$ and the boost by $\eta^i = \omega^i_0$ it follows that the generators $X^{\mu \nu}$ coincide with the $\sigma^{\mu \nu}$ matrices, which are given in the Weyl representation explicitely as

$$\sigma^{0i} = i \left( \begin{array}{cc} -\sigma^i & 0 \\ 0 & \sigma^i \end{array} \right) ,$$

(5.84)

and

$$\sigma^{ij} = \varepsilon^{ijk} \left( \begin{array}{cc} \sigma^k & 0 \\ 0 & \sigma^k \end{array} \right) .$$

(5.85)

In contrast to (5.81), the expression $S(\Lambda) = \exp \left( -i \sigma_{\mu \nu} \omega^{\mu \nu} / 4 \right)$ is valid in any representation of the gamma matrices (see problem 5.5).
5.2. Dirac equation

Finally, we want to derive the transformation law of the gamma matrices. A straightforward method is to use our knowledge that the Dirac equation is Lorentz invariant. The connection between spinors in different frames is given by

$$\psi'(x') = \psi'(Ax) = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x') \quad (5.86)$$

and

$$\psi(x) = S(\Lambda^{-1})\psi'(x') = S(\Lambda^{-1})\psi(Ax). \quad (5.87)$$

Inserting the underlined part of Eq. (5.86) for $\psi'(x')$ in Eq (5.87), we obtain the expected result

$$S^{-1}(\Lambda) = S(\Lambda^{-1}). \quad (5.88)$$

We insert $\psi(x) = S^{-1}(\Lambda)\psi'(x')$ in the Dirac equation valid in the inertial system $x$ and multiply from the left with $S(\Lambda)$,

$$\left[ i S(\Lambda)\gamma^{\mu}S^{-1}(\Lambda)\partial_{\mu} - m \right] \psi'(x') = 0. \quad (5.89)$$

Next we use $\partial_{\mu} = \Lambda_{\nu}^{\mu}\partial'_{\nu}$ and obtain

$$\left[ i S(\Lambda)\gamma^{\mu}S^{-1}(\Lambda)\Lambda_{\nu}^{\mu}\partial'_{\nu} - m \right] \psi'(x') = 0. \quad (5.90)$$

Hence the Dirac equation is form invariant under Lorentz transformations, if

$$S_{ab}(\Lambda)\gamma_{bc}S^{-1}_{cd}(\Lambda)\Lambda_{\mu}^{\nu} = \gamma_{\mu\nu} \quad (5.91)$$

or

$$S(\Lambda)\gamma^{\mu}S^{-1}(\Lambda) = \Lambda_{\nu}^{\mu}\gamma^{\nu}. \quad (5.92)$$

Since we have not used the condition $\det(\Lambda) = 1$, this equation should hold both for discrete and proper Lorentz transformations. You are asked in problem 5.6 to confirm that (5.92) is true using the properties of the Clifford algebra which is done easiest for an infinitesimal Lorentz transformation.

**Example:** Lorentz transformation properties of bilinear quantities:

Because of $(\sigma_{\mu\nu})^1 = -\frac{1}{4}[\gamma^{\mu},\gamma^{\nu}] = \gamma^{0}\sigma_{\mu\nu}\gamma^{0}$, we find $S^1 = \gamma^{0}S^{-1}\gamma^{0}$. Hence under a Lorentz transformation $x'^{\mu} = \Lambda^{\mu}_{\nu}x^{\nu}$, the spinor transforms as $\psi'(x') = S(\Lambda)\psi(x)$ and the adjoint Dirac spinor as

$$\bar{\psi}'(x') = \bar{\psi}'(x')\gamma^{0} = \bar{\psi}'(x)S^{\dagger}\gamma^{0} = \bar{\psi}'(x)\gamma^{0}S^{-1}\gamma^{0} = (\bar{\psi}(x))S^{-1}. \quad (5.93)$$

Thus $\bar{\psi}\psi$ transforms as a scalar under Lorentz transformations,

$$\bar{\psi}'(x')\psi(x') = \bar{\psi}(x)S^{-1}\psi(x) = \bar{\psi}(x)\psi(x), \quad (5.94)$$

and using (5.92) the current

$$j^{\mu}(x') = \bar{\psi}'(x')\gamma^{\mu}\psi(x') = \bar{\psi}(x)S^{-1}\gamma^{\mu}\psi(x) = \Lambda_{\nu}^{\mu}j^{\nu}(x) \quad (5.95)$$

as a four-vector.

Equation (5.91) illustrates nicely the transformation properties of the gamma matrices $\gamma_{\mu\nu}^{ab}$: The index $a$ transforms with $S(\Lambda)$ as $\psi$, the index $b$ transforms with $S^{-1}(\Lambda)$ as $\bar{\psi}$, while the index $\mu$ transforms with $\Lambda$. 

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Parity

A process seen in a mirror is described by a coordinate system $x^\mu \to x'^\mu = (x^0, -x)$. If the Dirac equation is mirror symmetric, or invariant under parity $P$, then both processes are possible solutions of it.

Since $\gamma^0$ anti-commutes with $\gamma^i$ and commutes with itself,

$$((\gamma^0 \gamma^0 + \gamma^0 \gamma^i)\psi = (\gamma^0 - \gamma^i)\gamma^0 \psi, \quad (5.93)$$

we can change the derivative from $\partial_\mu$ to $\partial'_\mu$,

$$\gamma^0 (i \gamma^\mu \partial_\mu - m) \psi(x) = (i \gamma^\mu \partial'_\mu - m) \gamma^0 \psi(x) = 0 \quad (5.94)$$

after multiplication of the Dirac equation with $\gamma^0$. More generally, we can allow for an unobservable phase factor $\eta = e^{i\phi}$, defining the action of the parity operation on a Dirac spinor as

$$\psi'(x') = P\psi(x) = \eta \gamma^0 \psi(x). \quad (5.95)$$

The parity operation $P$ applied to left and right chiral fields exchanges them,

$$P\psi_L = \eta \gamma^0 \frac{1}{2}(1 - \gamma^5)\psi = \frac{1}{2}(1 + \gamma^5)\eta \gamma^0 \psi = P\psi_R', \quad (5.96)$$

as we noticed already from $P(S_L) = S_R$ discussing the fundamental representation of the Lorentz group.

Example: Find the transformation property of $\bar{\psi}(x)\psi(x)$ and $\bar{\psi}(x)\gamma^5\psi(x)$ under parity. With $\psi(x') = P\psi(x) = \eta \gamma^0 \psi(x)$ and $\bar{\psi}(x') = \eta^* \bar{\psi}(x)\gamma^0$, we find

$$\bar{\psi}'(x')\psi'(x') = \eta^* \bar{\psi}(x)\gamma^0 \gamma^0 \psi(x) = \bar{\psi}(x)\psi(x).$$

and

$$\bar{\psi}'(x')\gamma^5\psi'(x') = \eta^* \bar{\psi}(x)\gamma^0 \gamma^5 \gamma^0 \psi(x) = -\bar{\psi}(x)\gamma^5 \psi(x).$$

Thus the bilinear $\bar{\psi}\psi$ transforms as a scalar, while $\bar{\psi}\gamma^5\psi$ transforms as a pseudo-scalar.

Analogously, the axial vector current $\bar{\psi}\gamma^\mu\gamma^5\psi$ and the axial tensor current $\bar{\psi}\sigma_{\mu\nu}\gamma^5\psi$ transform odd under parity. Hence such interaction terms can be omitted in theories that respect parity (like QED and QCD), but have to be included in weak interactions.

Charge conjugation

We already discussed possible couplings between a Dirac fermion and the photon field. The only one with a dimensionless coupling $e > 0$ that also respects parity is

$$\mathcal{L}_I = -e j^\mu A_\mu = -e \bar{\psi}\gamma^\mu \psi A_\mu. \quad (5.97)$$

Solving the Lagrange equations for $\mathcal{L}_0 + \mathcal{L}_I$ gives the Dirac equation including a coupling to the electromagnetic field as

$$[i\gamma^\mu (\partial_\mu + ieA_\mu) - m] \psi(x) = 0. \quad (5.98)$$

This corresponds to the “minimal coupling” prescription known from quantum mechanics.

Having defined the coupling to an external electromagnetic field, we can ask ourselves how the Dirac equation for a charged conjugated field $\psi_c$ should look like. In the case of a scalar
particle, complex conjugation transformed a positively charged particle into a negative one and vice versa. We try the same for the Dirac equation,

\[
-\mathbf{i}\gamma^\mu((\partial_\mu - ieA_\mu) - m)\psi^*(x) = 0. \tag{5.99}
\]

The matrix $\gamma^{\mu*}$ satisfies also the Clifford algebra. Hence we should find the unitary transformation $U^{-1}\gamma^\mu U = -\gamma^{\mu*}$ or setting $U \equiv C\gamma^0$

\[
(C\gamma^0)^{-1}\gamma^\mu C\gamma^0 = -\gamma^{\mu*}. \tag{5.100}
\]

If it exists, then the charge-conjugated field $\psi^c \equiv C\gamma^0\psi^*$ satisfies the Dirac equation with $q = -e$,

\[
[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]C\gamma^0\psi^*(x) = 0. \tag{5.101}
\]

Explicit calculation shows that we may choose $C = \gamma^2\gamma^0$, see problem 5.8.

In the chiral representation, $\psi^c = C\gamma^0\psi^* = i\gamma^2\psi^*$ becomes

\[
\psi^c = \begin{pmatrix}
0 & i\sigma^2 \\
-i\sigma^2 & 0
\end{pmatrix}
\begin{pmatrix}
\phi^*_L \\
\phi^*_R
\end{pmatrix} = \begin{pmatrix}
i\sigma^2\phi^*_R \\
-i\sigma^2\phi^*_L
\end{pmatrix}, \tag{5.102}
\]

which is in agreement with $\phi^c_L = i\sigma^2\phi^*_R$ and $\phi^c_R = -i\sigma^2\phi^*_L$ found earlier.

**Example:** Since the $\gamma^2$ has the same form in the Dirac and the chiral representation, we find applying $C$ on the spinors $u(p, \pm)$ and $v(p, \pm)$ immediately that

\[
u^c(p, s) = C\gamma^0\bar{u}(p, s) = iv(p, s)
\]

\[
u^c(p, s) = C\gamma^0\bar{v}(p, s) = -iu(p, s).
\]

From

\[
iS_F(x) = \int \frac{d^3p}{(2\pi)^32\omega_p} \sum_s \left[ u(p, s)\bar{u}(p, s)e^{-i(E_p t - px)}\vartheta(x^0) - v(p, s)\bar{v}(p, s)e^{i(E_p t - px)}\vartheta(-x^0) \right] \tag{5.103}
\]

this implies that

\[
S_F(x) = CS_F(-x)C^{-1}, \tag{5.104}
\]

### 5.3. Dirac, Weyl and Majorana fermions

Up to now we have discussed the Dirac equation, having in mind a massive particle carrying a conserved U(1) charge as e.g. lepton number that allows us to distinguish particles and anti-particles. We call such particles Dirac fermions. In this section we consider the case where one of these two conditions is not fulfilled. In the SM all particles except the neutrinos carry an electric charge and are thus Dirac fermions.
5. Fermions and the Dirac equation

Weyl fermions, $q \neq 0$ and $m = 0$: The (massive) Dirac equation in the chiral representation
\[
\begin{pmatrix}
-m & i(\partial_0 + \sigma \nabla) \\
i(\partial_0 - \sigma \nabla) & -m
\end{pmatrix}
\begin{pmatrix}
\phi_L(p) \\
\chi_R(p)
\end{pmatrix} = 0
\] (5.105)
decouples for $m = 0$ into two equations called Weyl equations,
\[
i(\partial_0 - \sigma \nabla)\phi_L(p) = 0 \tag{5.106a}
\]
\[
i(\partial_0 + \sigma \nabla)\chi_R(p) = 0. \tag{5.106b}
\]
A fermion described by the Weyl equations is called a Weyl fermion. In particular, until the 1990’s neutrino masses were experimentally consistent with zero and neutrinos were incorporated into the SM as Weyl fermions.

For a plane-wave $\chi_R(p) = \chi_{RE}^{-i \varepsilon p x}$ and $p^0 = |p| = E$, we recover (5.22) in the limit $m = 0$,
\[
(E - \sigma p)\chi_{RE}^{-i \varepsilon p x} = 0. \tag{5.107}
\]
Thus the dispersion relation of a right-chiral Weyl fermion is $E = \sigma |p|$ for both positive and negative energy solutions, $\varepsilon = \pm 1$. Since $\sigma$ has the eigenvalues $\pm 1$, only the solution with positive helicity is allowed for $\chi_R$. Remember also that helicity is frame independent for a massless particle; in this case positive helicity agrees with right chirality. Thus a Weyl fermion ($m = 0$, $q \neq 0$) has two degrees of freedom, consisting of the left-chiral 2-spinor $\chi_L(\sigma = -1)$ with negative helicity and the right-chiral 2-spinor $\chi_R(\sigma = +1)$ with positive helicity.

Majorana fermions, $m > 0$ and $q = 0$: The Dirac field $\psi$ has to be complex, because it transforms under the complex representation $S(\Lambda)$ of the Lorentz group. However a neutral fermion field $\psi_M$, where we cannot distinguish particles and antiparticles, should have only half of the degrees of freedom of a charged Dirac field. By analogy with the scalar case, we expect that we can halve the number of degrees of freedom by imposing a reality condition, $\psi_M = \psi_M^*$. But this condition can be Lorentz invariant only in a special representation of the gamma matrices where $S(\Lambda)$ is real. More generally, we halve the number of degrees of freedom by using a self-conjugated field $\psi = \psi_M$. A fermion described by a self-conjugated field $\psi_M$ is called a Majorana fermion and the corresponding spinor a Majorana spinor.

We can replace any Dirac field $\psi_D$ by a pair of self-conjugated fields,
\[
\psi_{M,1} = \frac{1}{\sqrt{2}} (\psi_D + \psi_D^c), \tag{5.108a}
\]
\[
\psi_{M,2} = \frac{1}{\sqrt{2}} (\psi_D - \psi_D^c). \tag{5.108b}
\]
and vice versa inverting these relations. Expressed by Weyl spinors, a Majorana spinor becomes
\[
\psi^c = \psi = \begin{pmatrix} i \sigma^2 \phi_L^* \\ -i \sigma^2 \phi_L^c \end{pmatrix}. \tag{5.109}
\]
The Majorana 4-spinor is composed of only one Weyl spinor, which shows again that it describes 2 degrees of freedom. Thus a Majorana fermion ($m > 0$, $q = 0$) has two degrees of freedom.

Most authors call $\psi_{L/R}$ and $\phi_{L/R}$ not left and right-chiral but left and right-handed, although this identification holds only for massless particles.
Dirac, Weyl and Majorana fermions

freedom, consisting either of a left-chiral 2-spinor \( \chi_R(\sigma) \) with positive and negative helicity or a right-chiral 2-spinor \( \chi_R(\sigma) \) with both helicities.

The condition \( \psi_M = \psi_M^* \) is Lorentz invariant only, if \( \sigma_{\mu\nu} = -\sigma_{\mu\nu} \). This defines the Majorana representation of the \( \gamma \) matrices in which all \( \gamma^\mu \) and thus \( \sigma_{\mu\nu} \) are imaginary, the charge conjugation matrix \( C \) is the unity matrix, \( C = 1 \), and Majorana spinors can be chosen to be real. Since the spinors are real in this representation, no phase invariance \( \psi(x) \rightarrow \psi'(x) = \exp(i\phi)\psi(x) \) as in (5.72) can be implemented for a Majorana fermion and thus they cannot carry any conserved U(1) charge.

**Dirac versus Majorana mass terms** Charge conjugated spinors were defined by

\[
\psi^c = C \gamma^0 \psi^* = C \psi^t, \quad \bar{\psi}^c = C \bar{\psi}^t.
\]

We define

\[
\psi^c_L \equiv (\psi_L)^c = \frac{1}{2}(1 + \gamma^5)\psi^c = (\psi^c)^R,
\]

which is consistent with our previous definition for Weyl spinors.

As we saw, a Dirac mass term

\[
-L_D = m_D \bar{\psi} \psi = m_D (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)
\]

connects L and R components of the same field and \( \psi = \psi_L + \psi_R \) is a mass eigenstate. We now use the observation that \( (\psi_L)^c = (\psi^c)^R \) allows us to obtain new mass terms\(^6\) called Majorana mass terms,

\[
-L_L = m_L (\bar{\psi}_L \psi_L + \bar{\psi}_L \psi_L^c) \quad \text{and} \quad -L_R = m_R (\bar{\psi}_R \psi_R + \bar{\psi}_R \psi_R^c)
\]

which connect the L and R components of conjugated fields.

The mass eigenstates \( \chi \) and \( \omega \) are the self-conjugated fields

\[
\chi = \psi_L + \psi_L^c = \chi^c \\
\omega = \psi_R + \psi_R^c = \omega^c
\]

with

\[
-L_L = m_L \bar{\chi} \chi, \quad -L_R = m_R \bar{\omega} \omega
\]

In the general case we expect that both Dirac and Majorana mass terms are present

\[
-L_{DM} = m_D \bar{\psi}_L \psi_R + m_D \bar{\psi}_R \psi_L + m_R \bar{\psi}_R \psi_R + \text{h.c.}
\]

\[
= \frac{1}{2} m_D (\bar{\chi} \omega + \bar{\omega} \chi) + m_L \bar{\chi} \chi + m_R \bar{\omega} \omega
\]

or

\[
-L_{DM} = \left( \begin{array}{c} m_L \\ m_D/2 \\ m_R \end{array} \right) \left( \begin{array}{c} \chi \\ \omega \end{array} \right)
\]

\(^6\)Note that this argument does not forbid that a Majorana fermion carries conserved charges which transform under a real representation of a symmetry group: An example are gluinos, the supersymmetric partners of the gluons, which are Majorana fermions and transform under the adjoint representation of SU(3).

\(^7\)Note that the terms \( \bar{\psi}_L \psi_L = \bar{\psi}_L C \psi_L \), etc., vanish because of \( C^t = -C \), if one does not already assumes on the classical level that fields are anticommuting variables.
5. Fermions and the Dirac equation

The eigenvalues of this matrix are

\[ m_{1,2} = \frac{1}{2} \left\{ (m_L + m_R) \pm \sqrt{(m_L - m_R)^2 + m_D^2} \right\} \]

and the eigenvectors

\[ \eta_1 = \cos \vartheta \chi - \sin \vartheta \omega \]
\[ \eta_2 = \sin \vartheta \chi + \cos \vartheta \omega \]

with \( \tan 2\vartheta = m_D/(m_L - m_R) \).

Seesaw model  The seesaw model tries to explain why neutrinos have much smaller masses than all other particles in the standard model. Let us assume that there exist both left- and right-chiral neutrinos and that they obtain Dirac masses as the other fermions, say of order \( m_D \sim 100 \text{ GeV} \). The right-chiral \( \nu_R \) does not participate in any SM interaction and suffers the same fate as a scalar particle: its mass will be driven by quantum corrections to a value close to the validity of the effective theory (the SM) we are using, so \( m_R \gg m_D \). Expanding

\[ m_{1,2} \approx \frac{1}{2} \left\{ m_R \pm m_R \sqrt{1 + \frac{m_D^2}{m_R^2}} \right\} \]

the two eigenvalues are \( m_1 \approx m_R^2/(2m_R) \) and \( m_2 \approx m_R \). For \( m_R \sim 10^{14} \text{ GeV} \), we find light neutrino mass in the eV or sub-eV range, as required by experimental data.

5.4. Spin-statistic connection and Graßmann variables

Spin-statistic connection  We have noted at several places that fermionic fields should anti-commute: Expressing the Feynman propagator as sum of solutions or as time-ordered vacuum expectation value of the fields, and writing down a Majorana mass term required that the quadratic form \( \bar{\psi}_a \psi_b \) is antisymmetric. In a relativistic quantum field theory the spin and the statistics of a field is coupled:

- The wave equations of bosons are second-order differential equations. Therefore bosonic fields have mass dimension 1 and their propagators \( \Delta(p) \) are even in the momentum \( p \). As a result, bosonic fields commute and satisfy Bose-Einstein statistics.
- In contrast, fermions satisfy first-order differential equations and have mass dimension 3/2. Therefore the fermion propagator \( S_F(p) \) is linear in \( p \) and \( S_F(x) \) is an antisymmetric function. This implies that fermions satisfy Fermion-Dirac statistics and are described by anti-commuting classical spinors or operators.

This leads to a practical and a principal question: First, the practical one: How do we implement that classical functions which enter the path integral do anti-commute? And second, does the anticommutation of fermionic variables lead to a consistent picture? In particular is the Hamiltonian of such a theory bounded from below?

We will start to address the latter question calculating the energy density \( \rho = \mathcal{H} \) of the Dirac field assuming that the classical spinors \( u \) and \( v \) do commute. We determine first the canonically conjugated momenta as

\[ \pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = i \psi^\dagger \]

(5.110)
and \( \bar{\pi} = 0 \). Thus the Hamilton density is

\[
\mathcal{H} = \hat{\pi} \dot{\psi} - \mathcal{L} = i\psi^\dagger \partial_t \psi - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = i\psi^\dagger \partial_t \psi,
\]

where we used the Dirac equation in the last step. To make this expression more explicit, we express now \( \psi \) by plane wave solutions,

\[
\psi(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[ b_s(p)u_s(p)e^{-ipx} + d_s^\dagger(p)v_s(p)e^{ipx} \right],
\]

and

\[
\psi^\dagger(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[ b_s^\dagger(p)u_s^\dagger(p)e^{ipx} + d_s(p)v_s(p)e^{-ipx} \right].
\]

Inserting these expressions into (5.111) gives schematically \((b^\dagger + d)(b - d^\dagger)\), where the relative minus sign comes from \( \partial_t \) acting on \( \psi \). Since the spinors \( u \) and \( v \) are orthonormal, cf. (5.48), only the diagonal terms survive, \((b^\dagger + d)(b - d^\dagger) \rightarrow b^\dagger b - dd^\dagger\). Hence the energy of a Dirac field is

\[
H = \int d^3x \mathcal{H} = \sum_s \int d^3p E_p \left[ b_s^\dagger(b_s - d_s) - d_s^\dagger d_s \right].
\]

If \( d \) and \( d^\dagger \) would be normal Fourier coefficients of an expansion into plane waves, the second term would be negative and the energy density of a fermion field could be made arbitrarily negative.

This conclusion is avoided if the fermion fields anticommute: In canonical quantisation, one promotes the Fourier coefficients to operators. Requiring then anti-commutation relations between the creation and annihilation operators,

\[
\{d_s(p), d_{s'}^\dagger(p')\} = \delta_{ss'} \delta(p - p'),
\]

compensates the sign in the second term.

If we restore units, then we have to add a factor \( \hbar \) on the RHS. Thus the RHS vanishes in the classical limit, \( \hbar \rightarrow 0 \), which implies that classical spinors should anti-commute. Thus we should perform the path integral of fermionic fields over anti-commuting numbers which are called Graßmann numbers. Either way, we find the amazing result that in a relativistic quantum field theory the spin determines the statistics in such a way the Hamiltonian is bounded from below.

**Graßmann variables** We now proceed to the practical question how the mathematics of anticommuting numbers works. We define a Graßmann algebra \( G \) requiring that for \( a, b \in G \) the anticommutation rules

\[
\{a, a\} = \{a, b\} = \{b, b\} = 0
\]

and thus \( a^2 = b^2 = 0 \) are valid. Then any smooth function \( f \) of \( a \) and \( b \) can be expanded into power-series as

\[
f(a, b) = f_0 + f_1 a + \tilde{f}_1 b + f_2 ab = f_0 + f_1 a + \tilde{f}_1 b - f_2 ba.
\]
5. Fermions and the Dirac equation

Defining the derivative as acting to the right, \( \partial \equiv \overrightarrow{\partial} \), it is
\[
\frac{\partial f}{\partial a} = f_1 + f_2 b \quad (5.117)
\]
\[
\frac{\partial f}{\partial b} = f_1 - f_2 a \quad (5.118)
\]
and
\[
\frac{\partial^2 f}{\partial a \partial b} = -\frac{\partial^2 f}{\partial b \partial a} = -f_2. \quad (5.119)
\]

Defining integration for Grassmann variables, we require linearity and that the infinitesimals \( da, db \) are also Grassmann variables,
\[
\Rightarrow \{a, da\} = \{b, db\} = \{a, b\} = \{da, db\} = 0. \quad (5.120)
\]
Multiple integrals are iterated,
\[
\int da db f(a, b) = \int da \left( \int db f(a, b) \right). \quad (5.121)
\]
We have to determine the value of \( \int da \) and \( \int daa \). For the first, we write
\[
\left( \int da \right)^2 = \left( \int da \right) \left( \int db \right) = \int dadb = -\int dbda = - \left( \int da \right)^2 \quad (5.122)
\]
and thus \( \int da = 0 \). We are left with \( \int daa \): Since there is no intrinsic scale—states are empty or occupied—we are free to set
\[
\int daa = 1. \quad (5.123)
\]
This implies also that there is no difference between definite and indefinite integrals for Grassmann variables. Moreover, differentiation and integration is equivalent for Grassmann variables.

Assume now that \( \eta_1 \) and \( \eta_2 \) are real Grassmann variables and \( A \in \mathbb{R} \). Then
\[
\int d\eta_1 d\eta_2 e^{\eta_2 A \eta_1} = \int d\eta_1 d\eta_2 (1 + \eta_2 A \eta_1) = \int d\eta_1 d\eta_2 \eta_2 A \eta_1 = \int d\eta_1 A \eta_1 = A. \quad (5.124)
\]
Next we consider a two-dimensional integral with an anti-symmetric matrix \( A \) and \( \eta = (\eta_1, \eta_2) \). Then
\[
A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \quad (5.125)
\]
and \( \eta^T A \eta = 2a \eta_1 \eta_2 \). Using an arbitrary matrix would lead to the same result, since its symmetric part cancels. Expanding again the exponential gives
\[
\int d^3 \eta \exp \left( \frac{1}{2} \eta^T A \eta \right) = a = (\det(A))^{1/2}. \quad (5.126)
\]
One can show that an arbitrary antisymmetric matrix can be transformed into block diagonal form, where the diagonal is composed of matrices of the type \( (5.125) \). Thus the last formula holds for arbitrary \( n \).
Finally, we introduce complex Grassmann variables $\eta = (\eta_1, \ldots, \eta_n)$ and their complex conjugates $\overline{\eta} = (\overline{\eta}_1, \ldots, \overline{\eta}_n)$. For any complex matrix $A$,

$$\int d^n\eta d^n\overline{\eta} \exp(\eta^\dagger A \eta) = \prod_{i=1}^{n} a_i = \det(A). \quad (5.127)$$

We can compare this to the result over commuting complex variables, $z_i = (x_i + i y_i)/\sqrt{2}$ and $\overline{z}_i = (x_i - i y_i)/\sqrt{2}$, with $dxdy = dzd\overline{z}$ and

$$\int d^nzd^n\overline{z} \exp(-z^\dagger A z) = (2\pi)^n \frac{\det(A)}{\det(A)}. \quad (5.128)$$

Thus for Grassmann variables the determinant appearing in the evaluation of a Gaussian integral is in the numerator, while it is in the denominator for real or complex valued functions.

**Path integral for fermions** The path integral for a free Dirac fermion without external sources is

$$Z[0] = \int D\psi D\overline{\psi} e^{iS[\psi,\overline{\psi}]} = \int D\psi D\overline{\psi} e^{i \int d^4x \overline{\psi}(i\partial /m)\psi}. \quad (5.129)$$

For its evaluation, we use (5.127) in the limit $n \to \infty$. Since it is quadratic in the fields, we can perform it formally,

$$Z[0] = \det(i\partial /m) = \exp \text{tr} \ln(i\partial /m). \quad (5.130)$$

Using the cyclic property of the trace, we write

$$\text{tr} \ln(i\partial /m) = \text{tr} \ln \gamma^5(i\partial /m)\gamma^5 = \text{tr} \ln(-i\partial /m) = \frac{1}{2} \left[ \text{tr} \ln(i\partial /m) + \text{tr} \ln(-i\partial /m) \right] = \frac{1}{2} \left[ \text{tr} \ln(\Box + m^2) \right]. \quad (5.131)$$

Thus $Z[0] = \exp[+\text{tr} \ln(\Box + m^2)/2]$; We have found the remarkable result that the zero-point energy of fermions has the opposite sign compared to the one of bosons. We arrive at the same conclusion, using anti-commutation relations

$$\{d_s(p), d_{s'}(p')\} = \delta_{s,s'} \delta(p - p') \quad (5.132)$$

in Eq. (5.113),

$$H = \sum_s \int d^3p \ 4E_p \left[ b^\dagger b + d^\dagger d - \delta^{(3)}(0) \right]. \quad (5.133)$$

With $\delta^{(3)}(0) = \int d^3x / (2\pi)^3$ we see that the last term corresponds to the negative zero-point energies of a fermion.

Note that this opens the possibility that the zero-point energies of (groups of) bosons and fermions cancel exactly, provided that i) the degrees of freedom of fermions and bosons agree. (For instance, the trace in Eq. (5.131) includes the trace over the 4x4 matrix in spinor space, leading to a factor four larger results than for a single scalar.) ii) Their masses are the same, $m_f = m_b$. iii) Their interactions match, so that also higher-order corrections are identical for fermions and bosons. The corresponding symmetry that guarantees automatically that the conditions i)-iii) are satisfied is called “supersymmetry.” In an unbroken supersymmetric theory, the cosmological constant would be zero.

Clearly, condition ii) is most problematic, since e.g. no bosonic partner of the electron has been found (yet). Hence supersymmetry must be a broken symmetry, but as long as the mass splitting $m_f^2 - m_b^2$ between fermions and bosons is not too large, it might be still “useful.”
5. Fermions and the Dirac equation

**Feynman rules** Next we add Graßmannian sources $\eta$ and $\bar{\eta}$ to the action, $S[\psi, \bar{\psi}] + \bar{\eta}\psi + \bar{\psi}\eta$. Then we have to complete the square,

$$\bar{\psi} A \psi + \bar{\eta}\psi + \bar{\psi}\eta = (\bar{\psi} + \bar{\eta} A^{-1}) A (\psi + A^{-1}\eta) - \bar{\eta} A^{-1} \eta,$$

obtaining

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i \int d^4x \bar{\psi} A \psi + \bar{\eta}\psi + \bar{\psi}\eta} = Z[0] e^{-i \bar{\eta} A^{-1} \eta} = Z[0] \exp \left( -i \int d^4x d^4x' \bar{\eta}(x) S_F(x - x') \eta(x') \right).$$

Here, $A^{-1}(x, x') = S_F(x - x') = -S^T_F(x' - x)$ which corresponds to the fact that the matrix $A$ is antisymmetric.

The propagator of a Dirac fermion is a line with an arrow representing the flow of the conserved charge which distinguishes particles and anti-particles. Thus a fermion line cannot split, and the arrow cannot change direction. In contrast, the arrow along a Majorana fermion line can change direction, since there is no difference between a particle and anti-particle in this case.

$$\quad = S_F(p) = \frac{i}{p - m + i\varepsilon}$$

We look at possible interaction terms of a Dirac fermion with scalars and photons, restricting ourselves to dimensionless coupling constants,

$$\mathcal{L}_I = -g_s S \bar{\psi} \psi - g_a P \bar{\psi} \gamma^5 \psi - e \bar{\psi} \gamma^\mu A_\mu.$$

Both interaction terms of the fermion with the scalars respect parity, if the field $S$ is a true scalar and the field $P$ a pseudo-scalar. Analogous to the $-i\lambda$ coupling in the case of a scalar self-interactions, we read off from the Lagrangian the following interaction vertices in momentum space,

**Fermion loops** A closed fermion loop with $n$ propagators corresponds to

$$\psi^\dagger(x_1) \psi(x_1) \psi^\dagger(x_2) \psi(x_2) ... \psi^\dagger(x_n) \psi(x_n)$$

Thus we have to anticommute $\psi^\dagger(x_1)$ and $\psi(x_n)$, generating a minus sign. Another way to understand the minus sign of fermion loops is to look at the generating functional for connected graphs setting the sources to zero, $iW[0] = \ln Z[0] = \ln \det A$. The generated...
graphs are single-closed loops with \( n \) Feynman propagators. The change from \( 1/\det A \) in \( Z \) for bosonic fields to \( \det A \) in \( Z \) for fermionic fields implies an additional minus sign for closed fermion loops.

More formally, we can compare a complex bosonic field \( \phi \) and fermionic field \( \psi \) coupled to an external field \( \sigma \). In the bosonic case, we find

\[
Z[\sigma] = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{i\int d^4x \phi^*(B-\sigma)\phi} = \frac{1}{\det(B-\sigma)} \tag{5.137}
\]

with \( B = - (\Box - m^2) \), while we obtain for fermions

\[
Z[\sigma] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i\int d^4x \bar{\psi}(A-\sigma)\psi} = \det(A-\sigma) \tag{5.138}
\]

with \( A = i\partial - m \). We rewrite now the generating functional for connected graphs for zero sources as

\[
iW[\sigma] = - \ln \det(B - \sigma) = - \text{Tr} \ln(B - \sigma) = - \text{Tr} \ln(1 - B^{-1}\sigma) + \text{const.} \tag{5.139}
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}(B^{-1}\sigma)^n + \text{const}. \tag{5.140}
\]

Evaluating the trace in the coordinate space results in

\[
\text{Tr}(B^{-1}\sigma) = \int d^4x \langle x|B^{-1}\sigma|x\rangle = \int d^4x i\Delta_F(0)\sigma(x) \tag{5.141a}
\]

\[
\text{Tr}(B^{-1}\sigma)^2 = \int d^4x d^4y \langle x|B^{-1}\sigma|y\rangle \langle y|B^{-1}\sigma|x\rangle = \int d^4x d^4y i\Delta_F(x-y)\sigma(y)\Delta_F(y-x)\sigma(x). \tag{5.141b}
\]

\[
\text{Tr}(B^{-1}\sigma)^3 = \ldots
\]

Thus the expansion generates the closed single loops with \( n \) propagators and symmetry factor \( 1/n! \). The same holds for the fermionic case, except that we obtain an additional overall minus sign since we start from \( iW[\sigma] = \ln \det(A - \sigma) \).

**Furry’s theorem** What is the relation between diagrams containing fermion loops with opposite orientation in QED? A fermion loop with \( n \) external photons attached corresponds to a trace over \( n \) fermion propagators separated by gamma matrices,

\[
G_1 = \text{tr}[\gamma_{\mu_1} S_F(y_1, y_n) \gamma_{\mu_2} S_F(y_2, y_3) \cdots \gamma_{\mu_2} S_F(y_2, y_1)]. \tag{5.142}
\]

If we insert \( CC^{-1} = 1 \) between all factors in the trace, use \( C\gamma^T C^{-1} = -\gamma^T \) and \( CS_F(-x)C^{-1} = S_F^T(x) \), then

\[
G_1 = (-1)^n \text{tr}[\gamma_{\mu_1}^T S_F^T(y_n, y_1) \gamma_{\mu_2}^T S_F^T(y_3, y_2) \cdots \gamma_{\mu_2}^T S_F^T(y_2, y_1)]. \tag{5.143}
\]

\[
= (-1)^n \text{tr}[\gamma_{\mu_1} S_F(y_1, y_2) \cdots \gamma_{\mu_n} S_F(y_n, y_1)] = (-1)^n G_2, \tag{5.144}
\]

where we used \( B^T A^T = (AB)^T \) in the last step. Except for the factor \( (-1)^n \), the last expression corresponds to the loop \( G_2 \) with opposite orientation. Hence for an odd number
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of propagators, the two contributions cancel, while they are equal for an even number of propagators. Alternatively, we can use a symmetry argument to convince us that all diagrams with an odd number of photons are zero in vacuum: Because the QED Lagrangian is invariant under charge conjugation, but \( \psi \rightarrow \psi^c = C\overline{\psi}^T \) and \( A_\mu \rightarrow A^c_\mu = -A_\mu \), all Green functions with an odd number of photons have to vanish.

Symmetry factors in QED

The last issue we want to address in this chapter is the question if the interactions in (5.136) lead to symmetry factors. We recall that drawing Feynman diagrams we should include only diagrams which are topologically distinct after the integration over internal coordinates. For instance, the two diagrams

are not topologically distinct, because a rotation around the \( x_1-x_2 \) axis interchanges them. Therefore, the corresponding two-point function \( G(x_1, x_2) \) has to be invariant under a change of the orientation of the fermion loop – as it is guaranteed by the Furry theorem. The two-point function \( G(x_1, x_2) \) consists of two identical diagrams obtained by exchanging the integration variables \( y_1 \) and \( y_2 \). Thus the factor 2! compensates the 1/2! from the Taylor expansion of \( \exp(-\mathcal{L}_{\text{int}}) \), if we draw only one diagram, and its symmetry factor is one.

Next we consider the four-point function \( G(x_1, x_2, x_3, x_4) \) which describes photon-photon scattering. Two diagrams describing this process are shown in Fig. 5.1. We can specify these diagrams by the order of the external lines connected to the fermion loop: Then the diagram on the LHS is denoted by \( (1, 2, 3, 4)^+ \), where the subscript specifies the orientation of the loop, while the diagram on the RHS is given by \( (1, 4, 3, 2)^- \). A rotation around the \( x_1-x_3 \) axis converts \( (1, 2, 3, 4)^+ \) into \( (1, 2, 3, 4)^- \) and hence the orientation of the fermion loop plays again no role. However, now topologically distinct diagrams exist, as we can not transform smoothly the diagram \( (1, 2, 3, 4)^+ \) into \( (1, 4, 3, 2)^- \).

The four-point function \( G(x_1, x_2, x_3, x_4) \) contains \( 4! \times 3! \) diagrams, obtained by permutating \( y_1, y_2, y_3, y_4 \) and \( x_2, x_3, x_4 \). After integration over the free \( y_i \) variables, the factor 4! compensates the 1/4! from the Taylor expansion of \( \exp(-\mathcal{L}_{\text{int}}) \). In configuration space, the 3! = 6 topologically distinct diagrams shown in Fig. 5.2 remain which carry no additional symmetry factor. The upper diagrams are permutation of the LHS diagram, the lower three diagrams of the RHS diagram.

Thus the resulting rule for QED is very simple: We do not need symmetry factors, if we draw all diagrams which are topologically distinct after the integration over internal coordinates.
Figure 5.2.: The six topologically distinct diagrams contributing to the photon four-point function.

Fermion loops with an odd number of fermions are zero and can be omitted. Independent of the type of interaction, any fermion loop leads to an additional minus sign.

**Summary**

The fundamental representation of the orthochronous Lorentz group for massive particles is given by left and right-chiral Weyl spinors. These two two-spinors are mixed by parity and thus one combines them into a Dirac four-spinor for parity conserving theories like electromagnetic and strong interactions.

Fermions satisfy first-order differential equations and have mass dimension $3/2$. Therefore the fermion propagator $S_F(p)$ is linear in $p$ and thus $S_F(x)$ is an antisymmetric function in $x$. As a result, fermions satisfy Fermion-Dirac statistics and are described either by Grassmann variables or by anti-commuting operators.

A Weyl fermion has $m = 0$ and $q \neq 0$ and satisfies the Weyl equation; its solution has two degrees of freedom, a left-chiral field with positive helicity and a right-chiral field with negative helicity. A Majorana fermion is a self-conjugated field with $m \neq 0$ and $q = 0$ which has therefore also only two degrees of freedom. It is described fully either by a left-chiral or a right-chiral 2-spinor $\chi_R(\sigma)$ with both helicities. In the Majorana representation, the spinor can be chosen to be real.

**Further reading**

The symmetries of the Dirac equation as well as of other relativistic wave equations are extensively discussed in W. Greiner *Relativistic Quantum Mechanics*. Two-component Weyl and Majorana spinor are discussed in more detail in Srednicki who introduces also the notation of dotted and undotted spinor indices.
5. Fermions and the Dirac equation

Problems

5.1 Lie algebra of the Poincaré group.
Derive the commutation relation (5.1) plus the missing ones involving translation using that the (hermetian) generator $T^a$ are connected to the Killing fields $V^a$ by $i T^a = V^a$. Rewrite the expressions in an Lorentz invariant form.

5.2 Pauli equation.
Linearise the Schrödinger equation following the logic in Sec. 5.1 and show that the Pauli equation follows.

5.3 $g_s$-factor of the electron.
Square the Dirac equation with an external field $A^\mu$ and show that the spin of the electron leads to the additional interaction term $\sigma_{\mu\nu} F^{\mu\nu}$ compared to the Klein-Gordon equation. Show that the Dirac equation predicts for the interaction of non-relativistic electron with a static magnetic field $H_{int} = (L + 2s)eB$, i.e. a $g_s$ equal to two for the electron.

5.4 Spin vector.
i) Show that the Pauli-Lubanski spin vector (3.25), setting $J^{\mu\nu} = \sigma^{\mu\nu}$, becomes in the rest-frame

$$W_i = -m\Sigma_i.$$  (5.145)

ii) Show that $W_\mu s^\mu$ can be used as relativistic spin operator.

5.5 Condition (5.82).
Show that $S(\Lambda) = \exp (-i\sigma_{\mu\nu} \omega^{\mu\nu}/4)$ is valid in any representation of the gamma matrices.

5.6 Condition (5.92).
Show that (5.92) is true for an infinitesimal Lorentz transformation.

5.7 Rotation of Dirac spinor.
Introducing the matrix $(I_X)^\nu_\mu$, parametrise as $S(\Lambda) = \exp (-i\omega I_\mu I_\nu \sigma_{\mu\nu}/4)$. Using $[(I_X)^\nu_\mu]^3 = (I_X)^\nu_\mu$ and show that...

5.8 Properties of $C$.
a.) Prove the following properties of the charge conjugation matrix $C$, $C^T = -C$ and $C^{-1} \gamma^\mu C = -\gamma^\mu$. b.) Show that $i\gamma^2\gamma^0$ has the required properties.

5.9 Majorana flip properties ♣.
Derive the properties of Majorana bilinears $\bar{\xi} \Gamma \eta = \pm \bar{\eta} \Gamma \xi$ under an exchange $\bar{\xi} \leftrightarrow \eta$.

5.10 Fermionic vacuum energy.
Calculate the contribution of a Dirac fermion to the vacuum energy density, following the steps (2.123) to (2.128) for a scalar.
6. Scattering processes

Most information about the properties of fundamental interactions and particles is obtained from scattering experiments. In a scattering process, the initial and final state contains widely separated particles which can be treated as free, real particles which are on mass-shell. By contrast, \( n \)-point Green functions describe the propagation of virtual particles. In order to make contact with experiments, we have to find therefore the link between Green functions and experimental results from scattering experiments. We introduce first the scattering matrix \( S \) which is an unitary operator mapping the initial state at \( t = -\infty \) on the final state at \( t = +\infty \). The unitarity of the \( S \)-matrix restricts the analytic structure of Feynman amplitudes, in particular it implies the optical theorem. Then we derive the connection between \( n \)-point Green functions and scattering amplitudes, justifying our Feynman rules for Feynman amplitudes, before we perform some explicit calculations of few tree-level processes. Finally, we consider the emission of additional soft particles and show that this process can be described by an universal factor.

The relation between Feynman amplitudes and cross sections or decay widths is essentially the same as in non-relativistic quantum mechanics; it is reviewed in appendix 6.A.

### 6.1. Unitarity of the S-matrix and its consequences

A scattering process is fully described in the Schrödinger picture by the knowledge how initial states \( |i,t\rangle \) at \( t \to -\infty \) are transformed into final states \( |f,t\rangle \) at \( t \to \infty \). This knowledge is encoded in the \( S \)-matrix elements

\[
|f,t = \infty\rangle = S_{fi} |i,t = -\infty\rangle .
\]

An intuitive, but mathematically delicate definition of the scattering operator \( S \) is the \( t \to \infty \) limit of the time-evolution operator \( U(t,-t) \)

\[
S = \lim_{t \to \infty} U(t,-t) .
\]

Thus the scattering operator \( S \) evolves an eigenstate \( |n,t\rangle \) of the Hamiltonian from \( t = -\infty \) to \( t = +\infty \),

\[
S |n,-\infty\rangle = |n,\infty\rangle .
\]

The unitarity of the scattering operator, \( S^\dagger S = SS^\dagger = 1 \), expresses the fact that we (should) use a complete set of states for the initial and final states in a scattering process,

\[
1 = \sum_n |n,+\infty\rangle \langle n,+\infty| = \sum_n S |n,-\infty\rangle \langle n,-\infty| S^\dagger = SS^\dagger
\]

\[
= \sum_n |n,-\infty\rangle \langle n,-\infty| = \sum_n S^\dagger |n,+\infty\rangle \langle n,+\infty| S = S^\dagger S .
\]
6. Scattering processes

**Optical theorem** We split the scattering operator $S$ into a diagonal part and the transition operator $T$, $S = 1 + iT$, and thus

$$1 = (1 + iT)(1 - iT^\dagger) = 1 + iT - T^\dagger + TT^\dagger$$

(6.6)

or

$$iT^\dagger = T - T^\dagger.$$  

(6.7)

Note that in perturbation theory the LHS is $O(g^n)$, while the RHS is $O(g^n)$. Hence this equation implies a non-linear relation between the transition operator evaluated at different order. At lowest order perturbation theory, the LHS vanishes and $T$ is real, $T = T^\dagger$. This corresponds to our observation that 2-point functions are imaginary for physical values of the energy.

We consider now matrix elements between the initial and final state,

$$\langle f | T - T^\dagger | i \rangle = T_{fi} - T^*_{if} = i \langle f | TT^\dagger | i \rangle = i \sum_n T_{fn} T^*_{in}.$$  

(6.8)

If we set $|i \rangle = |f \rangle$, we obtain a connection between the forward scattering amplitude $T_{ii}$ and the total cross section $\sigma_{\text{tot}}$ called the optical theorem,

$$2\Im T_{ii} = \sum_n |T_{in}|^2.$$  

(6.9)

The optical theorem informs us that the attenuation of a beam of particles in the state $i$, $dN_i \propto -|\Im T_{ii}|^2 N_i$, equals the total probability that they scatter into states $n$.

The RHS is given by the total cross section $\sigma_{\text{tot}}$ up to a factor depending on the flux of initial particles and possible symmetry factors. For the case of two particles in the initial state, comparison with (6.172,6.173) shows that

$$\Im A_{ii} = \frac{1}{2p_{\text{cms}} \sqrt{s}} \sigma_{\text{tot}}.$$  

(6.10)

Note also that the forward scattering amplitude $T_{ii}$ means *scattering* without change in any conserved quantum number, since we have separated already the identity part, $T_{ii} = (S_{ii} - 1)/i$.

**Imaginary part of the amplitude** Consider the Feynman amplitude $A$ as a complex function of the squared center-mass energy $s$. The threshold energy $\sqrt{s_0}$ in the cms equals the minimal energy for which the reaction is kinematically allowed. For $s < s_0$ and $s \in \mathbb{R}$, and, because of the optical theorem, also $A$ are real, $s = s^*$ and $A(s) = |A(s)|^*$. Thus

$$A(s) = |A(s^*)|^* \quad \text{for} \quad s < s_0.$$  

(6.11)

If $A(s)$ is an analytic function, then also the RHS is analytic and we can continue the relation into the complex $s$ plane. In particular, along the real axis for $s > s_0$

$$\Re A(s + i\varepsilon) = \Re A(s - i\varepsilon) \quad \text{and} \quad \Im A(s + i\varepsilon) = -\Im A(s - i\varepsilon).$$  

(6.12)

Thus starting from $s_0$, the amplitude $M$ has a branch cut along the real $s$ axis. How does an imaginary part in a Feynman diagram arise? From

$$\frac{1}{x \pm i\varepsilon} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x),$$  

(6.13)
we see that virtual particles which propagate on-shell lead to poles and to imaginary terms in the amplitude.

The second relation in \((6.12)\) allows us to obtain the imaginary part of a Feynman amplitude calculating its discontinuity,

\[
\text{disc}(\mathcal{A}) \equiv \mathcal{A}(s + i\varepsilon) - \mathcal{A}(s - i\varepsilon) = 2\Im\mathcal{A}(s + i\varepsilon) .
\]

The prototype of a function having a discontinuity and a branch cut (along \(R^-\)) is the logarithm,

\[
\text{Ln}(z) = \text{Ln}(re^{i\varphi}) = \ln(re^{i\varphi}) + 2k\pi i = \ln(r) + (\varphi + 2k\pi)i
\]

with \(\Im\ln(x + i\varepsilon) = -\pi\).

Example: Verify the optical theorem for \(\phi\phi \to \phi\phi\) scattering at \(O(\lambda^2)\):

The logarithmic terms in the scattering amplitude (2.147) for \(\phi\phi \to \phi\phi\) scattering at one-loop have the form

\[
f(q^2, m) = \int_0^1 dz \ln \left[ m^2 - q^2 z(1 - z) \right]
\]

with \(q^2 = \{s, t, u\}\). In the physical region, the relation \(q^2 > 4m^2\) holds only for the \(s\) channel diagram. The argument of the logarithm becomes negative for

\[
x_{1/2} = \frac{1 \pm \sqrt{1 - 4m^2/s^2}}{2} = \frac{1}{2} \pm \frac{1}{2} \beta
\]

with \(\beta = \sqrt{1 - 4m^2/s^2}\). Using now \(\Im[\ln(-x + i\varepsilon)] = \pm\pi\), the imaginary part follows as

\[
\Im(A) = \pi \frac{\lambda^2}{32\pi^2} \int_{\frac{1}{2} - \frac{1}{2} \beta}^{\frac{1}{2} + \frac{1}{2} \beta} dx = \frac{\lambda^2}{32\pi} \beta.
\]

The optical theorem implies thus that the total cross section \(\sigma_{\text{tot}}(\phi\phi \to \text{all})\) at \(O(\lambda^2)\) equals

\[
\sigma_{\text{tot}} = \frac{\Im\mathcal{A}_{\text{el}}}{2p_{\text{cme}} \sqrt{s}} = \frac{\lambda^2}{32\pi s},
\]

where we used \(2p_{\text{cme}} = \sqrt{s}\beta\). On the other hand, the Feynman amplitude at tree level is simply \(\mathcal{A} = \lambda\) and thus the elastic cross section for \(\phi\phi \to \phi\phi\) scattering follows as \(\sigma_{\text{el}} = \lambda^2/(32\pi s)\). At \(O(\lambda^2)\), the only reaction contributing to the total cross section is elastic scattering, and thus the two cross sections agree.

Note also the treatment of the symmetry factors: In the loop diagram, the symmetry factor \(S = 1/2!\) is already included, while the corresponding factor for the two identical particles in the final state is added only integrating the cross section.

6.2. LSZ reduction formula

We defined the generating functional \(Z[J]\) as the vacuum-vacuum transition amplitude in the presence of a classical source \(J\). Thus the path integral

\[
\langle 0, \infty|0, -\infty\rangle_J = Z[J] = \int \mathcal{D}\phi \exp i \int_\Omega d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J\phi \right)
\]
contains the boundary condition $\phi(x) \to 0$ for $t \to \pm \infty$. We can either try to connect the
Green functions derived from $Z[J]$ to $S$-matrix elements or define a new functional with the
appropriate boundary conditions. We choose the first way, restricting ourselves for simplicity
to the case of a real scalar field.

Let us start with the case of a $2 \to 2$ scattering process. We can generate a free two-particle
state composed of plane-waves by
$$|k_1, k_2\rangle = a^\dagger(k_1)a^\dagger(k_2)|0\rangle. \tag{6.21}$$

We obtain localized wave packets defining new creation operators with
$$a_1^\dagger = \int d^3k \, f_1(k)a^\dagger(k), \tag{6.22}$$
where $f_1(k)$ is e.g. a Gaussian centered around $k_1$,
$$f_1(k) \propto \exp[-(k - k_1)^2/(2\sigma^2)]. \tag{6.23}$$

We assume that our initial state at $t = -\infty$ can be described by freely propagating wave
packets,
$$|i\rangle = \lim_{t \to -\infty} a_1^\dagger(t)a_2^\dagger(t)|0\rangle = |k_1, k_2; -\infty\rangle, \tag{6.24}$$
and similarly the final state as
$$|f\rangle = \lim_{t \to \infty} a_1^\dagger(t)a_2^\dagger(t)|0\rangle = |k_1', k_2'; +\infty\rangle. \tag{6.25}$$

Green functions are the time-ordered vacuum expectation value of field operators. The first
property, time-ordering, is automatically satisfied for the transition amplitude $\langle f|i\rangle$, since we
can write
$$\langle f|i\rangle = \lim_{t \to \infty} \langle 0|a_1'(t)a_2'(t)a_1^\dagger(-t)a_2^\dagger(-t)|0\rangle
= \lim_{t \to \infty} \langle 0|T\{a_1'(t)a_2'(t)a_1^\dagger(-t)a_2^\dagger(-t)\}|0\rangle. \tag{6.26}$$

Thus we only have to re-express the creation and annihilation operators as (projected) field
operators. We define a scalar product for solutions of the Klein-Gordon equation as follows,
$$\langle \phi, \chi \rangle = i \int d^3x \, \phi^*(x)\overleftrightarrow{\partial} \chi(x) \equiv i \int d^3x \left[ \phi^*(x)\frac{\partial \chi(x)}{\partial t} - \frac{\partial \phi^*(x)}{\partial t}\chi(x) \right]. \tag{6.27}$$

Comparing this definition to Eq. (3.12), we see that the scalar product is the zero component
of the conserved current $J^\mu$. Thus the value of the scalar product $\langle \phi, \chi \rangle$ is time-independent
and corresponds to the number of particles minus the number of anti-particles.

For plane-wave components with definite momentum,
$$\phi_k(x) = \frac{1}{\sqrt{(2\pi)^32\omega_k}}\, e^{-ikx} = N_k \, e^{-ikx}, \tag{6.28}$$
the scalar product is given by
$$\langle \phi_k, \phi_{k'} \rangle = \delta(k - k'), \quad \text{and} \quad \langle \phi^*_k, \phi^*_{k'} \rangle = -\delta(k - k'). \tag{6.29}$$
and zero otherwise,
\[(\phi_k, \phi_{k'}^* ) = (\phi_{k'}^*, \phi_k ) = 0.\]

Thus we can invert the free field
\[
\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^32\omega_k}} \left[ a(k)e^{-ikx} + a^\dagger(k)e^{+ikx} \right],
\]
(6.30)
to obtain
\[
a^\dagger(k) = - (\phi_k, \phi) = -iN_k \int d^3x \ e^{-ikx} \frac{\partial}{\partial x} \phi(x).
\]
(6.31)

Using the identity
\[
a^\dagger(k, \infty) - a^\dagger(k, -\infty) = -i \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} a^\dagger(k, t)
\]
(6.32)
we insert first (6.31) assuming a wave-package localized around \(k_1\) and perform then the time differentiation,
\[
a^\dagger(k_1, \infty) - a^\dagger(k_1, -\infty) = -i \int d^3k f_1(k) \int d^4x \ \partial_t \left( e^{-ikx} \frac{\partial}{\partial x} \phi(x) \right)
\]
(6.33)
\[
= -i \int d^3k f_1(k) \int d^4x \ \left( e^{-ikx} \partial_t^2 \phi(x) - \phi(x) \partial_t^2 e^{-ikx} \right),
\]
where the two terms linear in \(\partial_t\) canceled. Then we use that the field is on-shell, \(k^2 = m^2\), for the replacement
\[\partial_t^2 e^{-ikx} = (\nabla^2 - m^2) e^{-ikx}.\]

Since the field is localized, we can perform two partial integrations moving thereby \(\nabla^2\) to the left, obtaining
\[
a^\dagger(k_1, \infty) - a^\dagger(k_1, -\infty) = -i \int d^3k f_1(k) \int d^4x \ e^{-ikx} \left( \Box + m^2 \right) \phi(x).
\]
(6.34)

In a free theory, \(\phi(x)\) satisfies the Klein-Gordon equation and the RHS would vanish. Since we aim to solve an interacting theory with e.g. \(\mathcal{L}_I = \lambda \phi^4/4!\), then \((\Box + m^2) \phi(x) = \lambda \phi^3/3! \neq 0\).

Now we can forget the wave-packet, \(\sigma \to 0\), and write simply
\[
a^\dagger(k, -\infty) = a^\dagger(k, \infty) + iN_k \int d^4x \ e^{-ikx} \left( \Box + m^2 \right) \phi(x).
\]
(6.35)

Taking the hermetian conjugate, we obtain for the annihilation operator
\[
a(k, \infty) = a(k, -\infty) + iN_k \int d^4x \ e^{ikx} \left( \Box + m^2 \right) \phi(x).
\]
(6.36)

When we insert these expressions into \(\langle f | i \rangle\), we obtain a four-point function combining the second terms from the RHS of (6.35) and (6.36). Including the terms \(a^\dagger(k, \infty)\) and \(a(k, -\infty)\) generates particles propagating from \(t = -\infty\) to \(t = +\infty\) with momenta unchanged, i.e. to terms corresponding to disconnected graphs. Hence we do not need to consider these.
6. Scattering processes

contributions, restricting ourselves to connected Green functions\(^1\). For \(n\) particles in the initial and \(m\) particles in the final state, we obtain

\[
\langle k_1, \ldots, k_n; +\infty \mid k_1, \ldots, k_n; -\infty \rangle = (i)^{n+m} \prod_{i=1}^{n+m} \int d^4 x_i \, N_{k_i} \, e^{-ik_i x_i} \left( \Box_{x_i} + m^2 \right) \\
\prod_{j=1}^{m} \int d^4 y_j \, N_{k_j} \, e^{ik_j y_j} \left( \Box_{y_j} + m^2 \right) \langle 0 \mid T\{\phi(x_1) \cdots \phi(y_{m+n})\} \mid 0 \rangle .
\]

(6.37)

This is the reduction formula of Lehmann, Symanzik and Zimmermann (LSZ): For each external particle we obtained the corresponding plane wave component and a Klein-Gordon operator. Since the latter is the inverse of the free 2-point functions, we can rephrase the content of the LZZ formula simply as follows: Replace the 2-point functions of external lines by appropriate wave functions, e.g. \(\phi_k(x)\) for scalar particles in the initial state and \(\phi_k^*(x)\) for scalar particles in the final state.

Since we started from field operators in the Heisenberg picture, the matrix element is in the Heisenberg picture, too. In the Schrödinger picture, it is

\[
\langle k_1, \ldots, k_n \mid iT \mid k_1, \ldots, k_n \rangle = iT_{f1},
\]

where we used also \(S = 1 + iT\) and the fact that we neglected disconnected parts.

Finally, we define the Fourier transformed \(n\)-point function as

\[
G(x_1, \ldots, x_n) = \int \prod_{i=1}^{n} \frac{d^4 k_i}{(2\pi)^4} \exp \left( -i \sum_i k_i x_i \right) G(k_1, \ldots, k_n).
\]

(6.38)

Then we obtain the LSZ reduction formula in momentum space,

\[
 iT_{f1} = (i)^{n+m} N_{k_1} \cdots N_{k_n} N_{k_1'} \cdots N_{k_m'} (k_1^2 - m^2) \cdots (k_n^2 - m^2)(k_1'^2 - m^2) \cdots (k_m'^2 - m^2) \\
G(k_1, \ldots, k_n, -k_1, \ldots, -k_m).
\]

(6.39)

The Green function \(G(k_1, \ldots, k_n, -k_1, \ldots, -k_m)\) is multiplied by zeros, since the external particles satisfy \(k^2 = m^2\). Thus \(T_{f1}\) vanishes, except when poles \(1/(k^2 - m^2)\) of \(G(k_1, \ldots, k_n, -k_1, \ldots, -k_m)\) cancel these zeros. In the case of external scalar particles, only their normalization factors are left. As they are not essential for the calculation of the transition amplitudes, one include these normalization factors into the phase space of final state particle and in the flux factor of initial particles. This explains our Feynman rule to replace the scalar propagator by one for amplitudes in momentum space.

The derivation of the LSZ formula for particles with spin \(s > 0\) proceeds in the same way. Their wave-functions contain additionally polarization vectors \(\varepsilon^\mu(k)\), tensors \(\varepsilon^{\mu\nu}(k)\), or spinors \(u(p)\) and \(\bar{u}(p)\) and their charge conjugated states. In the case of a photon (graviton), we have to add \(\varepsilon^\mu(k) (\varepsilon^{\mu\nu}(k))\) in the initial state and the complex conjugated \(\varepsilon^{\mu\nu}(k) (\varepsilon^{\nu\mu}(k))\) in the final state. In the case of Dirac fermions, we have to assign four different spinors to the four possible combinations of particle and anti-particles in the initial and final state. Having chosen \(u(p)\) as particle state with an arrow along the direction of time, the simple rule that a

\(^1\)You may wonder that we eliminate in this way also the forward scattering amplitudes \(T_{ii}\): Analyticity of scattering amplitudes guarantees that the amplitude calculated for, e.g., scattering angle \(\vartheta > 0\) contains also correctly the case \(\vartheta = 0\).
fermion line corresponds to a complex number $\bar{\psi} \cdots \psi$ fixes the designation of the other spinors as shown in Fig. 6.1. Connecting the upper fermion lines, $\bar{u}(p') \cdots u(p)$, corresponds to the scattering $e^-(p) + X \rightarrow e^-(p') + X'$, while $\bar{v}(q) \cdots u(p)$ describes the annihilation process $e^+(q) + e^-(p) \rightarrow X + X'$. Connecting the lower fermion lines, $\bar{v}(q) \cdots v(q)$, corresponds to the scattering $e^+(q) + X \rightarrow e^+(q') + X'$, while $\bar{u}(p') \cdots v(q)$ describes the pair creation process $X + X' \rightarrow e^+(q') + e^-(p')$. Note that the Feynman amplitude $A$ is defined omitting the normalisation factor $N_p = [2\omega_p(2\pi)^3]^{-1/2}$ from all wave-functions.

Figure 6.1.: Feynman rules for external particles in momentum space; initial state on the left, final state on the right.

Wave function renormalization. Up to now we have pretended that we can describe the fields in the initial and final state as free particles. Although e.g. Yukawa interactions between two, by assumption, widely separated particles at $t = \pm \infty$ are negligibly small, self-interactions persist. These interactions lead to a renormalization of the external wave-functions.

We can rephrase the problem as follows: If the creation operator $a^\dagger(k, -\infty)$ corresponds to the one of a free theory, then it can only connect one-particle states with the vacuum. Thus the expectation value between the vacuum and any many-particle state is zero. In contrast, the interacting field can also connect many-particle states to the vacuum and therefore its overlap with single-particle states is reduced,

$$a^\dagger(k, -\infty) |0\rangle = \sqrt{Z} |k\rangle + \sqrt{1-Z} \{ |k', k'', k'''\rangle + \ldots \}$$

$$= \sqrt{Z} a^\dagger_0(k_1) |0\rangle + \sqrt{1-Z} \{ |k', k'', k'''\rangle + \ldots \} \ldots (6.40)$$

We are lead to conclude that the free and the interacting fields are connected by

$$\phi(x) \rightarrow \sqrt{Z} \phi_0(x) (6.42)$$

for $t \rightarrow \pm \infty$. More precisely, this relation holds only for matrix elements of the fields, not for the field operators themselves.

As a result, we should include in all formulas derived above appropriate factors of the wave-function normalization constant $Z$. However, we will show later that these factors are canceled. A simpler short-cut is therefore to exclude all corrections in external legs and to use the LSZ formula as derived.
6.3. Specific processes

We consider now in detail a few specific processes. First, we derive the Klein-Nishina formula for the Compton scattering cross section using the standard “trace method”. Then we calculate polarized $e^+e^- \rightarrow \mu^+\mu^-$ and $e^+e^- \rightarrow \gamma\gamma$ scattering using helicity methods.

6.3.1. Trace method: Compton scattering and the Klein-Nishina formula

Matrix element The Feynman amplitude $A$ of Compton scattering $e^- (p) + \gamma(k) \rightarrow e^- (p') + \gamma(k')$ at $O(e^2)$ consists of two diagrams,

$$iA = -ie^2 \bar{u}(p') \left[ \frac{\not{p} + \not{k} + m}{(p + k)^2 - m^2} \not{\epsilon} + \frac{\not{p} - \not{k}' + m}{(p - k')^2 - m^2} \not{\epsilon}' \right] u(p). \quad (6.43)$$

Since the denominator is a non-zero light-like vector, we have omitted the $i\epsilon$. Note that the two amplitudes can be transformed into each other replacing $\epsilon \leftrightarrow \epsilon'$ and $k \leftrightarrow -k'$. This symmetry called crossing symmetry relates processes where a particle is replaced by an anti-particle with negative momentum on the other side of the reaction.

We evaluate the process in the rest-frame of the initial electron. Then $p = (m, 0)$ and choosing $\epsilon^\mu = (0, \epsilon)$ as well as $\epsilon'^\mu = (0, \epsilon')$, it follows

$$p \cdot \epsilon = p \cdot \epsilon' = 0. \quad (6.44)$$

Moreover, the photons are transversely polarized,

$$k \cdot \epsilon = k' \cdot \epsilon' = 0, \quad (6.45)$$

and we choose real polarization vectors. We anti-commute $\not{p}$ in the nominator to the right, $\not{p} \not{\epsilon}' = 2p \cdot \epsilon' - \not{\epsilon}' \not{p} = -\not{\epsilon}' \not{p}$ and use the Dirac equation, $\not{p} u(p) = m u(p)$. Moreover, we can simplify also the denominator with $p^2 = m^2$ and obtain

$$A = e^2 \bar{u}(p') \left[ \frac{\dot{\epsilon}' \not{k} \not{k}'}{2p \cdot k} + \frac{\dot{\epsilon}' \not{k} \not{k}'}{2p \cdot k'} \right] u(p). \quad (6.46)$$

Often the initial electron target is not polarized, and the spin of the final electron is not measured. Thus we sum the squared matrix element over the final and average over the initial electron spin,

$$|A|^2 = \frac{1}{2} \sum_{s,s'} |A|^2 = \frac{e^4}{2} \sum_{s,s'} \left| \bar{u}(p') \left( \frac{\dot{\epsilon}' \not{k} \not{k}'}{2p \cdot k} + \frac{\dot{\epsilon}' \not{k} \not{k}'}{2p \cdot k'} \right) u(p) \right|^2. \quad (6.47)$$

Calculating $|A|^2 = A^* A$ requires the knowledge of $A^* = A^\dagger$. For a general amplitude $A$ composed of spinors,

$$A = \bar{u}(p') \Gamma u(p), \quad (6.48)$$

with $\Gamma$ denoting a product of the basis elements given in Eq. (5.38), we have with $\bar{u} = u^\dagger \gamma^0$

$$A^* = \bar{u}(p) \gamma^0 \Gamma^\dagger \gamma^0 u(p') \equiv \bar{u}(p) \bar{\Gamma} u(p'). \quad (6.49)$$
and thus
\[ \Gamma = \gamma^0 \Gamma^\dagger \gamma^0 \] (6.50a)
\[ \Gamma^\mu = \gamma^\mu \] (6.50b)
\[ \gamma^5 = -\gamma^5 \] (6.50c)
\[ \not\!b \cdot \not\!\ell = \not\!b \cdot \not\!\ell \] (6.50d).

We now write out the spinor indices,
\[ |A|^2 = \frac{\epsilon^4}{2} \sum_{s,s'} \bar{u}_a(p') \left[ \frac{\not\!k \not\!k}{2p \cdot k} + \frac{\not\!k' \not\!k'}{2p' \cdot k'} \right]_{ab} u_b(p) \bar{u}_c(p) \left[ \frac{\not\!k \not\!k}{2p \cdot k} + \frac{\not\!k' \not\!k'}{2p' \cdot k'} \right]_{cd} u_d(p') \] (6.51)

Using the property (5.51) of the Dirac spinors,
\[ \sum_s u_a(p,s) \bar{u}_b(p,s') = (\not\!p + m)_{ab} \] (6.52)
we obtain
\[ |A|^2 = \frac{\epsilon^4}{8} \sum_s \{ (\not\!p + m) \} \left[ \frac{\not\!k \not\!k}{p \cdot k} + \frac{\not\!k' \not\!k'}{p' \cdot k'} \right] (\not\!p + m) \left[ \frac{\not\!k \not\!k}{p \cdot k} + \frac{\not\!k' \not\!k'}{p' \cdot k'} \right] \] (6.53)

Since \( \not\!p + m \) combines a spinor in \( \mathcal{M} \) and one in \( \mathcal{M}^\ast \), the result is a trace over gamma matrices,
\[ |A|^2 = \frac{\epsilon^4}{8} \sum_s \text{tr} \left\{ (\not\!p + m) \left[ \frac{\not\!k \not\!k}{p \cdot k} + \frac{\not\!k' \not\!k'}{p' \cdot k'} \right] (\not\!p + m) \left[ \frac{\not\!k \not\!k}{p \cdot k} + \frac{\not\!k' \not\!k'}{p' \cdot k'} \right] \right\} \] (6.54)

Useful identities for the evaluation of such traces are given in the appendix.

We simplify this trace by anti-commuting identical variables, such that they become neighbours. Then we can use \( \not\!\phi \not\!\phi = \alpha^2 \) and reduce thereby the number of gamma matrices in each step by two. Multiplying out the terms in the trace, we obtain three contributions that we denote by
\[ \text{tr} \{ \} = \frac{S_1}{(p \cdot k)^2} + \frac{S_2}{(p' \cdot k')^2} + \frac{2S_3}{(p \cdot k)(p' \cdot k')} \] (6.55)

We consider only the first term \( S_1 \) in detail. Starting from
\[ S_1 = \text{tr} \left\{ (\not\!p + m) \not\!k' \not\!k \not\!\phi \not\!\phi \not\!k' \not\!k' \right\} \] (6.56)
\[ = \text{tr} \left\{ \not\!p' \not\!k' \not\!k' \not\!k' \right\} + m^2 \text{tr} \left\{ \not\!k' \not\!k' \not\!k' \not\!k' \right\} \] (6.57)
\[ = 2 (k \cdot p) \text{tr} \left\{ \not\!p' \not\!k' \not\!\phi \not\!\phi \right\} - \text{tr} \left\{ \not\!p' \not\!k' \not\!\phi \not\!\phi \not\!k' \not\!k' \right\} \] (6.58)
we arrived at an expression with only six gamma matrices. We continue the work,
\[ S_1 = 2 (k \cdot p) \text{tr} \left\{ \not\!p' \not\!k' \not\!\phi \not\!\phi \not\!k' \not\!k' \right\} = -2 (k \cdot p) \text{tr} \left\{ \not\!p' \not\!k' \not\!\phi \not\!\phi \not\!k' \not\!k' \right\} \] (6.59)
\[ = 2 (k \cdot p) \text{tr} \left\{ \not\!p' \not\!k' \not\!\phi \not\!\phi \not\!k' \not\!k' \right\} = 2 (k \cdot p) \left[ 2 (k \cdot \not\!\epsilon') \text{tr} (\not\!p' \not\!\epsilon') - \text{tr} \left\{ \not\!p' \not\!k' \not\!\phi \not\!\phi \not\!k' \not\!k' \right\} \right] \] (6.60)
\[ = 8 (k \cdot p) \left[ 2 (k \cdot \not\!\epsilon') \left( p' \cdot \not\!\epsilon' \right) + (p' \cdot k) \right] \] (6.61)

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6. Scattering processes

We want to eliminate as next step the two scalar products that include $p'$. Because of four-momentum conservation $p' + k' = p + k$, it is

$$ (p' - k)^2 = (p - k')^2 $$  \hfill (6.62)

and thus

$$ p' \cdot k = p \cdot k' . $$  \hfill (6.63)

Multiplying the four-momentum conservation equation by $\varepsilon'$, it follows moreover

$$ p_1 + k_1 = p_2 + k_2 \Rightarrow \varepsilon' \cdot (p + k') = \varepsilon' \cdot (p' + k') . $$  \hfill (6.64)

Thus our final result for $S_1$ is

$$ S_1 = 8 (k \cdot p) \left[ 2 (k \cdot \varepsilon')^2 + k' \cdot p \right] . $$  \hfill (6.65)

$S_2$ can be obtained observing the crossing symmetry of the amplitude by the replacements $\varepsilon \leftrightarrow \varepsilon'$ and $k \leftrightarrow -k'$. The cross term $S_3$ has to be calculated and we give here only the final result for the combination of the three terms, where some terms cancel

$$ \left| \mathcal{A} \right|^2 = \frac{e^4}{\omega} \left[ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\varepsilon \cdot \varepsilon')^2 - 2 \right] . $$  \hfill (6.66)

Cross section  To obtain the cross section, we have to calculate the flux factor and to perform the integration over the phase space of the final state,

$$ d\sigma = \frac{1}{4I} (2\pi)^4 \delta^{(4)}(P_i - P_f)|A_{fi}|^2 \prod_{j=1}^n \frac{d^3 p_j}{2E_j (2\pi)^3} = \frac{1}{4I} |A_{fi}|^2 d\Phi^{(n)} , $$  \hfill (6.67)

with the final state phase space $d\Phi^{(n)}$. The flux factor $I$ in the rest system of the electron is simply

$$ I \equiv v_{rel} p_1 \cdot p_2 = m\omega . $$  \hfill (6.68)

Using Eq. (2.179),

$$ d^3 p' \frac{2E'}{2E} = d^4 p' \delta^{(4)}(p'^2 - m^2) \theta(p'_0) , $$  \hfill (6.69)

the phase space integration becomes

$$ d\Phi^{(2)} = \frac{1}{(2\pi)^2} \int d\Omega_{k'} \frac{|k'|^2 d^3 k'}{2|k'|} \int \frac{d^4 p'}{2E'} \delta^{(4)} (p' + k' - p - k) = $$

$$ = \frac{1}{8\pi^2} \int d\Omega_{k'} |k'| d^3 k' \delta \left( (p + k - k')^2 - m^2 \right) . $$  \hfill (6.70)

The argument of the delta function is

$$ (p + k - k')^2 - m^2 = m^2 + 2p \cdot k - 2p \cdot k' - 2k \cdot k' - m^2 $$

$$ = 2m|k| - 2m|k'| - 2|k||k'| (1 - \cos \vartheta) $$

$$ = 2m (\omega - \omega') - 2\omega \omega' (1 - \cos \vartheta) \equiv f(\omega') . $$  \hfill (6.71)

$$ (p + k - k')^2 - m^2 = m^2 + 2p \cdot k - 2p \cdot k' - 2k \cdot k' - m^2 $$

$$ = 2m|k| - 2m|k'| - 2|k||k'| (1 - \cos \vartheta) $$

$$ = 2m (\omega - \omega') - 2\omega \omega' (1 - \cos \vartheta) \equiv f(\omega') . $$  \hfill (6.72)
In order to use
\[ \int dx \delta \left[ f(x) \right] g(x) = \frac{g(x)}{|F(x)|} \bigg|_{f(x_0) = 0} \] (6.75)
we have to determine the derivative \( f'(\omega') \),
\[ f'(\omega') = -2m - 2\omega' (1 - \cos \vartheta) \] (6.76)
and the zeros of \( f(\omega') \),
\[ 0 = 2m (\omega - \omega') - 2\omega \omega' (1 - \cos \vartheta) . \] (6.77)
Solving for \( \omega' \) gives
\[ \omega' [\omega (1 - \cos \vartheta) + m] = m\omega \] (6.78)
and
\[ \omega' = \frac{\omega}{1 + \frac{\omega}{m} (1 - \cos \vartheta)} . \] (6.79)
This is the famous relation for the frequency shift of a photon found first experimentally in the scattering of X-rays on electrons by Compton 1921. The observed energy change of photons was crucial in accepting the quantum nature ("particle-wave duality") of photons. Combining everything, we obtain
\[ d\Phi(2) = \frac{1}{8\pi^2} \int d\Omega_{k'} |\omega'| d\omega' \delta (2m (\omega - \omega') - 2\omega \omega' (1 - \cos \vartheta)) \] (6.80)
and thus as differential cross section (Klein, Nishina 1928)
\[ \frac{d\sigma}{d\Omega} = \frac{1}{4m\omega} \frac{\omega'^2}{16\pi^2 m\omega} |\mathbf{A}|^2 = \alpha^2 \frac{\omega'^2}{4m^2 \omega^2} \left[ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\varepsilon \cdot \varepsilon')^2 - 2 \right] \] (6.82)
with \( \alpha \equiv e^2/(4\pi) \). For scatterings in the forward direction, \( \vartheta \to 0 \) and thus \( \omega' \to \omega \), the scattered photon retains (in the lab frame) its energy even in the ultra-relativistic limit \( \omega \gg m \). The same holds in the classical limit, \( \omega \ll m \), but now for all directions. Thus we obtain as classical limit of the Klein-Nishina formula the polarized Thomson cross section
\[ \frac{d\sigma}{d\Omega} \approx \alpha^2 \left( \frac{\varepsilon \cdot \varepsilon'}{m^2} \right)^2 = r_0^2 \left( \varepsilon \cdot \varepsilon' \right)^2 , \] (6.83)
with \( r_0^2 = \alpha/m \) as the classical electron radius.
Averaging and summing over the photon polarization vectors is simplest, if we choose the angle between \( \varepsilon \) and \( \varepsilon' \) as \( \vartheta \). Then
\[ \sum_{r,r'} \left( \varepsilon \cdot \varepsilon' \right)^2 = 1 + \cos^2 \vartheta . \] (6.84)
The integration over the scattering angle \( \vartheta \) can be done analytically. We use \( x = \cos \vartheta \) and set \( \tilde{\omega} \equiv \omega/m \),
\[ \sigma = \frac{\pi \alpha^2}{m^2} \int_{-1}^1 dx \left[ \frac{1}{[1 + \tilde{\omega}(1 - x)]^3} + \frac{1}{1 + \tilde{\omega}(1 - x)} - \frac{1 - x^2}{[1 + \tilde{\omega}(1 - x)]^2} \right] \] (6.85)
\[ = \frac{\pi \alpha^2}{2m^2} \left\{ \frac{1 + \tilde{\omega}}{\tilde{\omega}^3} \left[ 2\tilde{\omega}(1 + \tilde{\omega}) - \ln(1 + 2\tilde{\omega}) \right] + \ln(1 + 2\tilde{\omega}) \frac{1 + 3\tilde{\omega}}{2\tilde{\omega}} - \frac{1 + 3\tilde{\omega}}{(1 + 2\tilde{\omega})^2} \right\} . \] (6.86)
Since in the electron rest frame \( s = (p + k)^2 = m^2 + 2m\omega = m^2(1 + 2\tilde{\omega}) \), we can use \( \tilde{\omega} = (s/m^2 - 1)/2 \) to express \( \sigma \) in an explicit Lorentz invariant form.

Approximations for the non-relativistic and the ultra-relativistic limit are

\[
\sigma = \sigma_{\text{Th}} \times \begin{cases} 
1 - 2\tilde{\omega} + \mathcal{O}(\tilde{\omega}^2) & \text{for } \tilde{\omega} \ll 1, \\
\frac{3}{8\omega} (\ln(2\tilde{\omega}) + \frac{1}{2}) + \mathcal{O}(\tilde{\omega}^{-2}) & \text{for } \tilde{\omega} \gg 1,
\end{cases}
\]

(6.87)

where the Thomson cross section is given by \( \sigma_{\text{Th}} = \frac{8\pi\alpha^2}{3m^2} \). These approximations are shown together with the exact result in the left panel of Fig. 6.2.

In the ultra-relativistic limit \( s \gg m^2 \), the total cross section for Compton scattering decreases as \( \sigma \propto 1/s \). On the other hand, the differential cross section in the forward direction is constant. As a result, the relative importance of the forward region \( \vartheta \sim 0 \) increases for increasing \( s \) even faster than required by unitarity: While \( d\sigma/dx \) is symmetric around \( x = 0 \) in the classical limit \( \omega \rightarrow 0 \), it becomes more and more asymmetric with a a shrinking peak around the forward region at \( \vartheta \sim 0 \), cf. the right panel of Fig. 6.2.

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**Crossing symmetry** We noticed that the two amplitudes in Compton scattering can be transformed into each other replacing \( \varepsilon \leftrightarrow \varepsilon' \) and \( k \leftrightarrow -k' \). This is an example of a general symmetry of relativistic quantum field theories called crossing symmetry. Using the Feynman rules for in and out particles, it follows that matrix elements where an in-going particle is replaced by an out-going anti-particle or vice versa are related by the following substitutions,

- exchange the momentum \( k \leftrightarrow -k' \);
- exchange particle and anti-particle wave functions; thus in momentum space, \( 1 \leftrightarrow 1 \) for spinless particles, \( \varepsilon \leftrightarrow \varepsilon' \) for spin-1 and \( u \leftrightarrow v \) for fermions.
- multiply by \(-1\) for each exchanged fermion pair.

The additional minus for fermions is required, because the replacement \( p \leftrightarrow p' \) in a fermion spin sum gives

\[
\sum_s u(p,s)\bar{u}(p,s) = (\slashed{p} + m) = -(\slashed{p'} - m) = -\sum_s v(p',s)\bar{v}(p',s).
\]

(6.88)
Note that this symmetry allows us to obtain the matrix elements of different processes: For instance, we can relate the processes $e^-e^+ \rightarrow \mu^-\mu^+$ with $e^-\mu^+ \rightarrow e^-\mu^-$ and $\mu^-\mu^+ \rightarrow e^-e^+$.

In a more formal approach the crossing symmetry is derived not by relying on perturbation theory and the Feynman rules, but using the analytical properties of $S$-matrix elements in a relativistic quantum field theory. The LSZ reduction formula distinguishes in and out particles only by the sign of the momenta used in the Fourier transformation. If one can analytically continue the residue of a pole in an $S$-matrix element from $p^0$ to $-p^0$, then one converts the $S$-matrix for a particle with $\phi(p)$ into the one for an anti-particle with $\phi^*(-p)$. Remarkably, the basic properties of a relativistic quantum field theory, locality and causality, are sufficient to prove that this analytical continuation is possible.

### 6.3.2. Helicity method and polarized QED processes

The number of terms that have to calculated grows in the trace method as \( \sim n^2/2 \) with the number $n$ of diagrams. For large $n$, it should be therefore favorable to calculate the amplitude $A(s_1,\ldots,s_f)$ for fixed polarizations of the external particles: The amplitude is a complex number and can be trivially squared. An efficient way to calculate polarized amplitudes uses helicity wave functions, an approach used also in most modern computer programs as e.g. Madgraph.

**Massless fermions** We restrict our short introduction into helicity methods to massless particles. In the case of fermions, we know that then the use of Weyl spinors in the chiral representation is most convenient,

\[
u_L(p) = \begin{pmatrix} \phi_L(p) \\ 0 \end{pmatrix} \quad \text{and} \quad u_R(p) = \begin{pmatrix} 0 \\ \phi_R(p) \end{pmatrix}.
\]

We do not need to consider $v_{L,R}(p)$, since they correspond to particle spinors of opposite helicity, $u_{R,L}(p)$. Moreover, two out of the fours possible scalar products involving $u_{L,R}$ are zero for massless fermions,

\[	bar{u}_L(p)u_L(q) = \tbar{u}_R(p)u_R(q) = 0.
\]

This motivated Xu, Zhang and Chang to introduce a bracket notation for the helicity spinors as follows

\[	bar{u}_L(p) = \langle p, \quad \tbar{u}_R(p) = [p, \quad u_L(p) = p], \quad u_R(p) = p \rangle.
\]

Then the only non-zero Lorentz-invariant spinor products are given by a pair of brackets of the same type,

\[	bar{u}_L(p)u_R(q) = \langle pq \rangle \quad \text{and} \quad \tbar{u}_R(p)u_L(q) = [pq].
\]

We call the quantities on the RHS *angle* and *square brackets*. Next we consider the tensor product of the spinors,

\[p |p = u_R(p)\tbar{u}_R(p) = P_R\p, \quad \text{and} \quad p |p = u_L(p)\tbar{u}_L(p) = P_L\p. \]

These identities connect the massless spinors $p|$ and $|p$ to the light-like four-vector $p^\mu$.

We are now in position to derive some basic properties of the brackets. First, we can connect the two types of spinor products as

\[
\langle pq \rangle = \tbar{u}_L(p)u_R(q) = [\tbar{u}_R(p)u_L(q)]^* = [qp]^*.
\]
Next, multiplying $p\rangle [p$ and $q\rangle \langle q$ and taking the trace gives
\[ \langle pq\rangle [pq] = \operatorname{tr}[qpP_R] = 2p \cdot q, \]
so that
\[ |\langle pq\rangle|^2 = |[pq]|^2 = 2p \cdot q. \]
Finally, we express the spinor products through Weyl spinor s and use $u_R(p) = i\sigma^2 u_L^\dagger(p)$,
\[ \langle pq\rangle = \phi_L^\dagger(p)\phi_R(q) = \phi_L^\dagger(p)(i\sigma^2)_{ab}\phi^*_{Lb}(q). \]
Then the antisymmetry of $(i\sigma^2)_{ab} = \varepsilon_{ab}$ implies
\[ \langle pq\rangle = -\langle qp\rangle \quad \text{and} \quad [pq] = -[qp]. \]
Thus the brackets are square roots of the corresponding Lorentz vector products which are antisymmetric in their two arguments. Finally, we note that the Fierz identity applied to the sigma matrices,
\[ (\bar{\sigma}^\mu)_{ab}(\sigma^\mu)_{cd} = 2(i\sigma^2)_{ac}(i\sigma^2)_{bd}, \]
allows the simplification of contracted spinor expressions,
\[ \langle p\gamma^\mu q\rangle [k\gamma_\mu \ell] = 2\langle pk\rangle [\ell q], \quad \langle p\gamma^\mu q\rangle [k\gamma_\mu \ell] = 2\langle p\ell\rangle [kq]. \]
\( e^-e^+ \rightarrow \mu^-\mu^+ \) scattering It is now time to apply our new “bracket” formalism. We consider the tree-level amplitude for, e.g., $e^-_L(1)e^+_R(2) \rightarrow \mu^-_L(3)\mu^+_R(4)$ in QED, given by the single diagram shown in Fig. 6.3, considering all momenta as outgoing. Then the amplitude is
\[ iA = (-ie)^2 \frac{-1}{q^2} \bar{u}_L(3)\gamma^\mu u_L(4) \bar{u}_L(2)\gamma_\mu u_L(1) \]
\[ = \frac{-ie^2}{q^2} \langle 3\gamma^\mu 4\rangle [2\gamma_\mu 1] = 2\frac{ie^2}{q^2} \langle 32\rangle [14], \]
where we have used the Fierz identity (6.99) in the last step. Now, $\langle 32\rangle$ and $[14]$ are both square roots of
\[ s_{23} = (k_2 + k_3)^2 = (k_1 + k_4)^2 \]
6.3. Specific processes

which is just the Mandelstam invariant $u$. In the $e^+e^-$ center of mass frame, $u = -2E^2(1 + \cos \vartheta)$, and $q^2 = s = 4E^2$. Then

$$|\mathcal{A}|^2 = e^4(1 + \cos \vartheta)^2.$$  \hfill (6.104)

You should rederive this result using the more familiar trace formalism and compare the amount of algebra required in the two approaches (problem 6.4).

Massless gauge bosons In the next step, we incorporate massless gauge bosons as the photon into this framework. We claim that the polarization vectors for a final-state massless vector boson of definite helicity can be represented as

$$\epsilon^\mu_R(k) = \frac{1}{\sqrt{2}} \left[ \langle r\gamma^\mu k \rangle - \langle s\gamma^\mu k \rangle \right] = \frac{1}{\sqrt{2}} \left\{ -\langle r\gamma^\mu k \rangle\langle ks \rangle + \langle s\gamma^\mu k \rangle\langle kr \rangle \right\},$$

$$\epsilon^\mu_L(k) = -\frac{1}{\sqrt{2}} \left[ \langle r\gamma^\mu k \rangle - \langle s\gamma^\mu k \rangle \right]. \hfill (6.105)$$

Here $k$ is the momentum of the vector boson, and $r$ is a fixed lightlike 4-vector, called the reference vector. The only requirement on $r$ is that it cannot be collinear with $k$.

Now we show that this definition makes sense: First, we note that the vectors satisfy $[\epsilon^\mu_R(k)]^* = \epsilon^\mu_L(k)$. One can also check that the polarization vectors are correctly normalized. Moreover, the Dirac equation, $\frac{1}{\sqrt{k} k} = 0$, guarantees that the polarization vectors (6.105) are transverse, $k_{\mu\nu} \epsilon^\mu_R,L(k) = 0$. \hfill (6.106)

Finally, we have to show that a change from one reference vector $r$ to another lightlike vector $s$ corresponds to a gauge transformation and thus does not affect physics. The change of a polarization vector under a change of reference vector $r \rightarrow s$ is

$$\epsilon^\mu_R(k; r) - \epsilon^\mu_R(k; s) = \frac{1}{\sqrt{2}} \left[ \langle r\gamma^\mu k \rangle - \langle s\gamma^\mu k \rangle \right] = \frac{1}{\sqrt{2}} \left\{ -\langle r\gamma^\mu k \rangle\langle ks \rangle + \langle s\gamma^\mu k \rangle\langle kr \rangle \right\}. \hfill (6.107)$$

Now we use first the tensor products (6.93), and then the antisymmetry of the brackets,

$$\epsilon^\mu_R(k; r) - \epsilon^\mu_R(k; s) = \frac{1}{\sqrt{2}} \left[ \langle r\gamma^\mu k \rangle - \langle s\gamma^\mu k \rangle \right] = \frac{1}{\sqrt{2}} \left\{ -\langle r\gamma^\mu k \rangle\langle ks \rangle + \langle s\gamma^\mu k \rangle\langle kr \rangle \right\}. \hfill (6.108)$$

The last line finally follows from the Clifford algebra of Dirac matrices. Thus the difference of the polarization vectors induced by a change of the reference vector is a function proportional to the photon momentum,

$$\epsilon^\mu_R(k; r) - \epsilon^\mu_R(k; s) = f(r, s)k^\mu. \hfill (6.112)$$

Dotted into an on-shell photon amplitude, this expression will give zero by current conservation. Thus we can use the most convenient reference vector $s$ which can be chosen differently in any gauge-invariant set of Feynman diagrams.
### 6. Scattering processes

**e^{-} e^{+} \rightarrow \gamma \gamma \text{ scattering}**  
As second example, we consider now a scattering process with photons as external particles, $e^{-} e^{+} \rightarrow \gamma \gamma$, as illustration for the use of the polarization vectors. We label the momenta as in Fig. 6.4, taking all momenta as outgoing.

Then the amplitude for this process is

$$iA = (-ie)^2 \langle 2 | (\not{s} + 4) | 3 \rangle s_{24} \frac{i(2 + 3)}{s_{23}} \langle 3 | 4 \rangle 1 | | \frac{i(2 + 3)}{s_{23}} \langle 3 | 4 \rangle 1 | 2 \rangle = 0,$$

(6.113)

where we use the shorthand $(2 + 4)$ for $(k_2 + k_4)$ and define $s_{ij} = (i + j)^2$.

There are four possible choices for the photon polarizations. Exchanging the momenta 3 and 4 relates the cases $\gamma_R \gamma_L$ and $\gamma_L \gamma_R$, while parity connects $\gamma_R \gamma_R$ and $\gamma_L \gamma_L$. We start showing that the latter two amplitudes are zero in the massless limit we consider. For the case of $\gamma_R \gamma_R$, we choose as reference vector $r = 2$ for both polarization vectors,

$$\varepsilon^\mu(3) = \frac{1}{\sqrt{2}} \frac{[2\gamma^\mu 3]}{[23]}, \quad \varepsilon^\mu(4) = \frac{1}{\sqrt{2}} \frac{[2\gamma^\mu 4]}{[24]}.$$

(6.114)

When these choices are used in (6.113), we find, with the use of the Fierz identity (6.99),

$$\langle 2 \gamma^\mu \varepsilon_\mu(4) \rangle \propto \langle 2 \gamma^\mu [2\gamma_\mu 4] = 2(22)|4 = 0,$$

(6.115)

which vanishes because of $\langle 22 \rangle = 0$. A similar cancellation occurs with $\varepsilon(3)$. So the entire matrix element vanishes. The amplitude for the case $\gamma_L \gamma_L$ must then also vanish by parity; alternatively, we can find the same cancellation for that case by using $r = 1$ in both polarization vectors.

To compute the amplitude for the case $\gamma_R \gamma_L$, choose

$$\varepsilon^\mu(3) = \frac{1}{\sqrt{2}} \frac{[2\gamma^\mu 3]}{[23]} \quad \text{and} \quad \varepsilon^\mu(4) = -\frac{1}{\sqrt{2}} \frac{[1\gamma^\mu 4]}{[14]}.$$

(6.116)

Then the second diagram in Fig. 6.4 vanishes because of (6.115). Using the Fierz identity, the first diagram gives

$$iA = \frac{-ie^2}{s_{24}} \frac{2}{(-2)(23)[14]} \langle 24 | 1(2 + 4)2 | 31 \rangle.$$

(6.117)
Now we use the Dirac equation, $\mathcal{A}_L = 0$, and replace the vector $\mathcal{A}_L$ by an angle bracket,

$$
\langle iA \rangle = \frac{2ie^2}{s_{13}\langle 23 \rangle[14]}([24]\langle 14 \rangle[42][31])
$$

$$
= \frac{2ie^2}{s_{13}\langle 31 \rangle[23]}([24]\langle 14 \rangle[42][31]) = 2ie^2 \langle 24 \rangle^2 / \langle 23 \rangle^2.
$$

Here we used also four-momentum conservation to convert $s_{24}$ into such brackets that we end up with an expression containing only Mandelstam variables, $s_{23} = u$, $s_{13} = s_{24} = t$.

$$
|iA|^2 = 4e^4 \frac{t}{u} = 4e^4 \frac{1 - \cos \vartheta}{1 + \cos \vartheta}.
$$

### 6.4. Soft photons

The addition of an additional vertex introduces typically a factor $\alpha/\pi \sim 0.2\%$ into a QED cross section. Thus one may hope that perturbation theory in QED converges, at least initially, reasonably fast. An exception to this rule is the emission of an additional soft or collinear photon from an external line shown in Fig. 6.5. The denominator of the additional propagator goes for $k \to 0$ to

$$
\frac{1}{(p+k)^2 - m^2} \to \frac{1}{2p \cdot k} \sim \frac{1}{2E\omega(1 - \cos \vartheta)},
$$

where we assumed $|p| \gg m$ in the last step. The denominator can blow up in two different limits: Firstly, in case of emission of soft photons, $\omega \to 0$. Secondly, in case of collinear emission of photons, $\vartheta \to 0$, if the mass of the emitting particle can be neglected. We have seen in the example of Compton scattering that both cases correspond to the classical limit.

![Emission of an additional soft or collinear photon from an external line in the final state.](#)

**Universality and factorization** The fact that a photon sees in the soft limit $k \to 0$ a classical current should lead to considerable simplifications: In particular, interference effects should disappear and the amplitude $A_{n+1}$ for the emission of an additional soft photon should factor into an universal factor $\varepsilon^\mu S_\mu$ and the amplitude $A_n$ for the original process.
Let us start considering the emission of a soft photon by a spinless particle. If a scalar in the initial state with momentum \( p \) and charge \( q \) emits a photon with momentum \( k \), then\(^2\)

\[
A_{n+1} = q \frac{\epsilon_{\mu}(2p^\mu + k^\mu)}{(p-k)^2 - m^2 + i\varepsilon} A_n \rightarrow -q \frac{\epsilon \cdot p}{p \cdot k - i\varepsilon} A_n . \tag{6.122}
\]

For the emission of a soft photon from an initial state particle, the corresponding factor is \(+q \epsilon \cdot p / (p \cdot k + i\varepsilon)\). In the case of an internal line, in general no factor \((p \cdot k)^{-1}\) appears for \( k \rightarrow 0 \), since the virtual particle is off-shell.

For a spin-1/2 particle in the initial state, the emission of a soft photon adds the factor

\[
q \frac{\epsilon \bar{u}(p, s) \gamma^\mu(\not{p} + \not{k} - m)}{(p-k)^2 - m^2 + i\varepsilon} \rightarrow -q \frac{\epsilon \bar{u}(p, s) \gamma^\mu(\not{p} - m)}{-2p \cdot k - i\varepsilon} . \tag{6.123}
\]

to the amplitude \( A_n \). Now we replace \( \not{p} + m \) by the spin sum \( \sum_i u(p, s) \bar{u}(p, s_i) \), and use

\[
\bar{u}(p, s) \gamma^\mu u(p, s') = 2m p^\mu p_i \delta_{s, s'} = 2p^\mu \delta_{s, s'} . \tag{6.124}
\]

This relation can be checked by direct calculation, or by noting that the current \( q \bar{u}(p, s) \gamma^\mu u(p, s) \) should become \( q(p, \rho \nu) \) in the classical limit \( k \rightarrow 0 \). Thus we obtain the same universal factor describing the emission of a soft photon,

\[
S^\mu = -q \frac{p^\mu}{p \cdot k - i\varepsilon} , \tag{6.125}
\]
as in the case of a scalar. Moreover, we confirmed that the amplitude indeed factorizes, \( A_{n+1} = \epsilon_{\mu} S^\mu A_n \equiv \epsilon_{\mu} A^\mu_{n+1} \). If we allow for the emission of \( m \) soft photons from external particles with charge \( q_i \), then

\[
A^\mu_{n+m} \rightarrow \sum_{i=1}^m s_i q_i p^\mu_{i} A_n , \tag{6.126}
\]

where the signs are \( s_i = -1 \) for an initial and \( s_i = +1 \) for a final state particle.

We have seen that the polarization vector \( \epsilon_{\mu}(k) \) of a photon does not transform as a four-vector, cf. Eq. (4.23), but acquires a term proportional to \( k^\mu \). As we exploited already at various places, amplitudes containing polarization vectors \( \epsilon_{\mu}(k) \) of external photons therefore have to vanish when contracted with \( k^\mu \). Thus Eq. (6.126) implies in the limit \( k \rightarrow 0 \)

\[
k_{\mu} A^\mu_{n+m} \rightarrow \sum_{i=1}^m s_i q_i A_n = 0 . \tag{6.127}
\]

The prefactor of \( A_n \) is the total charge in the final state minus the total charge in the initial state. In order to obtain a Lorentz invariant matrix element for the soft emission of massless spin-1 particles, we have therefore to require that they couple to a conserved charge\(^3\). Thus Lorentz invariance is sufficient to guaranty the conservation of the electromagnetic current in the low-energy limit. While this argument does not rely on gauge invariance it does not say anything about the impact of “hard” photons.

\(^2\)We use the Feynman rule for a $\phi A^\mu$ vertex derived in problem 4.8.

\(^3\) Weinberg who developed this argument could show additionally the following [Wei65]: Massless spin-2 particles have to couple with universal strength to its source, the energy-momentum stress tensor. Thus any consistent theory of massless spin-2 particles implies a low-energy theory satisfying the equivalence principle. Consistent low-energy theories of interacting massless particles with \( s > 2 \) are not possible—thus such particle should decouple in the classical limit.
Bremsstrahlung  We discuss now as a concrete example the case of bremsstrahlung, i.e. the emission of a real photon in the scattering of a charged particle in the Coulomb field $A^0 = -Ze/(4\pi|x|)$ of a static charge. The $S$-matrix element of this process is

$$iS_{fi} = 2\pi\delta(E' + \omega - E)^{-1} \frac{Ze^3}{|q|^2} \bar{u}(p') \left[ \frac{p' + k + m}{2p' \cdot k} \gamma^0 + \gamma^0 \frac{p - k + m}{-2p \cdot k} \right] u(p),$$

(6.128)

where $1/|q|^2$ is the Fourier transform of $A^0$. Note that the external field breaks translation invariance and the momentum is not conserved. We commute now the momenta,

$$iS_{fi} \propto e^2 \bar{u}(p') \left[ \frac{2\varepsilon \cdot p' - (p' - m)\varepsilon + k\varepsilon}{2p' \cdot k} \gamma^0 + \gamma^0 \frac{2\varepsilon \cdot p - \varepsilon(p' - m) + k\varepsilon}{-2p \cdot k} \right] u(p),$$

(6.129)

such that we can use in the next step the Dirac equation. Neglecting additionally in the soft limit the $k$ term in the nominator, we find

$$iS_{fi} \propto e^2 \bar{u}(p') \gamma^0 u(p) \left[ \varepsilon \cdot p' \frac{p' \cdot k}{p' \cdot k} - \varepsilon \cdot p \frac{p \cdot k}{p \cdot k} \right].$$

(6.130)

As we have shown in the previous paragraph in general, the amplitude factorizes into the amplitude describing the “hard” process and the universal correction term. The latter consists of the two terms expected for the emission of a soft photon from an initial line with momentum $p$ and a final line with momentum $p'$. The probability for the emission of an additional soft photon is given integrating the square bracket over the phase space,

$$dP_{n+1} = \frac{d\sigma_{n+1}}{d\sigma_n} = \left[ \frac{\varepsilon \cdot p'}{p' \cdot k} - \frac{\varepsilon \cdot p}{p \cdot k} \right]^2 \frac{d^3k}{(2\pi)^3} \frac{d\omega}{\omega_k} \propto \frac{d\omega_k}{\omega_k}.$$  

(6.131)

This probability diverges for $\omega \to 0$ and therefore the process is called infrared (IR) divergent.

The resolution to this IR problem lies in the fact that soft photons with energy below the energy resolution $E_{\text{th}}$ of the used detector are not detectable. Therefore the amplitude for the emission of $n$ real soft photons with $E < E_{\text{th}}$ are indistinguishable from amplitudes including virtual photons. The IR divergences in the real and virtual corrections cancel, leading to a finite result. We will discuss a detailed example for how this cancellation works in chapter 14.2.

6.A. Appendix: Decay widths and cross sections

We establish first the connection between the correctly normalised transition matrix element $M$ and the Feynman amplitude $A$ where we do not include the normalisation factors of external particles. Then we derive decay widths and cross sections describing $1 \to n$ and $2 \to n$ processes.

**Normalisation**  We have split the scattering operator $S$ into a diagonal part and the transition operator $T$, $S = 1 + iT$. Taking matrix elements, we obtain

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^{(4)}(P_i - P_f)iM_{fi}$$

(6.132)
where we set also \( T_{fi} = (2\pi)^4 \delta^{(4)}(P_i - P_f)M_{fi} \).

We use as relativistic normalisation for all types of one-particle states
\[
\langle p | p' \rangle = 2 E_p (2\pi)^3 \delta(p - p') .
\] (6.133)

For a finite normalisation volume, which makes defining decay widths and cross sections easier, this becomes
\[
\langle p | p' \rangle = 2 E_p V \delta_{p,p'} ,
\] (6.134)

The Feynman amplitude \( A \) neglects all normalisation factors of external particles. Thus the transition between the matrix element \( M_{fi} \) and the Feynman amplitude \( A \) for a process with \( n \) particles in the initial and \( m \) in the final states is given by
\[
M_{fi} = \prod_{i=1}^{n} (2 E_i V)^{-1/2} \prod_{f=1}^{m} (2 E_f V)^{-1/2} A_{fi} .
\] (6.135)

6.A.1. Decay widths

We consider the decay of a particle into \( n \) particles in the final state. Squaring the scattering amplitude \( S_{fi} \) for \( i \neq f \) using \( (2\pi)^4 \delta^{(4)}(0) = VT \) gives as differential transition probability
\[
dW_{fi} = (2\pi)^4 \delta^{(4)}(P_i - P_f)VT |M_{fi}|^2 \prod_{f=1}^{n} \frac{Vd^3p_f}{(2\pi)^3} .
\] (6.136)

The decay rate \( d\Gamma \) is the transition probability per time,
\[
d\Gamma_{fi} = \lim_{T \to \infty} \frac{dW_{fi}}{T} = (2\pi)^4 \delta^{(4)}(P_i - P_f)V |M_{fi}|^2 \prod_{f=1}^{n} \frac{Vd^3p_f}{(2\pi)^3} .
\] (6.137)

Going over to the Feynman amplitude \( A \) eliminates the volume factors \( V \),
\[
d\Gamma_{fi} = (2\pi)^4 \delta^{(4)}(P_i - P_f) \frac{1}{2E_i} |A_{fi}|^2 \prod_{f=1}^{n} \frac{d^3p_f}{2E_f(2\pi)^3} .
\] (6.138)

Moreover, the phase space integrals in the final state are now Lorentz invariant, \( d^3p_f/(2E_f) \). Introducing the \( n \)-particle phase space volume
\[
d\Phi^{(n)} = (2\pi)^4 \delta^{(4)}(P_i - P_f) \prod_{f=1}^{n} \frac{d^3p_f}{2E_f(2\pi)^3} ,
\] (6.139)

the decay rate becomes
\[
d\Gamma_{fi} = \frac{1}{2E_i} |A_{fi}|^2 d\Phi^{(n)} .
\] (6.140)

Since both \( |M_{fi}|^2 \) and the phase space \( d\Phi^{(n)} \) are Lorentz invariant, the decay rate \( \Gamma \propto 1/E_i = 1/(\gamma_im_i) \) shows explicitly the time dilation effect for a moving particle.
We rewrite next the momentum integrals as energy integrals. The energy-momentum relation gives

\[
E_i = \sqrt{p_i^2 + m_i^2}
\]

where \(E_i\) is the cms momentum of the two final state particles.

Two-particle decays We evaluate the two particle phase space \(d\Phi^{(2)}\) in the rest frame of the decaying particle, \(d\Phi^{(2)} = (2\pi)^4 \delta(M - E_1 - E_2) \delta^{(3)}(p_1 + p_2) \frac{d^3 p_1}{2E_1(2\pi)^3} \frac{d^3 p_2}{2E_2(2\pi)^3}\). (6.141)

We perform the integration over \(d^3 p_1\) using the momentum delta function. In the resulting expression,

\[
d\Phi^{(2)} = \frac{1}{(2\pi)^2} \frac{1}{4E_1E_2} \delta(M - E_1 - E_2) \frac{d^3 p_2}{E_1E_2},
\] (6.142)

\(E_1\) is now a function of \(p_2\), \(E_1^2 = p_2^2 + m_1^2\). Introducing spherical coordinates, \(d^3 p_2 = d\Omega p_2 dp_2\),

\[
d\Phi^{(2)} = \frac{1}{(2\pi)^2} d\Omega \int_0^\infty \delta(M - E_1 - E_2) \frac{p_2^2 dp_2}{4E_1E_2}.
\] (6.143)

and evaluating the delta function with \(M = E_1 + E_2 = M - x\) and

\[
\frac{dp_2}{dx} = \frac{p_2x}{E_1E_2},
\]

(6.144)

gives

\[
d\Phi^{(2)} = \frac{|p_{\text{cms}}^2|}{4\pi^2} d\Omega,
\] (6.145)

where

\[
p_{\text{cms}}^2 = \frac{1}{4M^2} \left[ M^2 - (m_1 + m_2)^2 \right] \left[ M^2 - (m_1 - m_2)^2 \right]
\]

(6.146)
is the cms momentum of the two final state particles.

Three-particle decays The three particle phase space \(d\Phi^{(3)}\) is

\[
d\Phi^{(3)} = (2\pi)^4 \delta(M - E_1 - E_2 - E_3) \delta^{(3)}(p_1 + p_2 + p_3) \frac{d^3 p_1}{2E_1(2\pi)^3} \frac{d^3 p_2}{2E_2(2\pi)^3} \frac{d^3 p_3}{2E_3(2\pi)^3}.
\] (6.147)

We can use again the momentum delta function to perform the integration over \(d^3 p_3\),

\[
d\Phi^{(3)} = \frac{1}{(2\pi)^6} \delta(M - E_1 - E_2 - E_3) \frac{d^3 p_1 d^3 p_2}{8E_1E_2E_3},
\] (6.148)

To proceed we have to know the dependence of the matrix element on the integration variables. If there is no preferred direction (either for scalar particles or spin averaged), we obtain

\[
d\Phi^{(3)} = \frac{1}{8(2\pi)^5} \frac{4\pi p_1^2 dp_1}{E_1E_2E_3} 2\pi d\cos\theta dp_2 \delta(M - E_1 - E_2 - E_3)
\]

\[
= \frac{1}{32\pi^3} \frac{p_1 dp_1 (p_1 p_2 d\cos\theta)(p_2 dp_2)}{E_1E_2E_3} \delta(M - E_1 - E_2 - E_3).
\] (6.149)

We rewrite next the momentum integrals as energy integrals. The energy-momentum relation \(E_i^2 = m_i^2 + p_i^2\) gives \(E_i dE_i = p_i dp_i\) for \(i = 1, 2\). Furthermore,

\[
E_3^2 = (p_1 + p_2)^2 + m_3^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\theta + m_3^2
\]

(6.150)
6. Scattering processes

and thus $E_3 dE_3 = p_1 p_2 d \cos \theta$ for fixed $p_1, p_2$. Performing the angular integral, we obtain

$$d\Phi^{(3)} = \frac{1}{32 \pi^3} dE_1 dE_2 dE_3 \delta(M - E_1 - E_2 - E_3),$$

and finally

$$d\Phi^{(3)} = \frac{1}{32 \pi^3} dE_1 dE_2.$$

The last step is only valid, if the argument of the delta function is non-zero. Thus the remaining task is to determine the boundary of the integration domain. Introducing the invariant mass of the pair $(i, j)$

$$m^2_{23} = (p - p_1)^2 = (p_2 + p_3)^2 = M^2 - 2ME_1$$
$$m^2_{13} = (p - p_2)^2 = (p_1 + p_3)^2 = M^2 - 2ME_2$$
$$m^2_{12} = (p - p_3)^2 = (p_1 + p_2)^2 = M^2 - 2ME_3,$$


where the last column is valid in the rest frame of the decaying particle with mass $M$. With $E_1 + E_2 + E_3 = M$ one finds $m^2_{23} + m^2_{13} + m^2_{12} = M^2 + m^2_2 + m^2_3$. Therefore only two out of the three variables are independent. Let’s choose $m^2_{23}$ and $m^2_{13}$ as integration variables, with $m^2_{23}$ as the outer one. Then

$$(m_2 + m_3)^2 \leq m^2_{23} \leq (M - m_1)^2.$$

For given value of $m^2_{23}$, we now have to determine the allowed range of $m^2_{13}$. Inserting energy and momentum conservation into $E_3^2 = p_3^2 + m^2_3$, we obtain

$$(M - E_1 - E_2)^2 = m^2_3 + p_1^2 + p_2^2 + 2p_1 \cdot p_2.$$

The extrema correspond to

$$p_1 \cdot p_2 = \pm p_1 p_2 = \pm \sqrt{(E_1^2 - m^2_1)(E_2^2 - m^2_2)}.$$

(6.156) (6.157) (6.158)

6.A.2. Cross sections

We consider now the interaction of two particles in the rest system of either particle 1 or 2. For simplicity, we consider two uniform beams. They may produce $n$ final state particles. The total number of such scatterings is

$$dN \propto v_M n_1 n_2 dV dt$$

(6.159)

where $n_i$ is the density of particle type $i = 1, 2$. The Møller velocity $v_M$ is a quantity which coincides in the rest frame of particle 1 or 2 with the norm of $|v_2|$ and $|v_1|$, respectively. Therefore it is often denoted simply as their relative velocity $v_{rel}$. The proportionality constant has the dimension of an area and is called cross section $\sigma$. We define in the rest system of either particle 1 or 2

$$dN = \sigma v_M n_1 n_2 dV dt,$$

(6.160)

while we set in an arbitrary frame

$$dN = An_1 n_2 dV dt.$$

(6.161)
We determine now $A$. Since both $dN$ and $dV dt = d^4x$ are Lorentz invariant, the expression
$A n_1 n_2$ has to be Lorentz invariant too. Since the densities transform as

$$n_i = n_{i,0} \gamma = n_{i,0} \frac{E_i}{m_i},$$

(6.162)

the expression

$$A \frac{E_1 E_2}{p_1 \cdot p_2}$$

(6.163)

is also Lorentz invariant. In the rest system of particle 1, it becomes

$$A \frac{E_1 E_2}{E_1 E_2 - p_1 \cdot p_2} = A = \sigma v_{\text{Mol}}.$$  

(6.164)

Thus we found that $A$ in an arbitrary frame is given by

$$A = \sigma v_{\text{Mol}} \frac{p_1 \cdot p_2}{E_1 E_2}.$$  

(6.165)

A more handy expression for $A$ is obtained as follows: In the rest frame 1, we have

$$p_1 \cdot p_2 = m_1 E_2 = m_1 \frac{m_2}{\sqrt{1 - v_{\text{Mol}}^2}}.$$  

(6.166)

Thus the Møller velocity is given in general by

$$v_{\text{Mol}} = \sqrt{1 - \frac{m_1^2 m_2^2}{(p_1 \cdot p_2)^2}}.$$  

(6.167)

Since this expression is Lorentz invariant, we see that the notation of the Møller velocity as relative velocity is misleading.

Next we define

$$I \equiv v_{\text{Mol}} p_1 \cdot p_2 = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}.$$  

(6.168)

Inserting (6.165) together with the definition of $I$ into (6.161), we obtain

$$dN = \sigma I \frac{E_1 E_2 V}{E_1 E_2 V} (n_1 V)(n_2 dV) dt.$$  

(6.169)

Here, we regrouped the terms to make clear that after integration the total number $N$ of scattering events is proportional to the number $N_1 = n_1 V$ and $N_2 = \int n_2 dV$ of particles of type 1 and 2, respectively. The number $N$ of scattering events per time and per particles 1 and 2 is however simply the transition probability per time,

$$\frac{dW_{fi}}{T} = \frac{dN}{N_1 N_2 T} = d\sigma \frac{I}{E_1 E_2 V}.$$  

(6.170)

Inserting the expression (6.137) for $dW_{fi}$, we find

$$d\sigma = \frac{E_1 E_2 V^2}{I} (2\pi)^4 \delta^{(4)}(P_i - P_f) |M_{fi}|^2 \prod_{f=1}^{n} \frac{V q^3 p_f}{(2\pi)^3}.$$  

(6.171)
6. Scattering processes

Changing from the complete matrix element $\mathcal{M}$ to the Feynman amplitude $A$ introduces a factor $(2E_1V)^{-1}(2E_2V)^{-1}$ for the initial state and $\prod_f(2E_fV)^{-1}$ for the final state. Thus the arbitrary normalisation volume $V$ cancels and we obtain

$$d\sigma = \frac{1}{4I} (2\pi)^4 \delta^{(4)}(P_i - P_f)|\mathcal{M}_{fi}|^2 \prod_{f=1}^n \frac{d^3p_f}{2E_f(2\pi)^3} = \frac{1}{4I} |A_{fi}|^2 d\Phi^{(n)}$$

(6.172)

with the final state phase space $d\Phi^{(n)}$. The three pieces composing the differential cross section, the flux factor $I$, the Feynman amplitude $A$, and the final state phase space $d\Phi^{(n)}$, are each Lorentz invariant.

A symmetry factor $S = 1/n!$ has to be added to the total cross section, if there are $n$ identical particles in the final state.

2–2 scattering  The flux factor becomes in the cms

$$I^2 = (p_1 \cdot p_2)^2 - m_1^2m_2^2 = p_{\text{cms}}^2(E_1 + E_2)^2$$

(6.173)

or $I = p_{\text{cms}}\sqrt{s}$. Adding also the known expression for the 2-particle phase space gives

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2s} \frac{p'_{\text{cms}}}{p_{\text{cms}}} |A_{fi}|^2.$$  \hspace{1cm} (6.174)

The momenta $p_{\text{cms}}$ and $p'_{\text{cms}}$ are given by \((6.146)\) with the replacement $M \to s$.

Summary of chapter

The LSZ reduction formula shows that $S$-matrix elements are obtained from connected Green functions by a replacement of the propagators on external lines with the corresponding wavefunctions times the wavefunction renormalisation constant $\sqrt{Z}$. Cross sections and decay rates are calculated from squared Lorentz-invariant Feynman amplitudes $A$, the Lorentz-invariant final state phase space $d\Phi^{(n)}$ and a flux factor $I$. Squared Feynman amplitudes can be obtained using “Casimir’s trick”. If the number of diagrams increases, it is more convenient to calculate directly the amplitude using helicity methods.

Further reading/sources

The derivation of Boltzmann H-theorem follows closely the one in Weinberg. The LSZ formula for particles with spin $s > 0$ is derived in Greiner. For additional information about the helicity formalisms see Haber \[Hab94\] and Peskin \[Pes11\] from which our examples are taken.

Problems
6.A. Appendix: Decay widths and cross sections

6.1 Muon decay.
Derive the differential decay rate of the process $\mu^- (p_1) \to e^- (p_4) \nu_e (p_3) \bar{\nu}_\mu (p_2)$, via the exchange of a $W$-boson described by the vertex
$$-\frac{i g}{\sqrt{2}} \gamma_{\mu} (1 - \gamma^5) f W^\mu.$$ You can neglect all terms of order $m_e^2 / m_W^2$, $m_\mu^2 / m_W^2$.

6.2 Optical theorem.
Consider the theory of two light scalar fields $\phi_1$ and $\phi_2$ with mass $m$ coupled to one heavy scalar $\Phi$ with mass $M > 2m$, $\mathcal{L} = \mathcal{L}_0 + g \phi_1 \phi_2 \Phi$ where $\mathcal{L}_0$ is the free Lagrangian.

a.) Calculate the width $\Gamma$ of the decay $\Phi \to \phi_1 \phi_2$.
b.) Draw the Feynman diagram(s) and write down the Feynman amplitude $i \mathcal{M}$ for the scattering process $\phi_1 (p_1) \phi_2 (p_2) \to \phi_1 (p'_1) \phi_2 (p'_2)$. What is your interpretation of the behaviour of the amplitude for $s = (p_1 + p_2)^2 \to M^2$? c.) Consider the one-loop correction $i \mathcal{M}_{\text{loop}}$ to the mass of $\Phi$.

Write down $i \mathcal{M}_{\text{loop}}$ first for an arbitrary momentum $p$ of the external particle $\Phi$, then for its rest frame, $p = (M, 0)$. Find the poles of the integrand and use the theorem of residues to perform the $q^0$ part of the loop integral. Finally, use the identity
$$\frac{1}{x \pm i\varepsilon} = P \left(\frac{1}{x}\right) \mp i \pi \delta(x)$$
to find the imaginary part of the amplitude. You should find $M \Gamma (\Phi \to \phi_1 \phi_2) = \text{Im} \mathcal{M}_{\text{loop}}$, a special case of the optical theorem.

6.3 Identities for gamma matrices.
Derive the following identities $\gamma^\mu \phi \gamma_\mu = -2 \phi_\mu$, $\text{tr}[\bar{\phi} \phi] = 4 a \cdot b$, $\text{tr}[\bar{\phi} \phi \phi] = 4 [(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$.

6.4 Polarised cross sections in the trace formalism.
Consider the emission of two soft photons from the same external line and show that the matrix element $\mathcal{M}_{n+2}$ contains a product of the same factor as in the emission of single photons from two separate lines.

6.5 Emission of two soft photons.
Consider the emission of two soft photons from the same external line and show that the matrix element $\mathcal{M}_{n+2}$ contains a product of the same factor as in the emission of single photons from two separate lines.

6.6 Light deflection in gravity.
Consider the scattering of a scalar particle and a photon via graviton exchange. Calculate the scattering amplitude and the cross section in the static limit $p^0 = (M, 0)$ for small-angle scattering ($k^2 = \vartheta = 0$ in the nominator) and show that it agrees with Einstein’s prediction for light deflection by the Sun.
7. Gauge theories

We discuss in this chapter field theories in which the Lagrangian is invariant under a continuous group of local transformations in internal field space. The symmetry group of these transformations is called the gauge group and the vector fields associated to the generators of the group the gauge fields. We introduce as first step unbroken gauge theories, i.e. theories with massless gauge bosons, and defer the more complex case of broken gauge symmetries to the chapters 10 and 11. The Standard Model (SM) of particle physics contains with quantum electrodynamics (QED) with the abelian gauge group U(1) and quantum chromodynamics (QCD) as an non-abelian gauge theory with group SU(3) two examples for unbroken gauge theories. Non-abelian gauge theories were first studied by Yang and Mills and are therefore also often called Yang-Mills theories. The structure of Yang-Mills theories has many similarities with gravity. We use this property to introduce curvature of a space-time as analogon to the field-strength in Yang-Mills theories.

7.1. Electromagnetism as abelian gauge theory

In classical electromagnetism, the field-strength tensor $F$ is an observable quantity, while the potential $A$ is merely a convenient auxiliary quantity. From the definition

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , $$

(7.1)

it is clear that $F$ is invariant under the transformations

$$ A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x) . $$

(7.2)

Thus $A'_\mu(x)$ is for any $\Lambda(x)$ physically equivalent to $A_\mu(x)$, leading to the same field-strength tensor and thus e.g. to the same Lorentz force on a particle. The transformations (4.14) are called gauge transformations. Remember that a mass term $m^2 A_\mu A_\mu$ of the photon would break gauge invariance.

Consider now e.g. a free Dirac field $\psi(x)$ with electric charge $q$. We saw already that this field is invariant under global phase transformations $\exp[iq\Lambda] \in U(1)$, implying a conserved current via Noether’s theorem. Can we promote this global U(1) symmetry to a local one,

$$ \psi(x) \to \psi'(x) = U(x)\psi(x) = \exp[iq\Lambda(x)]\psi(x) $$

(7.3)

by making the phase $U$ space-time dependent as in (4.14)? The partial derivatives in the Dirac Lagrangian will lead to an additional term $\propto \partial_\mu U(x)$, destroying the invariance of the free Lagrangian. However, if we add a field $A_\mu(x)$ which transforms as defined in (4.14), the two gauge-dependent terms will cancel. Thus local U(1) gauge invariance requires the existence of a massless gauge boson and fixes its interaction with matter as

$$ \partial_\mu \to D_\mu = \partial_\mu + iqA_\mu . $$

(7.4)
Moreover, the gauge field has to transform as

\[ A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x) = A_\mu(x) - \frac{i}{q} U(x) \partial_\mu U^\dagger(x), \quad (7.5) \]

where we expressed the change \( \delta A_\mu(x) \) through the group elements \( U(x) \).

To summarise: The invariance of complex (scalar or Dirac) fields under \textit{global} phase transformations \( \exp[iq\Lambda] \in U(1) \) implies a conserved current, promoting it to a \textit{local} \( U(1) \) symmetry requires the existence of a massless \( U(1) \) gauge boson coupled via gauge-invariant derivatives to other fields.

### 7.2. Non-abelian gauge theories

#### 7.2.1. Gauge invariant interactions

We want to generalise now electromagnetism, using as symmetry group instead of the abelian group \( U(1) \) larger groups like \( SO(n) \) or \( SU(n) \). A group like \( SU(n) \) will describe the interactions of \( n^2 - 1 \) gauge bosons with matter, using as a single parameter the gauge coupling \( g \).

The gauge transformations will moreover mix fermions living in the same representation of the group, requiring that these fermions have the same interactions and the same mass if the symmetry is unbroken. In this way, non-abelian gauge theories lead to a partial \textit{unification} of matter fields and, separately, of interactions. Note the difference to an abelian symmetry: The emission of a photon does not change any quantum number (apart from the momentum) and thus does not "mix" different particles. Therefore there is also no connection between the electric charge of different particles.

The two non-abelian groups used in the SM are \( SU(2) \) for weak and \( SU(3) \) for strong interactions. A matrix representation for the fundamental representation of these two groups are the Pauli matrices, \( T^a = \sigma^a/2 \), and the Gell-Mann matrices, \( T^a = \lambda^a/2 \), respectively. Under the fundamental representation the fermions transform as doublets for \( SU(2) \), as triplets for \( SU(3) \), etc. Since the number of generators is \( m = n^2 - 1 \) for \( SU(n) \), the groups \( SU(2) \) contains three gauge bosons, while \( SU(3) \) contains eight bosons carrying strong interactions.

The most important difference of these non-abelian groups compared to \( U(1) \) is that the generators \( T^a \equiv T^a_{ij} \) of such groups do not commute with each other. As a result, we may expect that both the expression for the field-strength tensor, Eq. (4.13), and the transformation law for the gauge field, Eq. (4.14), becomes more complicated. In contrast, we postulate that the interaction law \( j_\mu A^\mu \) remains valid, with the sole difference that now \( A_\mu = A^a_\mu T^a \). Thus \( A_\mu \) is a Lorentz vector with values in the Lie algebra of the gauge group.

**Example:** \( SU(2) \) as gauge group for the weak interactions.

Charged current interactions lead e.g. in beta-decay to the transmutations \( n \leftrightarrow p \) and \( e^- \leftrightarrow \nu_e \). This suggests to consider \( e^- \) and \( \nu_e \) as different states of a doublet, (\( e^- , \nu_e \)) and \( SU(2) \) (or \( SO(3) \)) as symmetry group, if weak interactions are described by a gauge interaction. These groups have three generators, and predict therefore the existence of neutral currents. The success of the Fermi theory at energies \( s \ll G_F^{-1} \sim m_W^2/g^2 \) tells us that these gauge bosons are massive. Thus the gauge symmetry is broken, an effect we will have to understand latter.

We derive now the transformation laws and structure of the gauge sector, requiring that the transformation of the fermions and their interaction with the gauge field are locally invariant.
A local gauge transformation

\[ U(x) = \exp[ig \sum_{a=1}^{m} \vartheta^a(x) T^a] = \exp[ig \vartheta(x)] \]  \hspace{1cm} (7.6)

changes a vector of fermion fields \( \psi \) with components \( \{\psi_1, \ldots, \psi_n\} \) as

\[ \psi(x) \rightarrow \psi'(x) = U(x) \psi(x) . \]  \hspace{1cm} (7.7)

Here we assumed that the fermions transform under the fundamental representation of the group, i.e. as doublets for SU(2), triplets for SU(3), etc. Already global gauge invariance of the fermion mass term requires \( m_1 = m_2 = \ldots = m_n \) and for simplicity we set \( m_i = 0 \). We can implement local gauge invariance, if derivatives transform in the same way as \( \psi \). Hence we define a new covariant derivative \( D_\mu \) requiring

\[ D_\mu \psi(x) \rightarrow [D_\mu \psi(x)]' = U(x)[D_\mu \psi(x)] . \]  \hspace{1cm} (7.8)

The gauge field should compensate the difference between the normal and the covariant derivative,

\[ D_\mu \psi(x) = [\partial_\mu + igA_\mu(x)]\psi(x) . \]  \hspace{1cm} (7.9)

In the non-abelian case, the gauge field \( A_\mu \) is a matrix that is connected to its component fields by

\[ A_\mu = A^a_\mu T^a . \]  \hspace{1cm} (7.10)

We now determine the transformation properties of \( D_\mu \) and \( A_\mu \) demanding that (7.7) and (7.8) hold. Combining both requirements gives

\[ D_\mu \psi(x) \rightarrow [D_\mu \psi(x)]' = U(x)[D_\mu \psi(x)] , \]  \hspace{1cm} (7.11)

and thus the covariant derivative transforms as \( D'_\mu = UD_\mu U^{-1} \). Using its definition (7.9), we find

\[ [D_\mu \psi]' = [\partial_\mu + igA'_\mu]U \psi = U D_\mu \psi = U[\partial_\mu + igA_\mu]\psi . \]  \hspace{1cm} (7.12)

We compare now the second and the fourth term, after having performed the differentiation \( \partial_\mu (U \psi) \). The result

\[ [(\partial_\mu U) + igA'_\mu U] \psi = igU A_\mu \psi \]  \hspace{1cm} (7.13)

should be valid for arbitrary \( \psi \) and hence we arrive after multiplying from the right with \( U^{-1} \) at

\[ A_\mu \rightarrow A'_\mu = UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \]  \hspace{1cm} (7.14a)

\[ = UA_\mu U^{-1} - \frac{i}{g}U \partial_\mu U^{-1} . \]  \hspace{1cm} (7.14b)

Here we used also \( \partial_\mu (UU^{-1}) = 0 \). In most cases, the gauge transformation \( U \) is an unitary transformation and one sets \( U^{-1} = U^\dagger \). A term changing as \( U(x)D_\mu(x)U^\dagger(x) \) is called to transform homogenously, while the potential \( A_\mu \) is said to transforms inhomogenously.

---

1 We suppress in the following most indices; writing them out gives e.g. \( \psi_i(x) = U_{ij}(x) \psi_j(x) \) with \( U_{ij}(x) = \exp[ig \sum_{a=1}^{m} \vartheta^a(x) T^a_{ij}] \).
Example: We can determine the transformation properties of $A_\mu$ also by demanding that (7.3) defines the interaction term in a gauge invariant way. Replacing $\partial_\mu \to D_\mu$ in the free Lagrange density of fermions and inserting then $U^{-1}U = 1$ gives

$$
\mathcal{L}_f + \mathcal{L}_I = i\bar{\psi} \gamma^\mu D_\mu \psi = i\bar{\psi} \gamma^\mu \partial_\mu \psi - g\bar{\psi} \gamma^\mu A_\mu^a T^a \psi = \left[ i\bar{\psi} U^{-1} U \gamma^\mu \partial_\mu U^{-1} \psi - g\bar{\psi} U^{-1} \gamma^\mu A_\mu^a T^a U^{-1} U \psi \right].
$$

Using then $\psi' = U\psi$, we obtain

$$
\mathcal{L}_f + \mathcal{L}_I = i\bar{\psi}' \gamma^\mu U \partial_\mu U^{-1} \psi' - g\bar{\psi}' \gamma^\mu U A_\mu^a T^a U^{-1} \psi' = \left[ i\bar{\psi}' \gamma^\mu \partial_\mu \psi' - g\bar{\psi}' \gamma^\mu \left( U A_\mu^a T^a U^{-1} - \frac{i}{g} U (\partial_\mu U^{-1}) \right) \right] \psi'.
$$

The Lagrange density $\mathcal{L}_f + \mathcal{L}_I$ is thus invariant, if the gauge field transforms as given in Eqs. (7.14).

Specialising to infinitesimal transformations, $U(x) = \exp(ig\vartheta^a(x)T^a) = 1 + ig\vartheta(x) + \mathcal{O}(\vartheta^2)$, it follows

$$
A_\mu(x) \to A'_\mu(x) = A_\mu(x) - ig[A_\mu(x), \vartheta(x)] - \partial_\mu \vartheta(x).
$$

In the abelian U(1) case, the commutator term is not present and the transformation law reduces to the known $A_\mu \to A_\mu - \partial_\mu \vartheta$. For a (semi-simple) Lie group one defines

$$
[T^a, T^b] = if^{abc}T^c
$$

with structure constants $f^{abc}$ that can be chosen to be completely antisymmetric. Thus

$$
A_\mu^a(x) \to A'_\mu^a(x) = A_\mu^a(x) + gf^{abc} A_\mu^b(x) \vartheta^c(x) - \partial_\mu \vartheta^a(x) = A_\mu^a(x) - [\delta^{ac}\partial_\mu - g f^{abc} A_\mu^b(x)] \vartheta^c(x)
$$

$$
\equiv A_\mu^a(x) - D_\mu^a \vartheta^c(x),
$$

where we introduced in the last line the covariant derivative acting on the gauge fields which transform according the adjoint representation of the gauge group. In this way, the general substitution rule $\partial_\mu \to D_\mu$ can be used not only to derive the coupling of a gauge field to other fields but can be also applied to its self-interactions.

Finally, we have to derive the field strength tensor $F$ and the Lagrange density $\mathcal{L}_{YM}$ of the gauge field. The quantity $F^2$ requires now additionally a summation over the group index $a$,

$$
\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} = -\frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu},
$$

where the standard normalisation $\text{tr} T^a T^b = \delta^{ab}/2$ for the group generators $T^a$ was assumed. The last equation shows that it is sufficient for the gauge invariance of the action that the field-strength tensor transforms homogeneously,

$$
F(x) \to F'(x) = U(x) F(x) U^{-1}(x).
$$

\footnote{The $n$ complex fermion and $n^2 - 1$ real gauge fields of SU($n$) live in different representations of the group, as already the mismatch of their number indicates, see also Appendix B. Note also that the gauge transformations of the gauge fields have to be real, in contrast to the ones of the fermion fields.}
7. Gauge theories

There are several ways to derive the relation between $F$ and $A$. The field-strength tensor should be anti-symmetric. Thus we should construct it out of the commutator of gauge invariant quantities that in turn should contain $A$. An obvious try is $F_{\mu \nu} \propto [D_\mu, D_\nu]$. Requiring that we reproduce the standard result for U(1), we obtain

$$F_{\mu \nu} = F^a_{\mu \nu} T^a = \frac{1}{ig} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \quad (7.23)$$

In components, this equation reads explicitly

$$F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu. \quad (7.24)$$

Differential forms:

A surface in $\mathbb{R}^3$ can be described at any point either by its two tangent vectors $e_1$ and $e_2$ or by the normal $n$. They are connected by a cross product,

$$n^i = \varepsilon^{ijk} e_{1,j} e_{2,k}. \quad (7.25)$$

In four dimensions, the $\varepsilon$ tensor defines a map between 1-3 and 2-2 tensors. Since $\varepsilon$ is antisymmetric, the symmetric part of tensors would be lost; hence the map is suited only for antisymmetric tensors.

Antisymmetric tensors of rank $n$ can be seen also as differential forms: Functions are forms of order $n = 0$; differential of functions are an example of forms of order $n = 1$,

$$df = \frac{\partial f}{\partial x^i} dx^i. \quad (7.26)$$

Thus the $dx^i$ form a basis, and one can write in general

$$A = A_i dx^i. \quad (7.27)$$

For $n > 1$, the basis has to be antisymmetrized. Hence, a two-form as the field-strength tensor is given by

$$\mathbf{F} = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu. \quad (7.28)$$

with $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. Looking at $df$, we can define a differentiation of a form $\omega$ with coefficients $w$ and degree $n$ as an operation that increases its degree by one to $n + 1$,

$$d\omega = dw_{\alpha,\ldots,\beta} dx^{\alpha+1} \wedge dx^\beta. \quad (7.29)$$

Thus we have $\mathbf{F} = d\mathbf{A}$. Moreover, it follows $d^2 \omega = 0$ for all forms. Hence for an abelian gauge transformation $\mathbf{F}' = d(\mathbf{A} + d\chi) = \mathbf{F}$.

Since the field-strength tensor is anti-symmetric, we can use also the language of differential forms: There are two possibilities to form a two-form out of the one-form $\mathbf{A} = A_\mu dx^\mu$ (which has values in a non-abelian Lie group): Via exterior differentiation $d\mathbf{A}$ and via $\mathbf{A}^2 = [A_\mu, A_\nu] dx^\mu \wedge dx^\nu$. Hence we have to fix the coefficients $a$ and $b$ in $\mathbf{F} = ad\mathbf{A} + b\mathbf{A}^2$. (We become also for a moment mathematicians, using their normalisation $ig\mathbf{A} \rightarrow \mathbf{A}$.)

We apply $d$ to $\mathbf{A} \rightarrow \mathbf{A}' = U \mathbf{A} U^\dagger + UdU^\dagger$, obtaining

$$d\mathbf{A} \rightarrow d\mathbf{A}' = dU \mathbf{A} U^\dagger + Ud\mathbf{A} U^\dagger - U \mathbf{A} dU^\dagger + UdU^\dagger. \quad (7.30)$$
Here we used \( d^2 = 0 \) and the Leibniz rule for forms: Because of \( dx^i \wedge dx^j = -dx^j \wedge dx^i \), the rule is \( d(\omega \wedge \sigma) = (d\omega) \wedge \sigma + (-1)^n \omega \wedge d\sigma \), if \( \omega \) is a form of degree \( n \). Thus there is a minus commuting \( d \) with the one-form \( A \), but no minus commuting \( d \) with \( U \) that as a function is a form of degree zero. Next we consider the transformation of \( A^2 \),

\[
A^2 \rightarrow A'^2 = (UA^\dagger + UdU^\dagger)(UA^\dagger + UdU^\dagger) = UA^2U^\dagger + UdU^\dagger - dUAU^\dagger - UdU^\dagger, \tag{7.31}
\]

where we used again \( d(UU^\dagger) = 0 \). Adding the two equations using \( a = b = 1 \), we see that six terms cancels, and the two remaining ones are

\[
dA + A^2 \rightarrow U(dA + A^2)U^\dagger. \tag{7.32}
\]

Thus \( F = dA + A^2 \) transforms homogeneously as required but is not gauge invariant.

### 7.2.2. Gauge fields as connection

There is a close analogy between the covariant derivative \( \nabla_\mu \) introduced for a space-time containing a gravitational field and the gauge invariant derivative \( D_\mu \) required for a space-time containing a gauge field. In the former case, the moving coordinate basis in curved space-time, \( \partial_\mu e^\nu \neq 0 \), introduces an additional term in the derivative of vector components \( V^\mu = e^\mu \cdot V \). Analogously, a non-zero gauge field \( A^\mu \) leads to a rotation of the basis vectors \( e_i \) in group space which in turn produces an additional term \( \psi \cdot (\partial_\mu e_i) \) performing the derivative of a \( \psi_i = \psi \cdot e_i \).

Let us rewrite our formulas such that the analogy between the covariant gauge derivative \( D_\mu \) to the covariant space-time derivative \( \nabla_\mu \) becomes obvious. The vector \( \psi \) of fermion fields with components \( \{\psi_1, \ldots, \psi_n\} \) transforming under the fundamental representation of a gauge group can be written as

\[
\psi(x) = \psi_i(x) e_i(x). \tag{7.33}
\]

We can pick out the component \( \psi_j \) by mutlying with the corresponding basis vector \( e_j \),

\[
\psi_j = \psi \cdot e_j(x). \tag{7.34}
\]

If the coordinate basis in group space depends on \( x^\mu \), then the partial derivative of \( \psi_i \) acquires a second term,

\[
\partial_\mu \psi_i = (\partial_\mu \psi) \cdot e_i + \psi \cdot (\partial_\mu e_i). \tag{7.35}
\]

We can argue as in chapter 1 that \( (\partial_\mu \psi) \cdot e_i \) is an invariant quantity, defining therefore as gauge invariant derivative

\[
D_\mu \psi_i = (\partial_\mu \psi) \cdot e_i = \partial_\mu \psi_i - \psi \cdot (\partial_\mu e_i). \tag{7.36}
\]

The change \( \partial_\mu e_i \) of the basis vector in group space should be proportional to \( gA_\mu \). Setting

\[
\partial_\mu e_i = -ig(A_\mu)_{ij} e_j \tag{7.37}
\]

we are back to our old notation.
7. Gauge theories

![Parallelogram](image)

Figure 7.1.: Parallelogram used to calculate the rotation of a test field $\psi_1$ moved along a closed loop in the presence of a non-zero gauge field $A^\mu$.

**Gauge loops** The correspondence between the derivatives $\nabla_\mu$ and $D_\mu$ suggests that we can use the gauge field $A_\mu$ to transport fields along a curve $x^\mu(\sigma)$. In empty space, we can use the partial derivative $\partial_\mu \psi(x)$ to compare fields at different points,

$$\partial_\mu \psi(x) \propto \psi(x + dx^\mu) - \psi(x).$$  \hspace{1cm} (7.38)

If there is an external gauge field present, the field $\psi$ is additionally rotated in group space moving it from $x$ to $x + dx$,

$$\tilde{\psi}(x + dx) = \psi(x + dx) + igA_\mu(x)\psi(x)dx^\mu$$

$$= \psi(x) + \partial_\mu \psi(x)dx^\mu + igA_\mu(x)\psi(x)dx^\mu.$$  \hspace{1cm} (7.39)

Then the total change is

$$\tilde{\psi}(x + dx) - \psi(x) = [\partial_\mu + igA_\mu(x)]\psi(x)dx^\mu = D_\mu \psi(x)dx^\mu.$$  \hspace{1cm} (7.40)

Thus we can view\(^\text{3}\)

$$P_{dx}(x) = 1 - igA_\mu(x)dx^\mu$$  \hspace{1cm} (7.41)

as an operator which allows us to transport a gauge-dependent field the infinitesimal distance from $x$ to $x + dx$.

We ask now what happens to a field $\psi_1(x)$, if we transport it along an infinitesimal parallelogram, as shown in Fig. 7.1. Calculating the path 2, we find

$$P_{dy}(x + dx) = 1 - igA_\nu(x + dx)dy^\nu$$

$$= 1 - igA_\nu(x)dy^\nu - ig\partial_\nu A_\mu(x)dx^\mu dy^\nu,$$  \hspace{1cm} (7.42)

where we Taylor expanded $A_\nu(x + dx)$. Combining the paths 1 and 2, we arrive at

$$P_{dy}(x + dx)P_{dx}(x) = [1 - igA_\nu(x)dy^\nu - ig\partial_\nu A_\mu(x)dx^\mu dy^\nu][1 - igA_\mu(x)dx^\mu]$$

$$= 1 - igA_\mu(x)dx^\mu - igA_\nu(x)dy^\nu - ig\partial_\nu A_\mu(x)dx^\mu dy^\nu$$

$$- g^2 A_\nu(x)A_\mu(x)dx^\mu dy^\nu + O(dx^3).$$  \hspace{1cm} (7.43)

Instead of performing the calculation for a round trip $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$, we evaluate next $4 \rightarrow 3$ which we then subtract from $1 \rightarrow 2$. In this way, we can re-use our result for $1 \rightarrow 2$ after exchanging labels, $A_\mu dx^\mu \leftrightarrow A_\nu dy^\nu$, obtaining

$$P_{dx}(x + dx)P_{dy}(x) = 1 - igA_\nu(x)dy^\nu - igA_\mu(x)dx^\mu - ig\partial_\nu A_\mu(x)dx^\mu dy^\nu$$

$$- g^2 A_\nu(x)A_\mu(x)dx^\mu dy^\nu + O(dx^3).$$  \hspace{1cm} (7.44)

\(^3\)Note the sign change compared to the covariant derivative where we pull-back the field from $x + dx$ to $x$.

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The first three terms on the RHS’s of (7.44) and (7.45) cancel in the result $P(\square)$ for the round-trip, leaving us with

$$P(\square) \equiv P_{dy}(x + dx)P_{dx}(x) - P_{dx}(x + dy)P_{dy}(x) = -ig \{\partial_{\mu}A_{\nu} - \partial_{\mu}A_{\nu} + ig[A_{\mu}, A_{\nu}]\} dx^\nu dy^\nu.$$  \hspace{1cm} (7.46)

Maxwell’s equations inform us that the line integral of the vector potential equals the enclosed flux: The area of the parallelogram corresponds to $dx^\nu dy^\nu$, and the prefactor has to be therefore the field-strength tensor. If the enclosed flux is non-zero, then $P(\square)\psi_i \neq \psi_i$ and thus the field is rotated.

### 7.2.3. Curvature of space-time

#### Curvature and the Riemann tensor

We continue to work out the analogy between Yang-Mills theories and gravity further. Both the gauge field $A_{\mu}$ and the connection $\Gamma^\kappa_{\mu\rho}$ transform inhomogenously. Therefore we can not use them to judge if a gauge or gravitational field is present. In the gauge case, we introduced therefore the field-strength: It transforms homogeneously and thus the statement $F_{\mu\nu}(x) = 0$ holds in any gauge. This suggests to transform (7.23) into a definition for a tensor measuring the curvature of space-time,

$$(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})T^\mu_{\nu\cdots\kappa\rho} = [\nabla_{\alpha}, \nabla_{\beta}]T^\mu_{\nu\cdots\kappa\rho} \neq 0,$$ \hspace{1cm} (7.47)

Thus the curvature of space-time should be proportional to the area of a loop and the amount a tensor is rotated.

For the special case of a vector $V^\alpha$ we obtain with

$$\nabla_{\rho}V^\alpha = \partial_{\rho}V^\alpha + \Gamma^\alpha_{\beta\rho}V^\beta$$ \hspace{1cm} (7.48)

first

$$\nabla_{\sigma}\nabla_{\rho}V^\alpha = \partial_{\sigma}(\partial_{\rho}V^\alpha + \Gamma^\alpha_{\beta\rho}V^\beta) + \Gamma^\alpha_{\kappa\rho}(\partial_{\sigma}V^\kappa + \Gamma^\kappa_{\beta\rho}V^\beta) - \Gamma^\kappa_{\beta\rho}(\partial_{\kappa}V^\alpha + \Gamma^\alpha_{\beta\kappa}V^\beta).$$ \hspace{1cm} (7.49)

The second part of the commutator follows from the simple relabelling $\sigma \leftrightarrow \rho$,

$$\nabla_{\rho}\nabla_{\sigma}V^\alpha = \partial_{\rho}(\partial_{\sigma}V^\alpha + \Gamma^\alpha_{\beta\sigma}V^\beta) + \Gamma^\alpha_{\kappa\sigma}(\partial_{\rho}V^\kappa + \Gamma^\kappa_{\beta\sigma}V^\beta) - \Gamma^\kappa_{\beta\sigma}(\partial_{\kappa}V^\alpha + \Gamma^\alpha_{\beta\kappa}V^\beta)$$ \hspace{1cm} (7.50)

Now we subtract the two equations using that $\partial_{\rho}\partial_{\sigma} = \partial_{\sigma}\partial_{\rho}$ and $\Gamma^\alpha_{\beta\rho} = \Gamma^\alpha_{\rho\beta}$,

$$[\nabla_{\rho}, \nabla_{\sigma}]V^\alpha = \left[\partial_{\rho}\Gamma^\alpha_{\beta\sigma} - \partial_{\sigma}\Gamma^\alpha_{\beta\rho} + \Gamma^\alpha_{\kappa\rho}\Gamma^\kappa_{\beta\sigma} - \Gamma^\alpha_{\kappa\sigma}\Gamma^\kappa_{\beta\rho}\right]V^\beta \equiv R^\alpha_{\beta\rho\sigma}V^\beta.$$ \hspace{1cm} (7.51)

The tensor $R^\alpha_{\beta\rho\sigma}$ is called Riemann or curvature tensor. In problem 7.5 you are asked to show that the tensor $R_{\alpha\beta\rho\sigma} = g_{\alpha\gamma}R^\gamma_{\beta\rho\sigma}$ is antisymmetric in the indices $\rho \leftrightarrow \sigma$, antisymmetric in $\alpha \leftrightarrow \beta$ and symmetric against an exchange of the index pairs $(\alpha\beta) \leftrightarrow (\rho\sigma)$. Therefore, we can construct out of the Riemann tensor only one non-zero tensor of rank two, contracting $\alpha$ either with the third or fourth index, $R^\rho_{\alpha\beta\rho} = -R^\rho_{\alpha\beta\rho}$. We define the Ricci tensor for a pseudo-Riemannian metric by $R_{\alpha\beta} = R^\gamma_{\alpha\beta\gamma}$, while we set $R_{ab} = R^c_{abc}$ for a Riemannian

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For a comparison of various sign conventions see (A.6) in Appendix A.
metric (e.g. for the spatial part of the metric $ds^2 = dt^2 - dl^2$). Then the the Ricci tensor is given by

$$R_{\alpha\beta} = R^\rho_{\alpha\beta\rho} = -R^\rho_{\alpha\rho\beta} = \partial_\beta \Gamma^\rho_{\alpha\rho} - \partial_\rho \Gamma^\rho_{\alpha\beta} + \Gamma^\sigma_{\beta\rho} \Gamma^\rho_{\alpha\sigma} - \Gamma^\rho_{\alpha\beta} \Gamma^\rho_{\rho\sigma},$$  \hspace{1cm} (7.52)

A further contraction gives the curvature scalar,

$$R = R_{\alpha\beta} g^{\alpha\beta}. \hspace{1cm} (7.53)$$

**Example:** *Sphere* $S^2$. Calculate the Ricci tensor $R_{\alpha\beta}$ and the scalar curvature $R$ of the two-dimensional unit sphere $S^2$.

We have already determined the non-vanishing Christoffel symbols of the sphere $S^2$ as $\Gamma^\phi_\phi_\theta = \Gamma^\phi_\phi_\phi = \cot \theta$ and $\Gamma^\theta_\phi_\phi = -\cos \theta \sin \theta$. We will show later that the Ricci tensor of a maximally symmetric space as a sphere satisfies $R^{\alpha\beta} = Kg^{\alpha\beta}$. Since the metric is diagonal, the non-diagonal elements of the Ricci tensor are zero too, $R_{\phi\theta} = R_{\theta\phi} = 0$. We calculate with

$$R_{\alpha\beta} = R^c_{\alpha\beta c} = \partial_c \Gamma^c_{\alpha\beta} - \partial_{\alpha} \Gamma^c_{c\beta} + \Gamma^c_{\beta c} \Gamma^d_{\alpha d} - \Gamma^d_{\alpha c} \Gamma^c_{\beta d}$$

the $\theta\theta$ component,

$$R_{\phi\theta} = 0 - \partial_\theta (\Gamma^\phi_{\phi\theta} + \Gamma^\phi_{\theta\phi}) + 0 - \Gamma^d_{\phi\theta} \Gamma^c_{\phi d} = 0 + \partial_\theta \cot \theta - \Gamma^\phi_{\phi\theta} \Gamma^\phi_{\theta\phi}$$

$$= 0 - \partial_\theta \cot \theta - \cot^2 \theta = 1$$

From $R_{\alpha\beta} = Kg_{\alpha\beta}$, we find $R_{\phi\theta} = Kg_{\phi\theta}$ and thus $K = 1$. Hence $R_{\phi\phi} = g_{\phi\phi} = \sin^2 \theta$.

The scalar curvature is (diagonal metric with $g_{\phi\phi} = 1/\sin^2 \theta$ and $g_{\theta\theta} = 1$)

$$R = g^{\alpha\beta} R_{\alpha\beta} = g^{\phi\phi} R_{\phi\phi} + g^{\theta\theta} R_{\theta\theta} = \frac{1}{\sin^2 \theta} \sin^2 \theta + 1 \times 1 = 2.$$

Note that our definition of the Ricci tensor guarantees that the curvature of a sphere is also positive, if we consider it as subspace of a four-dimensional space-time.

**Comparison** We can push the analogy further by remembering that the field-strength defined in (7.23) is a matrix. Writing out the implicit matrix indices of $F_{\mu\nu}$ in Eq. (7.23) gives

$$(F_{\mu\nu})_{ij} = \partial_\mu (A_\nu)_{ij} - \partial_\nu (A_\mu)_{ij} + ig \left((A_\mu)_{ik}(A_\nu)_{kj} - (A_\nu)_{ik}(A_\mu)_{kj}\right). \hspace{1cm} (7.54)$$

Comparing this expression to

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\rho\mu} \Gamma^\rho_{\beta\nu} - \Gamma^\alpha_{\rho\nu} \Gamma^\rho_{\beta\mu}$$  \hspace{1cm} (7.55)

we see that the first two indices of the Riemann tensor, $\alpha$ and $\beta$, correspond to the group indices $ij$ in the field-strength tensor. This is in line with the relation of the potential $(A_\mu)_{ij}$ and the connection $\Gamma^\alpha_{\beta\nu}$ implied by (7.37).
7.3. Quantization of gauge theories

7.3.1. Abelian case

We discussed already in Sec. 4.2 that we can derive the photon propagator only fixing a gauge. Now we reconsider this problem, and ask how we should modify the Lagrange density in order to be able to derive the photon propagator. The Lagrange density that leads to the Maxwell equation is

\[ L_{\text{em}} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} = -\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{1}{2} A_{\mu} [\eta^{\mu \nu} \Box - \partial_\mu \partial_\nu] A_\nu \]

where we made as usual a partial integration dropping the surface term. Deriving the photon propagator requires to invert the term in the square bracket. In particular, we should find the inverse of the operator

\[ P_{\mu \nu}(k) = \eta_{\mu \nu} - k_\mu k_\nu / k^2. \] (7.56)

We have already seen that this operator projects any four-vector on the three-dimensional subspace orthogonal to \( k \). This ensures that the physical degrees of the photon contain no longitudinal component.

More formally, we see that \( P_{\mu \nu}(k) \) is a projection operator,

\[ P_{\mu \nu} P_{\lambda \nu} = P_{\mu \lambda}, \] (7.57)

and has thus as only eigenvalues 0 and 1. Since \( P_{\mu \nu}(k) \) does not look like the unit operator, it has at least one zero eigenvalue and is thus not invertible. Since its trace is

\[ P_\mu = \eta_{\mu \nu} P_{\mu \nu} = \delta_\mu^\mu - 1 = 3, \] (7.58)

we see that three eigenvalues are one and one eigenvalue is zero. The latter eigenvalue corresponds to \( k_\mu P_{\mu \nu} = 0 \).

We can invert the operator \( P_{\mu \nu}(k) \), if we choose a gauge such that the subspace parallel to \( k \) is included. The simplest choice is the Lorenz gauge. Imposing this gauge on the level of the Lagrangian means adding the term

\[ L \to L_\text{eff} = L + L_{\text{gf}} = L - \frac{1}{2} (\partial_\mu A_\mu)^2. \] (7.59)

More generally, we can add the term

\[ L_{\text{gf}} = -\frac{1}{2\xi} (\partial_\mu A_\mu)^2 \]

that depends on the arbitrary parameter \( \xi \). This group of gauges is employed in the proof of the renormalizability of gauge theories and is therefore called \( R_\xi \) gauge. The combined effective Lagrange density is thus

\[ L_\text{eff} = L_{\text{em}} + L_{\text{gf}} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2\xi} (\partial_\mu A_\mu)^2 = \frac{1}{2} A_\mu \left[ \eta^{\mu \nu} \Box - \left( 1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] A_\nu. \] (7.61)
To find the propagator we have to invert the term in the square brackets. Denoting this term in momentum space again by \( P_{\mu\nu}(k) \), we find

\[
P_{\mu\nu} = -k^2 \eta_{\mu\nu} + (1 - \xi^{-1}) k_\mu k_\nu.
\]  

(7.62)

This expression can be split into a transverse and a longitudinal part by factoring out terms proportional to the corresponding projection operators: Denoting now with \( P_{\mu\nu}^T = \eta_{\mu\nu} - k_\mu k_\nu / k^2 \) the original transverse projection operator from Eq. (7.58), we find

\[
P_{\mu\nu} = -k^2 \left( P_{\mu\nu}^T + \frac{k_\mu k_\nu}{k^2} \right) + (1 - \xi^{-1}) k_\mu k_\nu
\]  

(7.63)

where the reminder is proportional to the longitudinal projection operator \( P_{\mu\nu}^L = k_\mu k_\nu / k^2 \).

Special cases are the Feynman gauge \( \xi = 1 \), the Landau gauge \( \xi = 0 \), while \( \xi \to \infty \) corresponds to the unitary gauge. Gauge invariance implies current conservation, \( \partial_\mu J_\mu(x) = 0 \) or \( k_\mu J_\mu(k) = 0 \). Thus the arbitrary, \( \xi \)-dependent part of the photon propagator vanishes in physical quantities, because it is always matched between two conserved currents. For finite \( \xi \) we see that the propagator is proportional to \( k^{-2} \) and we expect that less problems arise in loop calculations compared to the unitary gauge.

7.3.2. Non-abelian case

In the non-abelian case, the gauge transformation (7.20) adds not only a term \( \partial_\mu \vartheta \) but mixes also the fields via the term \( f^{abc} A_\mu^b \vartheta^c \). Therefore, we cannot expect that the unphysical degrees of freedom simply decouple and the quantisation of non-abelian theories becomes more challenging.

We consider first as a toy model for the generating functional of a Yang-Mills theory the two-dimensional integral

\[
Z \propto \int dx dy \ e^{iS(x)}.
\]  

(7.66)

Since the integration extends from \(-\infty\) to \(\infty\), the \( y \) integration does not merely change the normalisation of \( Z \) but makes the integral ill-defined. We can eliminate the dangerous \( y \) integration by introducing a delta function,

\[
Z \propto \int dx dy \ \delta(y) e^{iS(x)}.
\]  

(7.67)

Since the value of \( y \) in the delta function plays no role, we can replace \( \delta(y) \) by \( \delta(y - f(x)) \) with an arbitrary function \( f(x) \). If \( y = f(x) \) is the solution of \( g(x, y) = 0 \), we obtain with

\[
\delta(g(x, y)) = \frac{\delta(y - f(x))}{|\partial g / \partial y|}
\]  

(7.68)
7.3. Quantization of gauge theories

assuming that $\partial g / \partial y > 0$

$$Z \propto \int dx dy \frac{\partial g}{\partial y} e^{iS(x)} . \tag{7.69}$$

Generalising this to $n$ dimensions, we need $n$ delta functions and have to include the Jacobian,

$$Z \propto \int d^n x d^n y \det \left( \frac{\partial g_i}{\partial y_j} \right) \prod_i \delta (g_i) e^{iS(x)} . \tag{7.70}$$

We now translate this toy example to the Yang-Mills case. The functions $g$ are the gauge fixing conditions that we choose as

$$g^a(x) = \partial^\mu A^a_\mu(x) − \omega^a(x) , \tag{7.71}$$

where the $\omega^a(x)$ are arbitrary functions. The discrete index $i$ corresponds to $\{x, a\}$, explaining why the gauge freedom results in an infinity: Although the integration measure of a compact gauge group is finite, the summation over $\mathbb{R}^4$ gives an infinite answer. Finally, we see that the infinitesimal generators $\vartheta^a$ correspond to the redundant coordinates $y_i$.

The generating functional for a Yang-Mills theory is thus with $DA \equiv \prod_{i=0}^3 D A_i$ as short-cut

$$Z \propto \int D A \det \left( \frac{\delta g^a}{\delta \vartheta^b} \right) \prod_{x,a} \delta (g^a) e^{iS_{YM}} , \tag{7.72}$$

where we omit for the moment the source term $L_s$.

Our task is to evaluate first $\delta g^a / \delta \vartheta^b$ and then to transform the determinant and the delta functionals into terms of similar form as our starting point, the action $\exp (iS_{YM})$: Only if we succeed to convert the determinant and the delta functionals into auxiliary fields, we can hope to use perturbation theory in the usual way.

From the connection between an infinitesimal gauge transformation and the covariant derivative given in Eq. (7.20), we obtain

$$g^a(x) \rightarrow g^a(x) − \partial^\mu D^a_\mu \vartheta^b(x) . \tag{7.73}$$

Thus the required functional derivative is

$$\frac{\delta g^a(x)}{\delta \vartheta^b(y)} = −\partial^\mu D^a_\mu \delta(x − y) . \tag{7.74}$$

We can eliminate the determinant remembering $\int d\eta d\bar{\eta} e^{\bar{\eta} A \eta} = \det A$ from Eq. (5.127), expressing the Jacobian as a path integral over Grassmann variables $c^a$ and $\bar{c}^a$,

$$\det \left[ \frac{\delta g^a(x)}{\delta \vartheta^b(y)} \right] \propto \int D c D \bar{c} e^{iS_{FP}} . \tag{7.75}$$

The corresponding Lagrangian is

$$L_{FP} = −\bar{c}^a \partial^\mu D^a_\mu c^b = (\partial^\mu c^a)(D^a_\mu c^b) \tag{7.76}$$

$$= \partial^\mu \bar{c}^a \partial_\mu c^a − g f^{abc} A^c_\mu \partial^\mu c^a \bar{c}^b , \tag{7.77}$$

where we made a partial integration and inserted the definition of the covariant derivatives acting on the gauge field, Eq. (7.20).
As a result, we have recast the determinant as the kinetic energy of complex scalar fields $c^a$ that interact with the gauge fields. Since we had to use for the scalar fields Grassmann variables $c^a$, their statistics is fermionic. Clearly, such fields are acceptable only as virtual particles in loop diagrams. They should never show up as real external particles and are therefore called Faddeev-Popov ghosts. In an abelian theory as $U(1)$, the interaction term in Eq. (7.77) is absent and ghost fields decouple. Since they change then only the normalisation of the path integral, they can be omitted all together in QED.

Next we have to eliminate the $\delta(g^a(x))$. They contain the arbitrary functions $\omega^a(x)$, but the path integral does not depend on them. Thus we have the freedom to multiply with an arbitrary function $f(\omega^a)$, thereby changing only the normalisation. Our aim is to generate after integrating over the delta functions a term $\exp(iS_{gf})$, as in the case of electrodynamics. Choosing

$$Z \to \exp \left( -\frac{i}{2\xi} \int d^4x \omega^a(x) \omega^a(x) \right) Z ,$$

integrating $\prod_{x,a} \delta(g^a) \exp \left( -\frac{i}{2\xi} \int d^4x \omega^a(x) \omega^a(x) \right)$ with the help of $\delta(g^a)$ and (7.73), we obtain as gauge-fixing term the desired

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} \partial^\mu A^a_\mu \partial^\nu A^a_\nu .$$

(7.79)

The complete Lagrange density $\mathcal{L}_{\text{eff}}$ of a non-abelian gauge theory consists thus of four pieces,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{gf} + \mathcal{L}_{\text{FP}} + \mathcal{L}_{s} .$$

(7.80)

where the last one couples sources linearly to the fields,

$$\mathcal{L}_{s} = J^\mu A^a_\mu + \bar{\eta} c + \bar{c} \eta .$$

(7.81)

We break both $\mathcal{L}_{\text{YM}}$ and $\mathcal{L}_{\text{FP}}$ into a piece of $\mathcal{O}(g^0)$ defining the free propagator, and pieces of $\mathcal{O}(g)$ corresponding to a three gluon and a two ghost-gluon vertex, respectively, and a four gluon vertex of $\mathcal{O}(g^2)$. After a partial integration of the free part, we obtain

$$\mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{FP}} = \frac{1}{2} A^a_\mu (\eta^{\mu\nu} \square - \partial^\mu \partial^\nu) A^a_\mu - gf^{abc} A^a_\mu A^b_\nu \partial^\mu A^c_\nu - \frac{1}{4} g^2 f^{abc} f^{cde} A^a_\mu A^b_\nu A^e_\sigma A^d_\rho - \bar{c}^a \square c^a - gf^{abc} A^c_\mu \partial^\mu c^a A^b .$$

(7.82)

The Feynman rules can now be read off after Fourier transforming into momentum space, cf. problem ???. Combining the resulting expression with $\mathcal{L}_{gf}$, we see that the gluon propagator is diagonal in the group indices and otherwise identical to the photon propagator in $R_\xi$ gauge.

The ghost propagator is the one of a mass-less scalar particle,

$$\Delta_{ab}(k) = \frac{\delta_{ab}}{k^2 + i\varepsilon} .$$

(7.83)

Being a fermion, a closed ghost loop introduces however a minus sign.
Non-covariant gauges  The introduction of ghost fields can be avoided, if one uses non-covariant gauges which depend on an arbitrary vector $n^\mu$. An example used often in QED is the Coulomb or radiation gauge,

\[ \partial_\mu A^\mu - (n_\mu \partial^\mu)(n_\mu A^\mu) = 0 \] (7.84)

with $n_\mu = (1,0,0,0)$. In QCD, one employs often the axial gauge,

\[ n_\mu A_\mu^a = 0, \quad a = 1, \ldots, 8 \] (7.85)

with $n^2 = 0$. Then the Fadeev-Popov determinant does not dependent on $A_\mu^a$, and can be absorbed in the normalisation of the path integral. While non-covariant gauges bypass the introduction of unphysical particles in loop graphs, the resulting propagators are unhandy. Moreover, they contain spurious singularities which require care. Therefore in most applications the use of the $R_\xi$ gauge is advantageous.

7.A. Appendix: Feynman rules for an unbroken gauge theory

The Feynman rules for a non-broken Yang-Mills theory as QCD are given; for the abelian case of QED set the structure constants $f_{abc} = 0$, $T = 1$ and replace $g_s \rightarrow e q_f$, where $q_f$ is the electric charge of the fermion in units of the elementary charge $e > 0$.

Propagators

\[ g_{\mu,a} g_{\nu,b} \]

\[ -i \delta_{ab} \left[ \frac{g_{\mu\nu}}{k^2 + i\epsilon} - (1 - \xi G) \frac{k_\mu k_\nu}{(k^2)^2} \right] \] (7.86)

\[ \omega \]

\[ \delta_{ab} \frac{i}{k^2 + i\epsilon} \] (7.87)

Triple Gauge Interactions

\[ -g_s f^{abc} \left[ g^{\mu\nu} (p_1 - p_2)^\rho + g^{\nu\rho} (p_2 - p_3)^\mu \right. \]

\[ + g^{\rho\mu} (p_3 - p_1)^\nu \left] \right. \] (7.88)

\[ p_1 + p_2 + p_3 = 0 \]
Quartic Gauge Interactions

\[-i g_s^2 \left[ f_{eab} f_{ecd} (g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \rho}) \right. \\
\left. + f_{eac} f_{edb} (g_{\mu \sigma} g_{\rho \nu} - g_{\mu \nu} g_{\rho \sigma}) \right. \\
\left. + f_{ead} f_{ebc} (g_{\mu \nu} g_{\rho \sigma} - g_{\mu \rho} g_{\nu \sigma}) \right] p_1 + p_2 + p_3 + p_4 = 0 \quad (7.89)\]

Fermion Gauge Interactions

\[-i g_s \gamma^\mu T_{ij} \quad (7.90)\]

Ghost Interactions

\[-g_s f^{abc} p_1^\mu \quad (7.91)\]

\[p_1 + p_2 + p_3 = 0\]

Summary of chapter

Requiring local symmetry under a gauge group as SU(n) or SO(n) specifies the self-interactions of massless gauge boson as well as their couplings to fermions and scalars. The presence of self-interactions implies that a pure Yang-Mills theory is non-linear. The gauge invariant derivative $D^\mu$ is the analogon to the covariant derivative $\nabla^\mu$ of gravity, while the field-strength corresponds to the Riemann tensor: Both measure the rotation of a vector parallel-transported along a closed loop.

The quantisation of Yang-Mills theories in the covariant $R_\xi$ gauge leads to ghost particles—fermionic scalars—which compensate the unphysical degrees of freedom contained in the vector fields $A_\mu$ even after gauge fixing $\partial_\mu A^\mu = 0$.

Further reading

The Feynman rules are taken from [RS12]. This article contains all Standard Model Feynman rules in a convention independent notation which allows an easy comparison of references with differing conventions.
Problems

7.1 Non-abelian Maxwell equations.
Derive the non-abelian analogue of the Maxwell equations. Do gauge invariant currents and charges exist i) locally and ii) globally?

7.2 Palatini approach.
Consider the Yang-Mills action as a functional of the potential and the field-strength,

$$S_{YM}[A_\mu, F_{\mu\nu}] = -\frac{1}{4} \int d^4x F^a_{\mu\nu} \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A^b_\mu A^c_\nu \right).$$

Derive the non-abelian Maxwell equations in the Palatini approach, i.e. by varying $A_\mu$ and $F_{\mu\nu}$ independently.

7.3 Adjoint representation.
\(a\.) Derive the Jacobi identity \([[[A,B],C] = [A,[B,C]] - [B,[A,C]]\) and show that the structure constants $f^{abc}$ of a Lie algebra satisfy the Lie algebra by themselves.

\(b\.) Insert $(T^a)_{bc} = -if^{abc}$ for the adjoint representation in the general definition (7.9) for the covariant derivative and show that the result agrees with (7.20).

7.4 Hyperbolic plane $H^2$.
The line-element of the Hyperbolic plane $H^2$ is given by $ds^2 = y^{-2}(dx^2 + dy^2)$ with $y \geq 0$.

a.) Write out the geodesic equations and deduce the Christoffel symbols $\Gamma^a_{bc}$.
b.) Calculate the Riemann (or curvature) tensor $R^a_{\mu\nu\rho}$ and the scalar curvature $R$.

7.5 Riemann tensor.
Derive the symmetry properties of the Riemann tensor and the number of its independent components for an arbitrary number of dimensions. (Hint: Simplest using an inertial system.)

7.6 Three and four gauge boson vertex.
Derive the tensor structure $V^{rst}(k^\rho_1, k^\sigma_2, k^\tau_3)$ of the three-gluon vertex by Fourier-transforming the part of $\mathcal{L}_I$ containing three gluon fields to momentum space,

$$F = \int d^4p_1 d^4p_2 d^4p_3 (2\pi)^4 \delta(p_1 + p_2 + p_3) \times f(A^\mu_{\rho_1}(p_1)A^\nu_{\sigma_2}(p_2)A^{\tau\nu}(p_3))$$

and then eliminating the fields by functional derivatives with respect to them

$$V^{rst}(k^\rho_1, k^\sigma_2, k^\tau_3) = \frac{\delta^3F}{\delta A^\rho_1_1(k_1) \delta A^\sigma_2_2(k_2) \delta A^{\tau\nu}_3(k_3)}.$$

Similarly, derive the tensor structure $V^{rst}(k^\rho_1, k^\sigma_2, k^\tau_3, k^\lambda_4)$ of the four-gluon vertex. Use alternatively symmetry arguments, if possible.
8. Renormalisation

We have encountered already three examples of divergent loop integrals discussing the $\lambda \phi^4$ theory. In these cases, it was possible to subtract the infinities in such a way that we obtained finite and physically sensible observables. Main aim of this chapter is to obtain a better physical understanding of this renormalisation procedure. We will learn that the $\lambda \phi^4$ theory as well as the electroweak and strong interactions of the SM are examples for renormalisable theories: For such theories, the renormalisation of the finite number of parameters contained in the initial classical Lagrangian is sufficient to make all observables finite in any order perturbation theory. While most of our discussion stays within our standard perturbative treatment, we introduce in the last section a non-perturbative approach based on ideas developed in solid-state physics and the renormalisation group.

Why renormalisation at all? We are using perturbation theory with free, non-interacting particles as asymptotic states as tool to evaluate non-linear quantum field theories. Interactions change however the parameters of the free theory, as we know already both from classical electrodynamics and quantum mechanics. In the former case, Lorentz studied 1904 the connection between the observed electron mass $m_{\text{phys}}$, its mechanical or inertial mass $m_0$ and its electromagnetic self-energy $m_{\text{el}}$ in a toy model. He described the electron as a spherically symmetric uniform charge distribution with radius $r_e$, obtaining

$$m_{\text{phys}} = m_0 + m_{\text{el}} = m_0 + \frac{4e^2}{5r_e} .$$

Special relativity forces us to describe the electron as a point particle: Taking thus the limit $r_e \to 0$, classical electrodynamics implies an infinite “renormalisation” of the “bare” electron mass $m_0$ by its electromagnetic self-energy $m_{\text{el}}$.

Another familiar example for renormalisation appears in quantum mechanics. Perturbation theory is possible, if the Hamilton operator $\hat{H}$ can be split into a solvable part $\hat{H}^{(0)}$ and an interaction $\lambda \hat{V}$,

$$\hat{H} = \hat{H}^{(0)} + \lambda \hat{V} ,$$

and the parameter $\lambda$ is small. Using then as starting point the normalised solutions $|n^{(0)}\rangle$ of $\hat{H}^{(0)}$,

$$\hat{H}^{(0)} |n^{(0)}\rangle = E^{(0)}_n |n^{(0)}\rangle ,$$

we can find the eigenstates $|n\rangle$ of the complete Hamiltonian $\hat{H}$ as a power-series in $\lambda$,

$$|n\rangle = |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + \ldots$$

Since we started with normalised states, $\langle n^{(0)}|n^{(0)}\rangle = 1$, the new states $|n\rangle$ are not longer correctly normalised. Thus going from free (or “bare”) to interacting states requires to renormalise the states,

$$R \langle n|n\rangle_R = 1 \Rightarrow |n\rangle_R \equiv Z^{1/2} |n\rangle .$$
8.1. Anomalous magnetic moment of the electron

A very similar problem we encountered introducing the LSZ formalism. In the parlance of field theory, we continue often to call this procedure wave-function renormalisation, although $Z$ renormalises field operators. Note also that in non-degenerate perturbation theory, $\langle n^{(0)}|n^{(1)}\rangle = 0$, so we see that $Z = 1 + O(\lambda^2)$.

The familiar process of renormalisation in quantum mechanics becomes for quantum field theories more mysterious by the fact that the renormalisation constants are infinite. Thus we have to regularise as first step, i.e. employing a method which makes our expressions finite, before we perform the renormalisation.

8.1. Anomalous magnetic moment of the electron

We start the chapter with the calculation of the anomalous magnetic moment of the electron. Apart from being the first successful loop calculation in the history of particle physics, this process illustrates the case that quantum corrections introduce new types of interactions: While an electron on the classical level interacts via $L_{\text{int}} = e\bar{\psi}\gamma^\mu\psi A_\mu$ with an electromagnetic field, loop graphs add an $(e^3/m)\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ interaction. Characteristic for renormalisable theories is that the loop integrals associated with these new interactions are finite.

**Vertex function** We want to write down the most general form $\Lambda^\mu$ of the coupling term between an external electromagnetic field and an on-shell Dirac fermion, consistent with Lorentz invariance, current conservation and parity. Since $p^2 = p'^2 = m^2$, the only non-trivial scalar variable in the problem is $p \cdot p'$. We choose to use the equivalent quantity $q^2 = (p - p')^2$ as the variable on which the arbitrary scalar functions in our ansatz for $\Lambda^\mu$ depend. Imposing parity conservation forbids the use of $\gamma^5$. Hence the most general ansatz compatible with Lorentz invariance and parity is

$$\Lambda^\mu(p, p') = A(q^2)\gamma^\mu + B(q^2)p^\mu + C(q^2)p'^\mu + D(q^2)\sigma^{\mu\nu}p_\nu + E(q^2)\sigma^{\mu\nu}p'_\nu. \quad (8.6)$$

Current conservation requires $q_\mu \Lambda^\mu(p, p') = 0$ and leads to $C = B$ and $E = -D$. Hence

$$\Lambda^\mu(p, p') = A(q^2)\gamma^\mu + B(q^2)(p^\mu + p'^\mu) + D(q^2)\sigma^{\mu\nu}q_\nu. \quad (8.7)$$

Hermeticity finally requires that $A, B$ are real and $D$ is purely imaginary.
**Gordon decomposition** We derive now an identity that allows us to eliminate one of the three terms in Eq. (8.7), if we sandwich $\Lambda^\mu$ between two spinors which are on-shell. We evaluate

\[ F^\mu = \bar{u}(p') [\slashed{p}' \gamma^\mu + \gamma^\mu \slashed{p}] u(p) \]  

first using the Dirac equation for the two on-shell spinors, finding

\[ F^\mu = 2m \bar{u}(p') \gamma^\mu u(p) . \]  

Secondly, we can use $\gamma^\mu \gamma^\nu = \eta^{\mu\nu} - i \sigma^{\mu\nu}$, obtaining

\[ F^\mu = \bar{u}(p') \left[ (p' + p)^\mu + i \sigma^{\mu\nu} (p' - p)_\nu \right] u(p) . \]  

Equating (8.9) and (8.10) gives the Gordon identity: It allows us to separate the Dirac current

\[ \text{secondly, we can use } \gamma^\mu \gamma^\nu = \eta^{\mu\nu} - i \sigma^{\mu\nu} , \text{ obtaining} \]

\[ \bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[ (p' + p)^\mu + \frac{i \sigma^{\mu\nu} (p' - p)_\nu}{2m} \right] u(p) . \]  

Moreover, the Gordon identity shows that the three terms in Eq. (8.7) are not independent. Depending on the context, we can eliminate therefore the most annoying term. We follow conventions and introduce the (real) form-factors $F_1(q^2)$ and $F_2(q^2)$ by

\[ \Lambda^\mu(p,p') = F_1(q^2) \gamma^\mu + F_2(q^2) \frac{i \sigma^{\mu\nu} q_\nu}{2m} = F_1(q^2) \frac{(p' + p)^\mu}{2m} + [F_1(q^2) + F_2(q^2)] \frac{i \sigma^{\mu\nu} q_\nu}{2m} . \]  

The form-factor $F_1$ is the coefficient of the electric charge, $e F_1(q^2) \gamma^\mu$, and should thus go to one for small momentum transfer, $F_1(0) = 1$. Therefore the magnetic moment of an electron is shifted by $1 + F_2(0)$ from the tree-level value $g = 2$ you derived in problem 5.3. The deviation $a = (g - 2)/2$ is called anomalous magnetic moment, the two form-factors are often called electric and magnetic form-factors.

Note the usefulness of the procedure to express the vertex function using only general symmetry requirements but not a specific theory for the interaction: Equation (8.12) allows experimentalists to present their measurements using only two scalar functions which in turn can be easily compared to predictions of specific theories.

**Anomalous magnetic moment** After having discussed the general structure of the electromagnetic vertex function, we turn now to its calculation in perturbation theory for the case of QED. The Feynman diagrams contributing to the matrix element at $O(\alpha^3)$ with wave-functions as external lines are shown in Fig. 8.1. We separate the matrix element into the tree-level part and the one-loop correction, $-ie \bar{u}(p') [\gamma^\mu + \Gamma^\mu] u(p)$. Using the Feynman gauge for the photon propagator, we obtain

\[ \Gamma^\mu(p,p') = \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 + i\varepsilon} \left(-i e \gamma^\nu\right) \frac{i}{p' - k - m + i\varepsilon} \gamma^\mu \frac{i}{p - k - m + i\varepsilon} (-i e \gamma^\nu) . \]  

This integral is logarithmically divergent for large $k$,

\[ \int \frac{d^4 k}{k^2} \propto \text{ln} \Lambda . \]
8.1. Anomalous magnetic moment of the electron

Before we perform the explicit calculation, we want to understand if this divergence is connected to a specific kinematical configuration of the momenta. We split therefore the vertex correction into an on-shell and an off-shell part,

\[ \Gamma^\mu(p, p') = \Gamma^\mu(p, p) + [\Gamma^\mu(p, p') - \Gamma^\mu(p, p)] \equiv \Gamma^\mu(p, p) + \Gamma^\mu_{\text{off}}(p, p'). \] (8.16)

Next we rewrite the first fermion propagator for small \( p' - p = q \rightarrow 0 \) as

\[ \frac{1}{p' - k - m} = \frac{1}{p - k - m + (p' - p)} = \frac{1}{p - k - m} - \frac{1}{p - k - m}(p' - p) \cdot \frac{1}{p - k - m} \ldots \] (8.17)

\[ = \frac{1}{p - k - m} + \frac{1}{p - k - m}(p' - p) + \ldots \] (8.18)

The first term of this expansion leads to the logarithmic divergence of the loop integral for large \( k \). In contrast, the reminder of the expansion that vanishes for \( p' - p = q \rightarrow 0 \) contains additional powers of \( 1/k \) and is thus convergent. Hence the UV divergence is contained solely in the on-shell part of the vertex correction, while the function \( \Gamma^\mu_{\text{off}}(p, p') = \Gamma^\mu(p, p') - \Gamma^\mu(p, p) \) is well-behaved. Moreover, we learn from Eq. (8.12) that the divergence is confined to \( F_1(0) \), while \( F_2(0) \) is finite. This is good news: The divergence is only connected to a quantity already present in the classical Lagrangian, the electric charge. Thus we can predict the function \( \Gamma^\mu(p, p') \) for all values \( p' \neq p \), after we have renormalised the electric charge in the limit of zero momentum transfer.

We now calculate the vertex function (8.14) explicitly. We set

\[ \Gamma^\mu(q) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{N^\mu(k)}{D} \] (8.19)

with

\[ N^\mu = \gamma^\mu(p' + k + m)\gamma^\mu(p + k + m)\gamma_\nu \] (8.20)

and

\[ D = [(p' + k)^2 - m^2][(p + k)^2 - m^2] k^2. \] (8.21)

Then we combine the propagators introducing Feynman parameter integrals,

\[ \Gamma^\mu(q) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{2N^\mu(k)}{\{k^2 + \alpha[(p' + k)^2 - m^2 - k^2] + \beta[(p + k)^2 - m^2 - k^2]\}^2}. \] (8.22)

The complete calculation of the vertex function (8.14) for arbitrary off-shell momenta is already quite cumbersome. In order to shorten the calculation, we restrict ourselves therefore to the part contributing to the magnetic moment \( F_2(0) \). Because of

\[ N^\mu(k) = [F_1(q^2) + F_2(q^2)] \gamma^\mu - F_2(q^2) \frac{(p' + p)\gamma^\mu}{2m} \] (8.23)

we can simplify the calculation of \( N^\mu(k) \), throwing away all terms proportional to \( \gamma^\mu \) which do not contribute to the magnetic moment. Moreover, we can consider the limit that the electrons are on-shell and the momentum transfer to the photon vanishes.

\[ \frac{1}{A + B} = \frac{1}{A} - \frac{1}{A} B + \ldots \]

1The identity can be checked by multiplying with \( A + B \).
Using the on-shell condition, \( p^2 = p'^2 = m^2 \), the two square brackets simplify to \( 2p' \cdot k \) and \( 2p \cdot k \), respectively,

\[
D = \left\{ k^2 + 2k \cdot (\alpha p' + \beta p) \right\}^3 .
\] (8.24)

Next we eliminate the term linear in \( k \) completing the square,

\[
D = \left\{ (k + \alpha p' + \beta p)^2 - (\alpha p' + \beta p)^2 \right\}^3 = \left\{ [l^2 - (\alpha^2 m^2 + \beta^2 m^2 + 2\alpha \beta p' \cdot p)] \right\}^3 .
\] (8.25)

Since the momentum transfer to the photon vanishes, \( q^2 = 2m^2 - 2p' \cdot p \to 0 \), we can replace \( p' \cdot p \to m^2 \) and obtain as final result for the denominator

\[
D = \{ l^2 - (\alpha + \beta)^2 m^2 \}^3 .
\] (8.26)

Now we move on to the evaluation of the nominator \( N^\mu(k) \). Performing the change of our integration variable from \( k = l - (\alpha p' + \beta p) \) to \( l \), the nominator becomes

\[
N^\mu(l) = \gamma^\nu (p' + l + m) \gamma^\mu (p + l + m) \gamma_\nu
\] (8.27)

with \( p' \equiv (1 - \alpha)p' - \beta \phi \) and \( p \equiv (1 - \beta)p - \alpha p' \).

Multiplying out the two brackets and ordering the result according to powers of \( m \), we observe first that the term \( \propto m^2 \) leads to \( \propto \gamma^\mu \) and thus does not contribute to \( F_2(0) \). Next we split further the term linear in \( m \) according to powers of \( l \): The term linear in \( l \) vanishes after integration, while the term \( m l^0 \) results in

\[
m(\gamma^\nu p' \gamma^\mu \gamma_\nu + \gamma^\nu \gamma^\mu p \gamma_\nu) = 4m(p'^\mu + p^\mu) = 4m[(1 - 2\alpha)p'^\mu + (1 - 2\beta)p^\mu].
\] (8.28)

Using the symmetry in the integration variables \( \alpha \) and \( \beta \), we can rewrite this expression as

\[
\to 4m[(1 - \alpha - \beta)(p'^\mu + p^\mu)].
\] (8.29)

We split the \( m^0 l^2 \) term in the same way according to the powers of \( l \). The \( m^0 l^2 \) term gives a \( \gamma^\mu \) term, the \( m^0 l^0 \) vanishes after integration, and the \( m^0 l^0 \) gives after some work

\[
\gamma^\nu p'^\mu \gamma^\nu p \gamma_\nu \to 2m[\alpha(1 - \alpha) + \beta(1 - \beta)](p' + p)^\mu .
\] (8.30)

Finally, the \( m^0 l^0 \) term contributes to the anomalous magnetic moment

\[
\to -2m(p' + p)^\mu [2(1 - \alpha)(1 - \beta)] .
\] (8.31)

Combining all terms, we find

\[
N^\mu = 4m(1 - \alpha - \beta)(p' + p)^\mu + 2m[\alpha(1 - \alpha) + \beta(1 - \beta)](p' + p)^\mu
\]

\[
- 4m[(1 - \alpha)(1 - \beta)](p' + p)^\mu
\]

\[
= 2m[(1 - \alpha - \beta)(\alpha + \beta)](p' + p)^\mu .
\] (8.32)

Thus

\[
\Gamma_2^{(3)}(0) = -2ie^2 \int d\alpha d\beta \int \frac{d^2l}{(2\pi)^2} \frac{N^\mu}{l^2 - (\alpha + \beta)^2 m^2} ,
\] (8.33)

where the subscript 2 indicates that we account only for the contribution to the anomalous magnetic moment.
We can perform now the $l$ integral using the formula (8.34) for $I(\omega, a)$ with $\omega = 2$ and $a = 3$,

$$ I(2, 3) = \frac{i}{32\pi^2} \frac{1}{(\alpha + \beta)^2 m^2 + i\varepsilon}, $$

(8.34)

obtaining as expected a finite result. As last step, we do the integrals over the Feynman parameters $\alpha$ and $\beta$,

$$ \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1 - \alpha - \beta}{\alpha + \beta} = \frac{1}{2} $$

(8.35)

and find thus

$$ \Gamma_2^e(0) = \frac{e^2}{8\pi^2} \frac{1}{2m} (p' + p)^\mu. $$

(8.36)

We have reproduced the result of the first successful calculation of a loop correction in a quantum field theory, performed by Schwinger 1948, $F_2(0) = \alpha/2\pi$. Together with Bethe’s previous estimate of the Lamb shift in the hydrogen energy spectrum, this stimulated the view that a consistent renormalisation of QED is possible.

The currently most precise experimental value for the electron anomalous magnetic moment $a_e \equiv F_2^e(0)$ is

$$ a_e^{\exp} = 0.001\, 159\, 652\, 180\, 73(28)[0.24\, \text{ppb}]. $$

(8.37)

The calculation of the universal (i.e. common to all charged leptons) QED contribution has been completed up to fourth order. There exists also an estimate of the dominant fifth order contribution,

$$ a_{\ell}^{\text{uni}} = 0.5 \left( \frac{\alpha}{\pi} \right) - 0.328\, 478\, 965\, 579\, 193\, 78 \ldots \left( \frac{\alpha}{\pi} \right)^2 + 1.181\, 241\, 456\, 587 \ldots \left( \frac{\alpha}{\pi} \right)^3 - 1.9144(35) \left( \frac{\alpha}{\pi} \right)^4 + 0.0(4.6) \left( \frac{\alpha}{\pi} \right)^5 $$

(8.38)

The three errors given in round brackets are: The error from the uncertainty in $\alpha$, the numerical uncertainty of the $\alpha^4$ coefficient and the error estimated for the missing higher order terms [Jeg07].

Comparing the measured value and the prediction using QED, we find an extremely good agreement. First of all, this is strong support that the methods of perturbative quantum field theory we developed so far can be successfully applied to weakly coupled theories as QED. Secondly, it means that additional contributions to the anomalous magnetic moment of the electron have to be tiny.

**Electroweak and other corrections** The lowest order electroweak corrections to the anomalous magnetic moment contain in the loop virtual gauge bosons ($W^\pm, Z$) or a higgs boson $h$ and are shown in Fig. 8.2. We will consider the electroweak theory describing these diagrams only later; for the present discussion it is sufficient to know that the weak coupling constant is $g \sim 0.6$ and that the scalar and weak gauge bosons are much heavier than leptons, $M \gg m$.

The second diagram corresponds schematically to the expression

$$ \sim g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \frac{A(m^2, k)}{|(p - k) - m^2|^2}. $$

(8.39)
As in QED, this integral has to be finite and we expect that it is dominated by momenta up to the mass $M$ of the gauge bosons, $k \lesssim M$. Therefore its value should be proportional to $g^2 m^2 / M^2$ (times a possible logarithm $\ln (M^2 / m^2)$) and electroweak corrections to the anomalous magnetic moment of the electron are suppressed by a factor $(m/M)^2 \sim 10^{-10}$ compared to the QED contribution. The property that the contribution of virtual heavy particles to loop processes is suppressed in the limit $|q^2| \ll M^2$ is called “decoupling”. Note the difference to the case of the mass of a scalar particle or the cosmological constant: In these examples, the loop corrections are infinite and we cannot predict these quantities. However, we consider it as unnatural that the measured value $\rho_{\text{vac}}$ is so much smaller than the expected minimal value $\Lambda^4 \gtrsim (\text{TeV})^4$. In contrast, the anomalous magnetic moment is finite but, as we include loop momenta up to infinity, depends in principal on all particles coupling to the electron, even if they are arbitrarily heavy. Only if these heavy particles “decouple,” we can calculate $\alpha_e$ without knowing e.g. the physics at the Planck scale. Thus the decoupling property is a necessary ingredient of any theory of high energy physics, otherwise nothing like chemistry or solid state physics would be possible before knowing the “theory of everything”.

Clearly, the contribution of heavy particles (either electroweak gauge and higgs bosons or other not yet discovered particles) is more visible in the anomalous magnetic moment of the muon than of the electron. Moreover, a relativistic muon lives long enough that a measurement of its magnetic moment is feasible. This is one example how radiative corrections (here evaluated at $q^2 = 0$) are sensitive to physics at higher scales $M$: If an observable can be measured and calculated with high enough precision, one can be sensitive to suppressed corrections of order $g^2 m^2 / M^2$. Other examples are rare processes like $\mu \rightarrow e + \gamma$ or $B_s \rightarrow \mu^+ \mu^-$ which are suppressed by a specific property of the SM which one does not expect to hold in general. The achieved precision in measuring and calculating such processes is high enough to probe generically scales of $M \sim 100\, \text{TeV}$, i.e. much higher than the mass scales that can be probed directly at current accelerators as LHC. For an overview see [Jeg07].

![Figure 8.2.: Lowest order electroweak corrections to the anomalous magnetic moment of fermions.](image)

### Finite versus divergent parts of loop corrections

We found that the vertex correction could be split into two parts

$$
\Lambda^\mu(p, p') = F_1(q^2) \gamma^\mu + F_2(q^2) \frac{i \sigma^{\mu\nu} q_\nu}{2m},
$$

(8.40)

where the form factor $F_2(q^2)$ is finite for all $q^2$, while the form factor $F_1(q^2)$ diverges for $q^2 \rightarrow 0$. The important observation is that $F_2(q^2)$ corresponds to a Lorentz structure that is not present in the original Lagrangian of QED. This suggests that we can require from a “nice” theory that
• all UV divergences are connected to structures contained in the original Lagrangian, all new structures are finite. The basic divergent structures are also called “primitive” divergent graphs.

• If there are no anomalies, then loop corrections respect the original (classical) symmetries. Thus, e.g., the photon propagator should be at all orders transverse, respecting gauge invariance. We will see that as consequence the high-energy behaviour of the theory improves.

In such a case, we are able to hide all UV divergences in a renormalisation of the original parameters of the Lagrange density.

8.2. Power counting and renormalisability

We try to make the requirements on a “nice” theory a bit more precise. Let us consider the set of $\lambda \phi^n$ theories in $d = 4$ space-time dimensions and check which graphs are divergent. We define the superficial degree $D$ of divergence of a Feynman graph as the difference between the number of loop momenta in the nominator and denominator of a Feynman graph. We can restrict our analysis to those diagrams called 1P irreducible (1PI) which cannot be disconnected by cutting an internal line: All 1P reducible diagrams can be decomposed into 1PI diagrams which do not contain common loop integrals and can be therefore analysed separately. Moreover, we are only interested in the loop integration and define therefore the 1PI Green functions as graphs where the propagators on the external lines were stripped off. In $d = 4$ space-time dimensions, the degree $D$ of divergence of a 1PI Feynman graph is thus

$$D = 4L - 2I,$$

(8.41)

where $L$ is the number of independent loop momenta and $I$ the number of internal lines. The former contributes a factor $d^4 p$, while the latter corresponds to a scalar propagator with $1/(p^2 - m^2) \sim 1/p^2$ for $p \to \infty$.

Momentum conservation at each vertex leads for an 1PI-diagram to

$$L = I - (V - 1),$$

(8.42)

where $V$ is the number of vertices and the $-1$ takes into account the delta function leading to overall momentum conservation. Thus

$$D = 2I - 4V + 4.$$  

(8.43)

Each vertex connects $n$ lines and any internal line reduces the number of external lines by two. Therefore the number $E$ of external lines is given by

$$E = nV - 2I.$$  

(8.44)

As result, we can express the superficial degree $D$ by the order of perturbation theory $(V)$, the number of external lines $E$ and the degree $n$ of the interaction polynomial $\phi^n$,

$$D = (n - 4)V + 4 - E.$$  

(8.45)

From this expression, we see that
• for $n = 3$, the coefficient of $V$ is negative. Therefore only a finite number of terms in the perturbative expansion are infinite. Such a theory is called super-renormalisable, the corresponding terms in the Hamiltonian are also called relevant.

• For $n = 4$, we find $D = 4 - E$. Thus the degree of divergence is independent of the order of perturbation theory being only determined by the number of external lines. Such theories contain an infinite number of divergent graphs, but they all correspond to a finite number of divergent structures—the so-called primitive divergent graphs. Such interactions are also called marginal and are candidates for a renormalisable theory.

• Finally, for $n > 4$ the degree of divergence increases with the order of perturbation theory. As result, there exists an infinite number of divergent structures, and increasing the order of perturbation theory requires more and more input parameter to be determined experimentally. Such a theory is called non-renormalisable, the interaction irrelevant.

In particular, the $\lambda \phi^4$ theory as an example for a renormalisable theory has only three divergent structures: i) the case $E = 0$ and $D = 4$ corresponding a contribution to the cosmological constant, ii) the case $E = 2$ and $D = 2$ corresponding to the self-energy, and iii) the case $E = 4$ and $D = 0$, i.e. logarithmically divergence, to the four-point function. As we saw in chapter 2, the three primitive divergent diagrams of the $\lambda \phi^4$ theory correspond to the following physical effects: Vacuum bubbles renormalise the cosmological constant. The effect of self-energy insertions is twofold: Inserted in external lines it renormalises the field, while self-energy corrections in internal propagators lead to a renormalisation of its mass. The vertex correction finally renormalises the coupling strength $\lambda$.

Let us move to the case of QED. Repeating the discussion, we obtain the analogue to Eq. (8.45), but accounting now for the different dimension of fermion and bose fields,

$$D = 4 - B - \frac{3}{2}F,$$  \hspace{1cm} (8.46)

where $B$ and $F$ count the number of external bosonic and fermionic lines, respectively. There are six different superficially divergent primitive graphs in QED: The photon and the fermionic contribution to the cosmological constant ($D = 4$), the vacuum polarisation ($D = 2$), the fermion self-energy ($D = 1$), the vertex correction ($D = 0$) and light-by-light scattering ($D = 0$).

In a gauge theory, the true degree of divergence can be smaller than the superficial one. For instance, light-by-light scattering corresponds to a term $\mathcal{L} \sim A^4$ that violates gauge invariance. Thus either the gauge symmetry is violated by quantum corrections or such a term is finite.

Because of the correspondence of the dimension of a field and the power of its propagator, we can connect the superficial degree of divergence of a graph to the dimension of the coupling constants at its vertices. The superficial degree $D(G)$ of divergence of a graph is connected to the one of its vertices $D_v$ by

$$D(G) - 4 = \sum_v (D_v - 4)$$ \hspace{1cm} (8.47)

which in turn depends as

$$D_v = \delta_v + \frac{3}{2} f_v + b_v = 4 - \lfloor g_v \rfloor$$ \hspace{1cm} (8.48)
8.3. Renormalisation of the $\lambda\phi^4$ theory

on the dimension of the coupling constant $g$ at the vertex $v$. Here, $f_v$ and $b_v$ are the number of fermion and boson fields at the vertex, while $\delta_v$ counts the number of derivatives.

Thus the dimension of the coupling constant plays a crucial role deciding if a certain theory is “nice” in the naive sense defined above. Clearly $D = 0$ or $[g] = 0$ is the border-line case:

- If at least one coupling constant has a negative mass dimension, $[g] < 0$ and $D_v > 4$, the theory is non-normalisable. Examples are the Fermi theory of weak interactions, $[G_F] = -2$, and gravitation, $[G_N] = -2$.
- If all coupling constants have positive mass dimension, $[g] > 0$ and $D_v < 4$, the theory is super-normalisable. An example is the $\lambda\phi^3$ theory in $D = 4$ with $[\lambda] = 1$.
- The remaining cases, with all $[g_i] = 0$, are candidates for renormalisable theories. Examples are Yukawa interactions, $\lambda\phi^4$, Yang-Mills theories that are unbroken (QED and QCD) or broken by the Higgs mechanism (electroweak interactions).

Theories with massive gauge bosons We have assumed that the propagators of bosons behave as $1/(p^2 - m^2) \sim 1/p^2$ for $p \to \infty$. This is true for scalars and for massless spin-1 particles like the photon and gluons. In contrast, the longitudinal part in the propagator of a massive spin-1 particles leads to a worse asymptotic behaviour. Including a mass term for gauge bosons breaks gauge invariance and leads to a non-renormalisable theory. A solution to this problem is the introduction of gauge boson masses via interactions that respect the gauge symmetry as in the Higgs mechanism.

Furry’s theorem We have not included in our list of primitive divergent graphs of QED loops with an odd number of fermions, i.e. the tadpole ($B = 1$ and $D = 3$) and “photon splitting” graph ($B = 3$ and $D = 1$). Such diagrams vanish in vacuum, as fact known as Furry’s theorem. Since the momentum flow in a loop graph plays no rule, we can write a loop as “$1/2$(clockwise+ anti-clockwise)”. Then we insert $CC^{-1} = 1$ between all factors in the trace, use $C\gamma^\mu C^{-1} = -\gamma^\mu T$ and $CS_F(-x)C^{-1} = S_F(x)$. Hence for an odd number of propagators, the two contributions cancel.

Alternatively, we can use a symmetry argument to convince us that all diagrams with an odd number of photons are zero: Because the QED Lagrangian is invariant under $C$, $\psi \to \psi^c = C\bar{\psi}^T$ and $A_\mu = \to -A_\mu$, all Green functions with an odd number of photons vanish.

8.3. Renormalisation of the $\lambda\phi^4$ theory

We have argued that a theory with dimensionless coupling constant is renormalisable, i.e. that a multiplicative shift of the parameters contained in the classical Lagrangian is sufficient to obtain finite Green functions. The simplest theory of this type in $D = 4$ is the $\lambda\phi^4$ theory for which we will discuss now the renormalisation procedure at one loop level. As starter, we examine the general structure of the divergences and review different regularisation options.

8.3.1. Renormalisation and regularisation

Structure of the divergences We learnt that the degree of divergence decreases increasing the number of external lines, since the number of propagators increases. The same effect has
8. Renormalisation

taking derivatives wrt external momenta $p$,

$$\frac{\partial}{\partial \hat{p}} \frac{1}{\hat{k} + \hat{p} - m} = -\frac{1}{(\hat{k} + \hat{p} - m)^2}.$$  

This means that

1. we can Taylor expand loop integrals, confining the divergences in the lowest order terms. Choosing e.g. $p = 0$ as expansion point in the fermion self-energy,

$$\Sigma(p) = A_0 + A_1 p + A_2 p^2 + \ldots$$

with

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \hat{p}^n} \Sigma(p),$$

we know that $A_0$ is (superficially) linear divergent. Thus $A_1$ can be maximally logarithmic divergent, while all other $A_n$ are finite.

2. We could choose a different expansion point, leading to different renormalisation conditions (within the same regularisation scheme).

3. The divergences can be subtracted by local operators, i.e. polynomials of the fields and their derivatives.

The last statement requires that non-local terms as e.g. $\ln(p^2/\mu^2)$ which can be generated by sub-loops in a diagram of order $g^n$ are cancelled by counter-terms of order $n' < n$. A sketch why this should be true goes as follows:

Green functions become singular for coinciding points, i.e. when the convergence factor $e^{-kx}$ in the Euclidean Green function becomes one. In the simplest cases (separate divergences in one diagram) as

$$\langle 0 \vert \phi(x') \phi(x) \vert 0 \rangle_{x' \to x} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x'-x)} \bigg|_{x' \to x} = \int \frac{d^3k}{(2\pi)^3 2\omega_k},$$

the infinities are eliminated by normal ordering. More complicated are overlapping divergences where two or more divergent loops share a propagator. Wilson suggested to expand the product of two fields as the sum of local operator $O_i$ times coefficient functions $C_i(x-y)$ as

$$\phi(x)\phi(y) = \sum_i C_i(x-y) O_i(x),$$

where the whole spatial dependence is carried by the coefficients. For a massless scalar field, dimensional analysis dictates that $C_i(x) \propto x^{-2+d_i}$, if the local operator $O_i$ has dimension $d_i$. Note that only the unity operator has a singular coefficient function $1/x^2$ corresponding to the massless scalar propagator. Similarly we can expand product of operators,

$$O_n(x)O_m(y) = \sum \, C_{nm}^{i}(x-y)O_i(x),$$

where now $C_{nm}^{i}(x) \propto x^{-d_n-d_m+d_i}$. Thus we can use this operator product expansion (or briefly “OPE”) to rewrite the overlapping divergences in terms of coefficient functions and finite local operators. Moreover, the sub-divergence occurring at order $k$, when $p < k$ points coincide, have the form found at order $p$. We conclude that non-local terms due to overlapping divergencies are cancelled by the counter terms found at lower order.
Regularisation schemes  Mathematical manipulations as shifting the integration variable in divergent loop integral are only well-defined, if we convert first these integrals into convergent ones. This process is called regularisation and can be done in a number of ways. In general, one reparametrises the integral in terms of a parameter \( \Lambda \) (or \( \varepsilon \)) called regulator such that the integral becomes finite for finite regulator while the limit \( \Lambda \to \infty \) (or \( \varepsilon \to 0 \)) returns to the original integral.

- We can avoid UV divergences evaluating loop-integrals introducing an (Euclidean) momentum cutoff,

\[
I \to I_\Lambda \equiv \int_0^\Lambda d^4k F(k) = A(\Lambda) + B + C \left( \frac{1}{\Lambda} \right).
\]

Somewhat more sophisticated, we could introduce instead of a hard cutoff a smooth function which suppresses large momenta. Using Schwinger’s proper-time representation

\[
\frac{1}{p^2 + m^2} = \int_0^\infty ds \exp\left(-s(p^2 + m^2)\right)
\]

we can cut-off large momenta setting

\[
\frac{1}{p^2 + m^2} \to \frac{\exp(-s(p^2 + m^2)/\Lambda^2)}{p^2 + m^2} = \int_0^\infty ds \exp\left(-s(p^2 + m^2)\right).
\]

Although conceptual very easy, both regularisation schemes violate generically all symmetries of our theory. This is not a principal flaw, since we should recover these symmetries in the limit \( \Lambda \to \infty \). But calculations become much thougher since we cannot use the symmetries at intermediate steps, and therefore these schemes are in practise not useful except for the simplest cases.

- Pauli-Villars regularisation is a scheme respecting gauge invariance of QED. Its basic idea is to add heavy particles coupled gauge invariantly to the photon,

\[
\frac{1}{k^2 - m^2 + i\varepsilon} \to \frac{1}{k^2 - m^2 + i\varepsilon} + \sum_i \frac{a_i}{k^2 - M^2 + i\varepsilon}.
\]

For \( k^2 \ll M^2 \), physics is unchanged, while for \( k^2 \gg M^2 \) and \( a_i < 0 \) the combined propagators scale as \( M^2/k^4 \) and the convergence of loop integrals improves. Since the heavy particles enter with the wrong sign, they are unphysical and serve only as a mathematical tool to regularise loop diagrams.

- Dimensional Regularisation (DR) is one of the most useful and least intuitive regulators. The reason for its usefulness is that it preserves gauge invariance. In DR, we replace our integrals with

\[
I \to I_d \equiv \int_0^\infty d^d k F(k)
\]

where \( d \) is the dimension of our measure. As the example \( \int d^d k \ k^{-2} = 0 \) which we will discuss later shows the “measure” we implement by physical requirements with DR is not a positive measure—as a mathematician would require. Nevertheless, we can
performs integrals in \( d \neq 4 \) space-time dimensions. Then we can replace \( d \to 4 - \varepsilon \) in the result and take the limit \( \varepsilon \to 0 \), splitting the result into poles and finite parts.

Evaluating fermion traces in the denominator using DR, we have to extend the Clifford algebra to \( d \) dimensions. A natural choice is \( \text{tr}(\gamma^\mu \gamma^\nu) = d \eta^\mu\nu \). Problematic is however the treatment of \( \gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \) relying heavily on \( d = 4 \). As a result DR breaks chiral symmetry. The same is true for supersymmetry.

- In lattice regularisation one replaces the continuous space-time by a discrete lattice. The finite lattice spacing \( a \) introduces a momentum cutoff, eliminating all UV divergences. Moreover the (Euclidean) path-integral becomes well-defined and can be calculated numerically without the need to do perturbation theory. Thus this approach is particularly useful in predicting many results of low-energy QCD, including hadron masses and form-factors, which otherwise are only calculable in low-energy approximations to QCD. Note that lattice regularisation for finite \( a \) respects the gauge symmetry, but spoils translation and Lorentz symmetry of the underlying QFT. Nevertheless, one recovers in the limit \( a \to 0 \) a relativistic QFT. A longstanding problem of lattice theory was how to implement correctly chiral fermions. This question has been recently solved and thus the SM can be now in a mathematically consistent, non-perturbative way defined as a lattice theory.

- Various other regularisation schemes as e.g. zeta function regularisation or point splitting methods exists.

Even fixing a regularisation scheme, e.g. DR, we can choose various renormalisation conditions. Three popular choices are

- on-shell renormalisation. In this scheme, we choose the subtraction such that the on-shell masses and couplings coincide with the corresponding values measured in processes with zero momentum transfer. For instance, we define the renormalised electric charge via the Thomson limit of the Compton scattering amplitude. While this choice is very intuitive, it is not practical for QCD.

- the minimal subtraction (MS) scheme, where we subtract only the divergent \( 1/\varepsilon \) poles.

- the modified minimal subtraction \( \overline{\text{MS}} \) (read em-es-bar) scheme, where we subtract additionally the \( \ln(\gamma_\varepsilon) + 4\pi \) term appearing frequently. This scheme gives more compact expressions than the others and is most often used in theoretical calculations. As drawback, quantities like \( m_\text{MS} \) have to be translated into the physical mass \( m_e \).

### 8.3.2. The \( \lambda \phi^4 \) theory at one-loop

There are two equivalent ways to perform perturbative renormalisation. In the one we use first called often “conventional” perturbation theory we use the “bare” (unrenormalised) parameters in the Lagrangian,

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\text{int} = \frac{1}{2} (\partial_\mu \phi_0)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!} \phi_0^4 .
\]  

(8.52)

Then we introduce a renormalised field \( \phi_R = Z^{-1/2}_\phi \phi_0 \) and choose the parameters \( Z_\phi, m_0 \) and \( \lambda_0 \) as function of the regularisation parameter \( (\varepsilon, \Lambda, \ldots) \) such that the field \( \phi_R \) has finite
Green functions. In the following we discuss the renormalisation procedure at one-loop level for the Green functions of the $\lambda\phi^4$ theory in this scheme. Since any 1P reducible diagram can be decomposed into 1PI diagrams which do not contain common loop integrals, we can restrict our analysis again to 1PI Green functions.

**Mass and wave-function renormalisation** The Feynman propagator in momentum space is

$$i\Delta_F(p) = \int d^4x \ e^{ipx} \langle 0| T\{\phi_0(x)\phi_0(0)\} | 0 \rangle = \frac{i}{p^2 - m_0^2 + i\varepsilon}$$

at lowest order perturbation theory. The full propagator $i\Delta_F(p)$ is the sum of an infinite chain of self-energy insertions,

$$i\Delta_F(p) = \frac{i}{p^2 - m_0^2 + i\varepsilon} + \frac{i}{p^2 - m_0^2 + i\varepsilon} \left(-i\Sigma(p^2)\right) \frac{i}{p^2 - m_0^2 + i\varepsilon} + \ldots$$

This expression looks like an infinite tower of independently propagating particles with different masses. But if we sum up the terms of this binomial series, we obtain

$$i\Delta_F(p) = \frac{i}{p^2 - m_0^2 + i\varepsilon} \left[\frac{1}{1 + i\Sigma(p^2)}\right] = \frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\varepsilon}.$$  

(8.55)

Now we see that the effect of the interactions results in a shift of the particle mass, $m_0^2 \rightarrow m_0^2 - \Sigma(p^2)$. Next we have to show that $\Sigma(p^2)$ is finite after renormalisation. The 1-loop expression is

$$-i\Sigma(p^2) = \frac{-i\lambda_0}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_0^2 + i\varepsilon},$$

i.e. quadratically divergent. As a particularity of the $\phi^4$ theory, the $p^2$ dependence of the self-energy $\Sigma$ shows up only at the 2-loop level.

We perform a Taylor expansion of $\Sigma(p^2)$ around the arbitrary point $\mu$,

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2 - \mu^2)\Sigma'(\mu^2) + \Sigma(p^2),$$

(8.57)

where $\Sigma(\mu^2) \propto \Lambda^2$, $\Sigma'(\mu^2) \propto \ln \Lambda$ and $\Sigma(p^2)$ is the finite remainder. A term linear in $\Lambda$ is absent, since we cannot construct a Lorentz scalar out of $p^\mu$. Note also $\Sigma(\mu^2) = \Sigma'(\mu^2) = 0$.

Now we insert (8.57) into (8.55),

$$\frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\varepsilon} = \frac{i}{p^2 - m_0^2 - \Sigma(\mu^2) - (p^2 - \mu^2)\Sigma'(\mu^2) - \Sigma(p^2) + i\varepsilon},$$

(8.58)

where we see that we can identify $\mu$ with the physical mass given by the pole of the propagator.

We aim at rewriting the remaining effect for $p^2 \rightarrow \mu^2$ of the self-energy insertion, $\Sigma(\mu^2)$, as a multiplicative rescaling. In this way, we could remove the divergence from the propagator by a rescaling of the field. At leading order in $\lambda$, we can write

$$\tilde{\Sigma}(p^2) = [1 - \Sigma'(\mu^2)] \Sigma(p^2) + O(\lambda^2)$$

(8.59)
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and thus

\[ i\Delta_F(p) = \frac{1}{1 - \Sigma'(\mu^2)} \left\{ \frac{i}{p^2 - \mu^2 - \Sigma(p^2) + i\epsilon} \right\} = \frac{iZ_\phi}{p^2 - \mu^2 - \Sigma(p^2) + i\epsilon} \] (8.60)

with the wave-function renormalisation constant

\[ Z_\phi = \frac{1}{1 - \Sigma'(\mu^2)} = 1 + \Sigma'(\mu^2). \] (8.61)

Close to the pole, the propagator is the one of a free particle with mass \( \mu \),

\[ i\Delta_F(p) = \frac{iZ_\phi}{p^2 - \mu^2 + i\epsilon} + O(p^2 - \mu^2). \] (8.62)

Thus the renormalisation constant \( Z_\phi \) is the same \( Z \) appearing in the LSZ formula. As far as \( S \)-matrix elements are concerned, the renormalisation of external lines has therefore the sole effect of cancelling the \( Z^{-1} \) factor from the LSZ formula. Therefore we can obtain \( S \)-matrix elements replacing propagators on external lines directly with on-shell wave-functions, excluding at the same time all diagrams where external lines are renormalised.

We define the renormalised field \( \phi = Z_\phi^{-1/2} \phi_0 \) such that the renormalised propagator

\[ i\Delta_R(p) = \int d^4x e^{ipx} \langle 0| T\{\phi(x)\phi(0)\} |0\rangle = Z_\phi^{-1} i\Delta(p) = \frac{i}{p^2 - \mu^2 - \Sigma(p^2) + i\epsilon} \] (8.63)

is finite. Similarly, we define renormalised \( n \)-point functions by

\[ G_R^{(n)}(x_1, \ldots, x_n) = \langle 0| T\{\phi(x_1) \cdots \phi_n(x_n)\} |0\rangle = Z_\phi^{-n/2} G_0^{(n)}(x_1, \ldots, x_n). \] (8.64)

Since the 1PI \( n \)-point Green functions miss \( n \) field renormalisation constant compared to connected \( n \)-point Green functions, the connection between renormalised and bare 1PI \( n \)-point functions becomes

\[ \Gamma_R^{(n)}(x_1, \ldots, x_n) = Z_\phi^{n/2} \Gamma_0^{(n)}(x_1, \ldots, x_n). \] (8.65)

This induces a corresponding sign change in the RGE for 1PI Green functions.

**Coupling constant renormalisation** We can choose an arbitrary point inside the kinematical region, \( s + t + u = 4\mu^2 \) and \( s \geq 4\mu^2 \), to define the coupling. For our convenience and less writing work, we choose instead the symmetric point

\[ s_0 = t_0 = u_0 = \frac{4\mu^2}{3}. \]

The bare four-point 1PI Green function is (see section 2.7.3)

\[ \Gamma_0^{(4)}(s, t, u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u), \] (8.66)

the renormalised four-point function at \( (s_0, t_0, u_0) \) is

\[ \Gamma_R^{(4)}(s_0, t_0, u_0) = -i\lambda. \] (8.67)
Next we expand the bare 4-point function around \( s_0, t_0, u_0 \),

\[
\Gamma^{(4)}_0(s, t, u) = -i\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \quad (8.68)
\]

where the \( \tilde{\Gamma}(x) \) are finite and zero at \( x_0 \). Now we define a vertex (or coupling constant) renormalisation constant by

\[
-iZ^{-1}_\lambda \lambda = -i\lambda_0 + 3\Gamma(s_0) \quad (8.69)
\]

Inserting this definition in (8.68) we obtain

\[
\Gamma^{(4)}_0(s, t, u) = -iZ^{-1}_\lambda \lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \quad (8.70)
\]

which simplifies at the renormalisation point to

\[
\Gamma^{(4)}_0(s, t, u) = -iZ^{-1}_\lambda \lambda \quad (8.71)
\]

We use now the connection between renormalised and bare Green functions,

\[
\Gamma^{(4)}_R(s, t, u) = Z^2_\phi \Gamma^{(4)}_0(s, t, u) \quad (8.72)
\]

and thus

\[
-i\lambda = Z^2_\phi Z^{-1}_\lambda (-i\lambda_0) \quad (8.73)
\]

The relation between the renormalised and bare coupling in the \( \lambda \phi^4 \) theory is thus

\[
\lambda = Z^2_\phi Z^{-1}_\lambda \lambda_0 \quad (8.74)
\]

Now we have to show that \( \Gamma^{(4)}_R(s, t, u) \) is finite.

\[
\Gamma^{(4)}_R(s, t, u) = Z^2_\phi \Gamma^{(4)}_0(s, t, u) \\
= -iZ^2_\phi Z^{-1}_\lambda \lambda_0 + Z^2_\phi [\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)] \\
= -i\lambda + Z^2_\phi [\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)] \quad (8.75)
\]

Since \( Z_\phi = 1 + \mathcal{O}(\lambda^2) \), this is equivalent to

\[
\Gamma^{(4)}_R(s, t, u) = -i\lambda + [\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)] + \mathcal{O}(\lambda^3) \quad (8.76)
\]

consisting only of finite expressions. This completes the proof that at one-loop order all Green functions in the \( \lambda \phi^4 \) theory are finite, renormalising the field \( \phi \), its mass and coupling constant as

\[
\phi = Z^{-1/2}_\phi \phi_0 \quad (8.77a)
\]

\[
\lambda = Z^2_\phi Z^{-1}_\lambda \lambda_0 \quad (8.77b)
\]

\[
\mu^2 = m^2_0 + \Sigma(\mu^2) = m^2_0 + \delta m^2 \quad (8.77c)
\]
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**Renormalised perturbation theory** We can use the conditions \(^{(8.77)}\) in order to formulate perturbation theory using directly only renormalised quantities. We set

\[
L_0 = L + L_{ct}
\]

where \(L\) has the same structure as \(L_0\) but is expressed through renormalised quantities,

\[
L = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4
\]

and thus

\[
L_{ct} = \frac{1}{2} (Z_\phi - 1) (\partial_\mu \phi)^2 - \frac{1}{2} \delta m^2 \phi^2 + (Z_\lambda - 1) \frac{\lambda}{4!} \phi^4.
\]

The term \(L_{ct}\) is called counter term Lagrangian and contains the divergent renormalisation constants. The latter are all of order \(\lambda\) and thus we can treat \(L_{ct}\) as a perturbation. Applying renormalised perturbation theory consists of the following steps:

1. Starting from \(^{(8.78)}\), one derives propagator and vertices.
2. One calculates 1-loop 1PI diagrams, finds the divergent parts which determine the counter terms in \(L_{ct}^{(1)}\) at order \(O(\lambda)\).
3. The new Lagrangian \(L + L_{ct}^{(1)}\) is used to generate 2-loop 1PI diagrams and \(L_{ct}^{(2)}\).
4. The procedure is iterated moving to higher orders.

**Renormalisation group equations** Consider two renormalisation schemes \(R\) and \(R'\). In the two schemes, the renormalised field will in general differ, being \(\phi_R = Z^{-1/2}_\phi(R) \phi_0\) and \(\phi_{R'} = Z^{-1/2}_\phi(R') \phi_0\), respectively. Hence the connection between the two renormalised fields is

\[
\phi_{R'} = \frac{Z^{-1/2}_\phi(R')}{Z^{-1/2}_\phi(R)} \phi_R \equiv Z^{-1/2}_{\phi(R', R)} \phi_R.
\]

As both \(\phi_R\) and \(\phi_{R'}\) are finite, also \(Z_\phi(R', R)\) is finite. The transformations \(Z^{-1/2}_\phi(R', R)\) form a group, called the renormalisation group.

If we consider

\[
G_0^{(n)}(x_1, \ldots, x_n) = Z^{n/2}_\phi G_R^{(n)}(x_1, \ldots, x_n),
\]

we know that the bare Green function is independent of the renormalisation scale \(\mu\). Taking the derivative with respect to \(\mu\), the LHS thus vanishes. For the simplest case of a massless theory, we find thus

\[
0 = \frac{d}{d \ln \mu} \left[ Z^{n/2}_\phi G_R^{(n)}(x_1, \ldots, x_n) \right]
\]

\[
= \left[ \frac{\partial}{\partial \ln \mu} + \frac{\partial \lambda}{\partial \ln \mu} \frac{\partial}{\partial \lambda} + \frac{n}{2} \frac{\partial \ln Z_\phi}{\partial \ln \mu} \right] G_R^{(n)}(x_1, \ldots, x_n)
\]

\[
\equiv \left[ \frac{\partial}{\partial \ln \mu} + \beta \frac{\partial}{\partial \lambda} + \frac{n}{2} \gamma \right] G_R^{(n)}(x_1, \ldots, x_n).
\]

Here we introduced in the last step the beta function

\[
\beta(\lambda) = \mu \frac{\partial \lambda(\mu)}{\partial \mu},
\]

\[\text{(8.83)}\]
8.3. Renormalisation of the $\lambda\phi^4$ theory

which determines the logarithmic change of the coupling constant and the anomalous dimension

$$
\gamma(\mu) = \mu \frac{\partial \ln Z_\phi(\mu)}{\partial \mu}
$$

of the field $\phi$. The physical meaning of $\gamma(\mu)$ is the topic of problem 8.6. Knowing the two functions $\beta(\lambda)$ and $\gamma(\mu)$, we can calculate the change of any Green function under a change of the renormalisation scale $\mu$. The general solution of (8.82) can be found by the method of characteristics or by the analogy of $d/d\ln \mu$ with a convective derivative, cf. problem 8.7.

Equations of the type (8.82) are called generically renormalisation group equations or briefly RGE. They come in various flavors, carrying the name of their inventors: Stückelberg–Petermann, Callan–Symanzik, Gell-Man–Low, . . .

Asymptotic behaviour of the beta-function The behaviour of the beta-function $\beta(\mu)$ in the limit $\mu \to 0$ and $\mu \to \infty$ provides a useful classification of quantum field theories. Consider e.g. the example shown in the left panel of Fig. 8.3. This beta function has a trivial zero at zero coupling, as we expect it in any perturbative theory, and an additional zero at $g_c$. How does the beta-function $\beta(\mu)$ evolve in the UV limit $\mu \to \infty$?

- Starting in the range $0 \leq g(\mu) < g_c$ implies $\beta > 0$ and thus $dg/d\mu > 0$. Therefore $g$ grows with increasing $\mu$ and the coupling is driven towards $g_c$.
- Starting with $g(\mu) > g_c$ implies $\beta < 0$ and thus $dg/d\mu < 0$. Therefore $g$ decreases for increasing $\mu$ and we are driven again towards $g_c$.

Fixed points $g_c$ approached in the limit $\mu \to \infty$ are called UV fixed points, while IR fixed points are reached for decreasing $\mu$. The range of values $[g_1 : g_2]$ which is mapped by the RGE flow on the fixed point is called its basin of attraction.

To see what happens for $\mu \to 0$, we have only to reverse the RGE flow, $d\mu \to -d\mu$, and are thus driven away from $g_c$: If we started in $0 \leq g(\mu) < g_c$, we are driven to zero, while the coupling goes to infinity for $g(\mu) > g_c$. Thus $g = 0$ is an IR fixed point. The distinction between IR and UV fixed points is sketched in the right panel of Fig. 8.3. If the beta function
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has several zeros, the theory consists of different phases which are not connected by the RG flow.

Working through problem 8.5 you should show that the beta-function of the $\lambda \phi^4$ theory at one loop is given by

$$\beta(\lambda) = b_1 \lambda^2 + b_2 \lambda^3 + \ldots = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3),$$

(8.85)

so that the running coupling satisfies

$$\lambda(\mu) = \frac{\lambda_0}{1 - 3\lambda_0/16\pi^2 \ln(\mu/\mu_0)}.$$  

(8.86)

Thus the theory has $\lambda = 0$ as an IR fixed point and the use of free particles as asymptotic states is sensible (but requires renormalisation). On the other hand, the coupling increases for $\mu \to \infty$ formally as $\lambda \to \infty$. Clearly, we cannot predict the behaviour of $\lambda(\mu)$ in the strong-coupling limit, because perturbation theory breaks down. The solution (8.86) suggests however that the coupling explodes already for a finite value of $\mu$: The beta function has a pole for a finite value of $\mu$ called Landau pole where the denominator of (8.86) becomes zero.

**The large logs** Although we derived the beta functions only at $\mathcal{O}(\lambda^2)$, the running coupling contains arbitrary powers of $[\lambda_0 \ln(\mu^2/\mu_0^2)]^n$, since $\lambda(\mu) = \lambda_0 \sum_n [b_1 \lambda_0 \ln(\mu^2/\mu_0^2)]^n / n!$. If the expansion parameter $b_1 \lambda_0$ is small, we managed to sum up the leading terms from higher order corrections. If on the other hand the expansion parameter $b_1 \lambda_0$ is large, our perturbative approach fails anyways in this regime.

8.4. Vacuum polarisation

We now turn from the $\lambda \phi^4$ theory to the case of unbroken gauge theories as QED and QCD. There are several complications compared to the simpler case of a scalar theory: First, the complexity of the calculations will increase together with the number of vertices. Second, we have to ensure that the gauge symmetry is not spoiled by loop corrections. From a more technical point of view, we note that in processes involving gauge bosons small errors can have drastic consequences, because the leading terms in a gauge-invariant set of Feynman diagrams often cancel. Finally, theories containing massless spin 1 or 2 particles have IR divergencies additional to UV divergencies, showing up in the emission of real or virtual soft photons, gluons or gravitons. Our discussion aims not to be complete. Instead we pick out a single process, the vacuum polarisation from which we can deduce the salient point of gauge theories: asymptotic freedom.

Using “conventional” perturbation theory, we define wave-function renormalisation constants for the electron and photon as

$$\psi_0 \equiv Z_2^{1/2} \psi$$

(8.87)

$$A_0^\mu \equiv Z_3^{1/2} A^\mu$$

(8.88)

Analogous to Eq. (8.74), we expect that the electric coupling is renormalised by

$$e(\mu) = \frac{Z_2 Z_3^{1/2}}{Z_1} e_0,$$

(8.89)
where $Z_1$ is the charge renormalisation constant, and $Z_2$ and $Z_3^{1/2}$ take into account that two electron fields and one photon field enter the 3-point function.

As it stands, the renormalisation condition (8.89) creates two major problems: First, the factor $Z_2^{1/2}$ will vary from fermion to fermion. For instance, the wave-function renormalisation constant of a proton includes strong interactions while the one of the electron does not. As a result, it is difficult to understand why the electric charge of an electron and an proton are renormalised such that they have the same value for $q^2 \to 0$, while they would disagree for $q^2 \neq 0$. In particular, we would expect that the universe is not electrically neutral, even assuming the same number density of both particles, $n_e = n_p$. Second, we see that the renormalised covariant derivative

$$D_\mu = \partial_\mu + i \frac{Z_1}{Z_2} e(\mu) A_\mu$$

remains only gauge-invariant, if $Z_1 = Z_2$. Clearly, this condition would also ensure the universality of the electric charge. We postpone the proof that $Z_1 = Z_2$ holds in suitable renormalisation schemes to section 8.5.2, where we will derive the Ward-Takahashi identities. In a non-abelian theory as QCD, where we have to ensure that the gauge coupling in all terms of Eq. (7.82) remains after renormalisation the same, several constraints of the type $Z_1 = Z_2$ arise.

### 8.4.1. Vacuum polarisation in QED

We calculate next the one-loop correction to the photon propagator, the so called vacuum polarisation tensor. Using the Feynman rules for QED, we obtain for the contribution of one fermion species with mass $m$

$$i\Pi^{\mu\nu}(q) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\text{tr}[\gamma^\mu(i(k + m)\gamma^\nu(k + \not{q} + m))]}{(k^2 - m^2)[(k + q)^2 - m^2]} = -e^2 \int \frac{d^4k}{(2\pi)^4} N^{\mu\nu}. \quad (8.91)$$

Our first task is to show that the vacuum polarisation tensor respects gauge invariance, i.e. has the structure $\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$. For this it is sufficient to show that $q_\mu \Pi^{\mu\nu}(q) = 0$. We write

$$q = (q + k - m) - (k - m) \quad (8.92)$$

and obtain

$$q_\mu N^{\mu\nu} = \text{tr}\{(i\not{q} + \not{k} - m) - (\not{k} - m)\gamma^\nu(\not{k} + \not{q} + m)\}$$

$$= [q + k]^2 - m^2] \text{tr}\{(\not{k} + m)\gamma^\nu\} - (k^2 - m^2) \text{tr}\{\gamma^\nu(\not{k} + \not{q} + m)\} \quad (8.93)$$

$$= [(q + k)^2 - m^2] \text{tr}\{(\not{k} + m)\gamma^\nu\} - (k^2 - m^2) \text{tr}\{\gamma^\nu(\not{k} + \not{q} + m)\} \quad (8.94)$$

where we used the cyclic property of the trace. Employing dimensional regularisation (DR) with $d = 4 - 2\varepsilon$ in order to obtain well-defined integrals,

$$q_\mu i\Pi^{\mu\nu}(q) = -e^2 \mu^{2\varepsilon} \int \frac{d^4k}{(2\pi)^d} \left\{ \text{tr}\{(\not{k} + m)\gamma^\nu\} - \text{tr}\{\gamma^\nu(\not{k} + \not{q} + m)\} \right\}, \quad (8.95)$$

we are allowed to shift the integration variable in one of the two terms. Thus $q_\mu \Pi^{\mu\nu}(q) = 0$ and hence the vacuum polarisation tensor at order $O(e^2)$ is transverse as required by gauge invariance. In Eq. (8.159), we will extend this result to full photon propagator, i.e. we will show that it holds in particular at any order perturbation theory.
Let us pause a moment and summarise before we start with the evaluation of $\Pi(q^2)$: In our power-counting analysis we found as superficial degree of divergence $D = 2$. This result was based on the assumption that the numerator $\mathcal{N}$ behaves as a constant. But the only constant available is $m^2$ which would lead to a mass term of the photon, $e^2 A_\mu \Pi^{\mu\nu} A_\nu \propto e^2 m^2 A_\mu \eta^{\mu\nu} A_\nu$. Thus the transversality of $\Pi^{\mu\nu}$ implies that the $m^2$ term in the numerator will disappear at some step of our calculation. Thereby the convergence of the polarisation tensor improves, becoming a “mild logarithmic” one.

Next we introduce as new integration variable $\gamma = q / \gamma$ and $k = 2 \omega, \alpha$. Evaluating the trace in the denominator using DR, we have to extend the Clifford algebra to $d = 2\omega$ dimensions. A natural choice is $\text{tr}(\gamma^\mu \gamma^\nu) = d \eta^{\mu\nu}$, giving with $\gamma^\mu \gamma_\mu = d$ and $\gamma^\mu \gamma_\mu = (2 - d) \eta$ as result for the trace

$$\mathcal{N} = \mathcal{N}_{\mu}^{\mu} = d(2 - d)k \cdot (k + q) + dm^2 = d \{(2 - d)[K^2 - q^2 x(1 - x)] + dm^2\}. \quad (8.100)$$

In the last step, we performed the shift $k \to K = k - qx$ omitting linear terms in $K$ that vanish after integration. Combining our results for $\mathcal{N}$ and $D$ we arrive at

$$(d - 1)q^2 \Pi(q^2) = -e^2 \mu^2 \epsilon d \int_0^1 \frac{dx}{\frac{(2\pi)^d}{(2 - d)K^2 + dm^2 - (2 - d)q^2 x(1 - x)}} \left. \frac{(2 - d)K^2 + dm^2}{(K^2 - a)^2} \right|_{a = m^2 - q^2 x(1 - x) > 0}.$$ \quad (8.101)

Finally, we want to use our results for the Feynman integrals $I_0(\omega, \alpha)$ and $I_2(\omega, \alpha)$ which were obtained performing a Wick rotation. The latter is possible as long as we do not pass a singularity. Since the prefactor $x(1 - x)$ of $q^2$ has as maximum 1/4, this requires $q^2 < 4m^2$. 

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We start to look for the first two terms where we expect a cancellation of the \( m^2 \) term in the nominator,

\[
\int \frac{d^dK}{(2\pi)^d} \frac{(2-d)K^2 + dm^2}{(K^2 - a)^2} = (2-d)I_2(\omega, 2) + dm^2I(\omega, 2) \tag{8.102}
\]

\[
= \frac{i}{(4\pi)^2\Gamma(2)} \left[ -2\omega(1-\omega)\Gamma(1-\omega) a^{\omega-1} + 2\omega m^2 \Gamma(2-\omega) a^{\omega-2} \right] \tag{8.103}
\]

\[
= \frac{i}{(4\pi)^2} 2\omega \Gamma(2-\omega)(-a + m^2) a^{\omega-2} \tag{8.104}
\]

\[
= \frac{4i}{(4\pi)^2} \Gamma(\varepsilon) (4\pi)^\varepsilon \frac{q^2 x(1-x)}{[m^2 - q^2 x(1-x)]^{\varepsilon}}. \tag{8.105}
\]

Hence the \( m^2 \) term dropped indeed out of the nominator and the whole expression is proportional to \( q^2 \), as required by the LHS of (8.101). Evaluating the third term in the same way we obtain

\[
- \int \frac{d^dK}{(2\pi)^d} \frac{(2-d)q^2 x(1-x)}{(K^2 + a)^2} = -(2-d)q^2 x(1-x)I(\omega, 2) \tag{8.106}
\]

\[
= \frac{2i}{(4\pi)^2} \Gamma(\varepsilon) (4\pi)^\varepsilon \frac{q^2 x(1-x)}{[m^2 - q^2 x(1-x)]^{\varepsilon}}. \tag{8.107}
\]

Adding the two contributions, we arrive at

\[
\Pi(q^2) = -\frac{8\varepsilon^2}{(4\pi)^2} \Gamma(\varepsilon) \int_0^1 dx x(1-x) \left[ \frac{4\pi \mu^2}{m^2 - q^2 x(1-x)} \right]^{\varepsilon}, \tag{8.108}
\]

where the factor \( i q^2 \) cancelled. We included also the factor \( (4\pi \mu^2)^\varepsilon \) into the last term, which becomes thereby dimensionless. Now we expand the Gamma function, \( \Gamma(\varepsilon) = \frac{1}{\varepsilon - \gamma + O(\varepsilon)} \), around \( d = 4 - 2\varepsilon \) and because of the resulting 1/\( \varepsilon \) term all other \( \varepsilon \) dependent quantities,

\[
\Pi(q^2) = -\frac{\varepsilon^2}{12\pi^2} \left\{ \frac{1}{\varepsilon - \gamma + \ln(4\pi)} - 6 \int_0^1 dx x(1-x) \ln \left[ \frac{m^2 - q^2 x(1-x)}{\mu^2} \right] \right\}. \tag{8.109}
\]

The prefactor \( x(1-x) \) has its maximum 1/4 for \( x = 1/2 \). Thus the branch cut of the logarithm starts at \( q^2 = 4m^2 \), i.e. when the virtuality of the photon is large enough that is can decay into a fermion pair of mass 2\( m \). This is a nice illustration of the optical theorem: The polarisation tensor is real below the pair creation threshold, and acquires an imaginary part above (which equals the pair creation cross section of a photon with mass \( m^2 = q^2 \), cf. problem 8.8).

The \( x \) integral can be integrated by elementary functions, but we display the result only for the two limiting cases of small and large virtualities, \( q^2/m^2 \to 0 \) and \( |q|^2/m^2 \to \infty \). In the first case, we obtain with \( \ln(1-x) \sim x \)

\[
\Pi(q^2) = -\frac{\varepsilon^2}{12\pi^2} \left[ \frac{1}{\varepsilon - \gamma + \ln(4\pi) + \ln(\mu^2/m^2) + \frac{q^2}{5m^2} + \ldots} \right], \tag{8.110}
\]

while the opposite limit gives

\[
\Pi(q^2) = -\frac{\varepsilon^2}{12\pi^2} \left[ \frac{1}{\varepsilon - \gamma + \ln(4\pi) - \ln(|q|^2/\mu^2) + \ldots} \right], \tag{8.111}
\]

\(^2\text{Note we use here the conventions } d = 2\omega = 4 - 2\varepsilon.\)
In the MS scheme, we obtain the renormalisation constant $Z_3$ for the photon field as the coefficient of the pole term,

$$Z_3 = 1 - \frac{e^2}{12\pi^2\varepsilon}.$$  

(8.112)

More often the on-shell renormalisation scheme is used in QED. Here we require that quantum corrections to the electric charge vanish for $q^2 = 0$, i.e. we choose $Z_3$ such that $\Pi_{\text{on}}(q^2 = 0) = 0$. This is obviously achieved setting

$$\Pi_{\text{on}}(q^2) = \Pi(q^2) - \Pi(0) = \frac{e^2}{60\pi^2} \frac{q^2}{m^2} + \ldots ,$$  

(8.113)

for $|q^2| \ll m^2$. This $q^2$ dependence leads to a modification of the Coulomb potential, which can be measured e.g. in the Lamb shift.

**Beta function**  We can derive the scale dependence of the renormalised electric charge from

$$e_0 = \frac{\mu^\varepsilon}{Z_3^{1/2}} e,$$  

(8.114)

where we used $Z_1 = Z_2$. Then the beta function is given as

$$\beta(e) \equiv \mu \frac{\partial e}{\partial \mu} = \mu \frac{\partial}{\partial \mu} \left( \mu^{-\varepsilon} Z_3^{1/2} e_0 \right).$$  

(8.115)

Since the bare charge $e_0$ is independent of $\mu$, we have to differentiate only $\mu$ and $Z_3$,

$$\beta(e) = \mu \frac{\partial e}{\partial \mu} = -\varepsilon \mu^{-\varepsilon} Z_3^{1/2} e_0 + \mu \mu^{-\varepsilon} \frac{1}{2} Z_3^{-1/2} \frac{\partial Z_3^3}{\partial \mu} e_0$$  

(8.116)

$$= -\varepsilon e + \frac{\mu}{2Z_3} \frac{\partial Z_3}{\partial \mu} e.$$

(8.117)

Inserting $Z_3 = 1 - e^2/(12\pi^2\varepsilon)$ and thus

$$\frac{\partial Z_3}{\partial \mu} = -\frac{1}{12\pi^2\varepsilon} \frac{2e\partial e}{\partial \mu}$$

(8.118)

gives

$$\beta(e) = -\varepsilon e - \frac{\mu}{12\pi^2\varepsilon Z_3} \frac{\partial e}{\partial \mu} e^2 = -\varepsilon e - \frac{1}{12\pi^2\varepsilon Z_3} \beta(e) e^2 .$$

(8.119)

Note that $Z_3$ is scheme-dependent, while the beta-function remains scheme independent up to two loop (problem 8.5). Solving for $\beta$ and neglecting higher order terms in $e^2$, we find in the limit $\varepsilon \to 0$

$$\beta(e) = -\varepsilon e \left( 1 - \frac{e^2}{12\pi^2\varepsilon} + \mathcal{O}(e^4) \right) = \frac{e^3}{12\pi^2} .$$

(8.120)

Thus the beta function contains indeed the renormalised charge on the RHS. Finally we note that the beta function can be re-expressed as

$$\beta(e^2) \equiv \mu^2 \frac{\partial e^2}{\partial \mu^2} = e \beta(e) = \frac{e^4}{12\pi^2} .$$

(8.121)
8.4. Vacuum polarisation

Its solution,\[ e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{6\pi^2} \ln \left( \frac{\mu}{\mu_0} \right)} \] (8.122)

shows not only explicitly the increase of \( e^2 \) with \( \mu \), but moreover that the electric coupling diverges for a finite value of \( \mu \). This singularity called Landau pole happens at

\[ \mu = \mu_0 \exp(6\pi^2/\epsilon^2(\mu_0)) = m_e \exp(3\pi/2\alpha(m_e)) \sim 10^{56} \text{ GeV} \gg M_{Pl} \] (8.123)

and has therefore no physical relevance.

8.4.2. Vacuum polarisation in QCD

We only sketch the calculations in QCD, stressing the new or different points compared to QED. In Fig. 8.4, we show the various 1-loop diagrams contributing to the vacuum polarisation tensor in a non-abelian theory as QCD. Most importantly, the three-gluon vertex allows in addition to the quark loop now also a gluon loop. Since a fermion loop has an additional minus sign, we expect that the gluon loop gives a negative contribution to the beta function. This opens the possibility that non-abelian gauge theories are in contrast to QED asymptotically free, if the number of fermion species is small enough.

**Quark loop**  The vertex changes from \(-ie\gamma^\mu\) in QED to \(-ig_s T^a \gamma^\mu\) in QCD. Since the quark propagator is diagonal in the group index, a quark loop contains for each flavor additionally the factor

\[ \text{tr}\{T^a T^a\} = \frac{1}{2} \delta^{aa} = 4 \, . \] (8.124)

Thus we have only to replace \( e \rightarrow 4n_f g_s \) in the QED result, where \( n_f \) counts the number of quark flavors. For the three light quarks, u, d and s, it is an excellent approximation to set \( m = 0 \). In contrast, the masses of the other three quarks (c, b and t) can not be neglected. As the calculation for massive quarks shows is the effect of particle masses well approximated including in the loop only particles with mass \( 4m^2 < \mu^2 \), making \( n_f \) scale dependent.
Loop with three-gluon vertex Since the three-gluon vertex connects identical particles, we have to take into account symmetry factors similar as in the case of the $\lambda\phi^4$ theory. We learnt that the imaginary part of a Feynman diagram corresponds to the propagation of real particles. Thus the imaginary part of the gluon vacuum polarisation can be connected to the total cross section of $g \rightarrow gg$ scattering. This cross section contains a symmetry factor $1/2!$ to account for two identical particles in the final state. Therefore the same symmetry factor should be associated to the vacuum polarisation with a gluon loop\(^3\). Applying the Feynman rule for the three-gluon and using the Feynman-t’Hooft gauge, we find

\[
i\Pi_{ab,3}(q^2) = \frac{1}{2}(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{N}_{ab}^{\mu\nu}}{((k + q)^2 + i\epsilon)((k + q)^2 + i\epsilon)}
\] (8.125)

with

\[
\mathcal{N}_{ab}^{\mu\nu} = \frac{1}{2}(\eta_{\mu\rho}(q + k)^\rho + \eta_{\nu\rho}(2k - q)^\rho + \eta_{\rho\sigma}(2q - k)^\rho] \times \frac{1}{2}(\eta_{\mu\rho}(q + k)^\rho + \eta_{\nu\rho}(2k - q)^\rho + \eta_{\rho\sigma}(2q - k)^\rho].
\] (8.126)

Evaluating the colour trace,

\[
f_{acd}f_{cdb} = N_c \delta_{ab}
\] (8.128)

and extracting in the usual way the pole part using DR one obtains for $d = 4 - \epsilon$

\[
i\Pi_{ab,3}(q^2) = -i \frac{N_c g^2}{16\pi^2\epsilon} \left(\frac{\mu^2}{q^2}\right)^{\epsilon/2} \left(\frac{1}{3} q^\mu q^\nu - \frac{19}{6} \eta^{\mu\nu} q^2 \right) + O(\epsilon^0).
\] (8.129)

Thus the contribution from the three-gluon vertex alone is not transverse—demonstrating again that a covariant gauge requires the introduction of ghost particles.

Ghost loop This diagram has the same dependence on the structure constants as the previous one,

\[
i\Pi_{ab,3}(q^2) = -(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{f_{bcd}(k - q)\nu f_{acd}k^{\nu}}{(k^2 + i\epsilon)((k + q)^2 + i\epsilon)}
\] (8.130)

and can thus be combined with the three-gluon loop. Evaluating the integral results in

\[
i\Pi_{ab,3}(q^2) = -i \frac{N_c g^2}{16\pi^2\epsilon} \left(\frac{\mu^2}{q^2}\right)^{\epsilon/2} \left(\frac{1}{3} q^\mu q^\nu + \frac{1}{6} \eta^{\mu\nu} q^2 \right) + O(\epsilon^0).
\] (8.131)

and summing the three-gluon and ghost loops gives the expected gauge-invariant expression. Moreover, the sum has the opposite sign as the quark loop and can thus lead to the opposite behaviour of the beta function as in QED.

Four-gluon loop and tadpole diagrams The loop with the four-gluon vertex contains a massless propagator and does not depend on external momenta,

\[
i\Pi_{ab,4}(q^2) \propto \int \frac{d^d k}{k^2 + i\epsilon}.
\] (8.132)

Our general experience with DR tells us that this graph is zero, as gluons are massless. However, this loop integral is also in DR ambiguous: For any space-time dimension $d$, the

\(^3\)This argument does not apply to the quark loop, since cutting leads in this case to a distinguishable $\bar{q}q$ state.
integral is either UV or IR divergent. To proceed, we split therefore the integrand introducing the arbitrary mass \( M \) as

\[
\frac{1}{k^2 + i\varepsilon} = \frac{1}{k^2 - M^2 + i\varepsilon} - \frac{M^2}{(k^2 + i\varepsilon)(k^2 - M^2 + i\varepsilon)}. \tag{8.133}
\]

Now the IR and UV divergence is separated, and we can use \( d < 2 \) in the first term and 2 < \( d < 4 \) in the second. By dimensional analysis, both terms have to be proportional to a power of the arbitrary mass \( M \). As the LHS is independent of \( M \), the only option is that the two terms on the RHS cancel, as an explicit calculation confirms. The two remaining tadpole diagrams (5 and 6 of Fig. 8.4) vanish by the same argument.

**Asymptotic freedom**  Deriving the beta function in QCD, we should evaluate

\[
g(\mu) = \frac{Z_2 Z_3^{1/2}}{Z_1} g_0, \tag{8.134}
\]

where \( Z_1 \) is the charge renormalisation constant, and \( Z_2 \) and \( Z_3 \) are quark and gluon renormalisation constants, respectively. Combining all contributions to the \( 1/\varepsilon \) poles as 1-loop contribution \( b_1 \) to the beta function of QCD gives

\[
\beta = \mu^2 \frac{\partial g^2}{\partial \mu^2} = -\frac{\alpha_s^2}{4\pi} (b_1 + b_2 \alpha_s + b_3 \alpha_s^2 + \ldots) \tag{8.135}
\]

with \( \alpha_s \equiv g^2/(4\pi) \),

\[
b_1 = \frac{11}{3} N_c - \frac{2}{3} n_f \tag{8.136}
\]

and \( n_f \) as number of quark flavors. For \( n_f < 16 \), the beta function is negative and the running coupling decreases as \( \mu \to \infty \). Asymptotic freedom of QCD explains the apparent paradox that protons are interacting strongly at small \( Q^2 \), while they can be described in deep-inelastic scattering as a collection of independently moving quarks and gluons.

Let us consider now the opposite limit, \( \mu \to 0 \). The solution of (8.135) at one loop,

\[
\frac{1}{\alpha_s(\mu^2)} = \frac{1}{\alpha_s(\mu_0^2)} + b_1 \ln \left( \frac{\mu^2}{\mu_0^2} \right) \tag{8.137}
\]

shows that the QCD coupling constant becomes formally infinite for a finite value of \( \mu \). We define \( \Lambda_{QCD} \) as the energy scale where the running coupling constant of QCD diverges, \( \alpha_s^{-1}(\Lambda^2_{QCD}) = 0 \). Experimentally, the best measurement of the strong coupling constant has been performed at the Z resonance at LEP, giving \( \alpha_s(m_Z^2) \sim 0.1184 \). Thus at one-loop level,

\[
\Lambda_{QCD} = m_Z \exp \left( \frac{1}{2b_1 \alpha_s(m_Z^2)} \right). \tag{8.138}
\]

\( \Lambda_{QCD} \) depends on the renormalisation scheme and, numerically more importantly, on the number of flavors used in \( b_1 \): For instance, \( \Lambda_{QCD}^{\overline{MS}} \sim 220 \text{ MeV} \) for \( n_f = 3 \). The fact that the running coupling provides a characteristic energy scale is called dimensional transmutation: Quantum corrections lead to the break-down of scale-invariance of classical QCD with massless quarks and to the appearance of massive bound-states, the mesons and baryons with masses of order \( \Lambda_{QCD} \). Note also that we are able to link exponentially separated scales by dimensional transmutation.
**Coupling constant unification** While the strong coupling $\alpha_s \equiv \alpha_3$ decreases with increasing $\mu^2$, the electromagnetic coupling $\alpha_{em} \equiv \alpha_1$ increases. Since two lines in a plane meet at one point, there is a point with $\alpha_1(\mu_*) = \alpha_3(\mu_*)$ and one may speculate that at this point a transition to a “unified theory” happens. Since the running is only logarithmic, unification happens at exponentially high scales, $\mu_* \sim 10^{16}$ GeV, but interestingly still below the Planck scale $M_{Pl}$. The problems becomes more challenging, if we add to the game the third, the weak coupling $\alpha_2$. The situation in 1991 assuming the validity of the SM is shown in the left panel of Fig. 8.5. The width of the lines indicates the experimental and theoretical error, and the three couplings clearly do not meet within these errors. The right panel of the same figure assume the existence of supersymmetric partners to all SM particles with an “average mass” of around $M_{SUSY} \sim 200$ GeV. As a result, the running changes above $\mu = 200$ GeV, and now the three couplings meet for $2 \times 10^{16}$ GeV.

![Figure 8.5.](image)

*Figure 8.5.: The measurements of the gauge coupling strengths at LEP do not evolve towards a unified value in the SM (left), but meet at $2 \times 10^{16}$ GeV assuming low-scale supersymmetry (right).*

### 8.5. Effective action and Ward identities

In this section we introduce first the effective action as the generating functional of 1PI Green functions. Then we will use the developed formalism to derive the Ward identities which imply e.g. that the exact photon propagator is transverse and that the renormalisation of the electric charge is universal. Later on in Sec.10.2 we will see that the discussion of the renormalisation of theories with spontaneous symmetry breaking simplifies using the effective action.

#### 8.5.1. Effective action

To start, recall the definition for the generating functionals of a real scalar field,

$$Z[J] = \int \mathcal{D}\phi \exp \{i \int d^4x (\mathcal{L}(x) + J(x)\phi(x))\} = e^{i W[J]}.$$  \hfill (8.139)
Next we define the classical field \( \phi_c(x) \) as \( \phi_c(x) = \delta W[J]/\delta J(x) \). Performing the functional derivative in its definition, we see immediately why this definition makes sense,

\[
\phi_c(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{1}{iZ} \delta Z[J] = \frac{1}{Z} \int D\phi \phi(x) \exp i \int d^4y (\mathcal{L} + J\phi) = \frac{\langle 0|\phi(x)J|0\rangle}{\langle 0|0\rangle} = \langle \phi(x) \rangle_J.
\] (8.140a)

Thus the classical field \( \phi_c(x) \) is defined as the vacuum expectation value of the field \( \phi(x) \) in the presence of the source \( J(x) \).

Now we define the effective action \( \Gamma[\phi_c] \) as the Legendre transform of \( W[J] \),

\[
\Gamma[\phi_c] = W[J] - \int d^4x J(x)\phi_c(x),
\] (8.141)

where (8.140a) should be used to replace \( J(x) \) by \( \phi_c(x) \) on the RHS. This procedure is completely analogous to the construction of the Hamilton function from the Lagrange function in classical mechanics: It will allow us to answer the question “which source \( J(x) \) produces a given \( \phi_c(x) \)?” because we can use \( \phi_c \) as independent variable in \( \Gamma[\phi_c] \). We compute the functional derivative w.r.t. \( \phi_c \) of this new quantity,

\[
\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(y)} = \int d^4x \frac{\delta J(x)}{\delta \phi_c(y)} \delta W - \int d^4x \frac{\delta J(x)}{\delta \phi_c(y)} \phi_c(x) - J(y).
\]

Using the definition \( \frac{\delta W}{\delta J(x)} = \phi_c(x) \), the first and second term cancels and we end up with

\[
\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(y)} = -J(y).
\] (8.142)

This is the desired relation determining the source \( J(x) \) producing a given classical field \( \phi_c(x) \).

To see why \( \Gamma[\phi_c] \) is called effective action, we consider a free scalar field. The free generating functional for connected Green functions is

\[
W_0[J] = -\frac{1}{2} \int d^4x d^4x' J(x)\Delta_F(x - x')J(x'),
\] (8.143)

and the classical field becomes

\[
\phi_c(x) = \frac{\delta W}{\delta J(x)} = -\int d^4x' \Delta_F(x - x')J(x').
\] (8.144)

If we apply the Klein-Gordon operator to the classical field,

\[
(\Box + m^2)\phi_c(x) = -\int d^4x'(\Box + m^2)\Delta_F(x - x')J(x') = \int d^4x' \delta(x - x') J(x') = J(x),
\] (8.145a)

we see that \( \phi_c \) is a solution of the classical field equation. Now we have all we need to write down an explicit expression for the free effective action,

\[
\Gamma_0[\phi_c] = W_0[J] - \int d^4x J(x)\phi_c(x) = \frac{1}{2} \int d^4x d^4x' J(x)\Delta_F(x - x')J(x').
\] (8.146a)
Inserting the expression (8.145) for \( J(x) \) and integrating partially twice, we obtain
\[
\Gamma_0[\phi_c] = \frac{1}{2} \int d^4x d^4x' \left[ (\Box + m^2)\phi_c(x) \right] \Delta_F(x-x') \left[ (\Box' + m^2)\phi_c(x') \right] 
\]
\[
= -\frac{1}{2} \int d^4x \phi_c(x) (\Box + m^2)\phi_c(x) = S_0[\phi_c].
\]

Thus the effective action of a free field equals the classical action of the field, justifying its name.

To gain some more information on the meaning of \( \Gamma[\phi] \), we can evaluate it perturbatively and expand it in \( \phi \), giving us a new set of \( n \)-point Green functions,
\[
\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \Gamma^{(n)}(x_1, \ldots, x_n) \phi_c(x_1) \cdots \phi_c(x_n).
\] (8.147)

The functions \( \Gamma^{(n)} \) are the one-particle-irreducible Green functions. Hence, the effective action \( \Gamma_0[\phi_c] \) is the generating functional for 1PI Green functions, the proof we postpone to section 10.2.

**Example:** Show that a) \( \Gamma^{(2)} \) is equal to the inverse propagator or inverse 2-point function, and b) derive the connection of \( \Gamma^{(3)} \) to the connected 3-point function.

a.) We write first
\[
\delta(x_1 - x_2) = \frac{\delta\phi(x_1)}{\delta\phi(x_2)} = \int d^4x \frac{\delta\phi(x_1)}{\delta\phi(x_2)} \frac{\delta J(x)}{\delta J(x)}
\]
using the chain rule. Next we insert \( \phi(x) = \delta W/\delta J(x) \) and \( J(x) = -\delta\Gamma/\delta\phi(x) \) to obtain
\[
\delta(x_1 - x_2) = -\int d^4x \frac{\delta^2 W}{\delta J(x) \delta J(x_1)} \frac{\delta^2 \Gamma}{\delta \phi(x) \delta \phi(x_2)}.
\]
Setting \( J = \phi = 0 \), it follows
\[
\int d^4x iG(x, x_1) \Gamma^{(2)}(x, x_2) = -\delta(x_1 - x_2)
\] (8.148)
or \( \Gamma^{(2)}(x_1, x_2) = iG^{-1}(x, x_1) \).

b.) The connected 3-point function \( G^{(3)}(x_1, x_2, x_3) \) is obtained by appending propagators to the irreducible 3-point vertex function \( \Gamma^{(3)}(x_1, x_2, x_3) \), one for each external leg,
\[
G^{(3)}(x_1, x_2, x_3) = i \int d^4x_1 d^4x_2 d^4x_3 \Gamma^{(2)}(x_1, x_1') \Gamma^{(2)}(x_2, x_2') \Gamma^{(3)}(x_3, x_3')
\]
This generalises to \( n > 3 \) and therefore one calls the \( \Gamma^{(n)} \) also amputated Green functions.

### 8.5.2. Ward-Takahashi identities

The redundancy implied by gauge invariance leads us to believe that not all of Green functions of a gauge theory are independent: Gauge invariance implies relations between Green functions called Ward or Ward-Takahashi identities in QED and Slavnov-Taylor identities in the non-abelian case. We use now the formalism of the effective action we developed to derive these for QED: Consider the generating functional
\[
Z[J^\mu, \eta, \bar{\eta}] = \int DAD\bar{\psi}D\psi \exp\{i \int d^4x \mathcal{L}_{\text{eff}}\},
\] (8.149)
8.5. Effective action and Ward identities

where $J^\mu$ is a four-vector source, $\eta$ and $\bar{\eta}$ are Grassmannian sources and the effective Lagrangian is composed of a classical term, a source term and a gauge fixing term,

$$L_{\text{eff}} = L_{\text{cl}} + L_s + L_{gf}, \quad (8.150a)$$

$$L_{\text{cl}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + i \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi, \quad (8.150b)$$

$$L_s = J^\mu A_\mu + \bar{\psi} \eta + \bar{\eta} \psi, \quad (8.150c)$$

$$L_{gf} = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2. \quad (8.150d)$$

We consider now the renormalised version of the Lagrangian $L_{\text{eff}}$, where the renormalised covariant derivative $D_\mu$ is given by Eq. (8.90) with $Z_1 \neq Z_2$ in general. This implies that an infinitesimal gauge transformation has the form

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda \quad (8.151a)$$

$$\psi \rightarrow \psi' = \psi - i \bar{\epsilon} \Lambda \psi \quad \text{with} \quad \bar{\epsilon} \equiv \frac{Z_1}{Z_2} e. \quad (8.151b)$$

As $L_{\text{cl}}$ is gauge invariant by construction, the variation of $L_{\text{eff}}$ under an infinitesimal gauge transformation consists only of

$$\delta \int d^4x \left( L_{gf} + L_s \right) = \int d^4x \left[ -\frac{1}{\xi} (\partial_\mu A^\mu) \Box \Lambda + J^\mu \partial_\mu \Lambda + i \bar{\epsilon} \Lambda (\bar{\psi} \eta - \bar{\eta} \psi) \right]. \quad (8.152)$$

Now we integrate partially the first term twice and the second term once, to factor out the arbitrary function $\Lambda$,

$$\delta \int d^4x L_{\text{eff}} = \int d^4x \left[ -\frac{1}{\xi} (\partial_\mu A^\mu) - \partial_\mu J^\mu + i \bar{\epsilon} \Lambda (\bar{\psi} \eta - \bar{\eta} \psi) \right] \Lambda. \quad (8.153)$$

Thus the variation of the generating functional $Z[J^\mu, \eta, \bar{\eta}]$ is

$$\delta Z[J^\mu, \eta, \bar{\eta}] = \int \mathcal{D} A \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp \left( i \int d^4x L_{\text{eff}} \right) i \delta \int d^4x L_{\text{eff}}. \quad (8.154)$$

The fields $A_\mu$, $\bar{\psi}$ and $\psi$ are however only integration variables in the generating functional. The gauge transformation (8.151) is thus merely a change of variables which cannot affect the functional $Z[J^\mu, \eta, \bar{\eta}]$. Thus its variation has to vanish, $\delta Z[J^\mu, \eta, \bar{\eta}] = 0$.

If we substitute fields by functional derivatives of their sources, the part from Eq. (8.153) can be moved outside the functional integral, leading to

$$0 = \left[ -\frac{1}{\xi} \Box \frac{\delta}{\delta J_\mu} - \partial_\mu J^\mu + i \bar{\epsilon} \left( \frac{\delta}{\delta \eta} \psi - \bar{\eta} \frac{\delta}{\delta \bar{\eta}} \psi \right) \right] \exp \{ i W \}
\quad = -\frac{1}{\xi} \Box \left( \partial_\mu \frac{\delta W}{\delta J_\mu} \right) - \partial_\mu J^\mu + i \bar{\epsilon} \left( \frac{\delta W}{\delta \eta} - \bar{\eta} \frac{\delta W}{\delta \bar{\eta}} \right). \quad (8.155)$$

Differentiating this equation with respect to $J_\nu(y)$ and setting then the sources $J^\mu$, $\eta$ and $\bar{\eta}$ to zero gives us our first result,

$$-\frac{1}{\xi} \Box \left( \partial_\mu \frac{\delta^2 W}{\delta J_\mu(x) \delta J_\nu(y)} \right) = \partial_\mu g^{\mu\nu} \delta(x - y). \quad (8.156)$$
The second derivative of $W$ w.r.t. to the vector sources $J^\mu$ is the full photon propagator $D_{\mu\nu}(x-y)$. If we go to momentum space, we have

$$\frac{i}{\xi}k^2k^\mu D_{\mu\nu}(k) = k^\nu. \quad (8.157)$$

Splitting the propagator into a transverse part and a longitudinal part as in (7.64), the transverse part immediately drops out and we find

$$\frac{i}{\xi}k^2k^\mu D_L(k^2) = k^\nu. \quad (8.158)$$

Thus the longitudinal part of the exact propagator agrees with the longitudinal part of the tree level propagator,

$$iD_L(k^2) = iD^L(k^2) = \xi k^2. \quad (8.159)$$

This implies that higher order corrections do not affect the longitudinal part of the photon propagator. Since we can expand all relations as power series in the coupling constant $e$, this holds also at any order in perturbation theory.

Let us go back to the constraint for the variation of the generating functional $Z$ under gauge transformations, Eq. (8.155). We aim to derive identities between 1PI Green functions and want therefore to transform it into a constraint for the effective action $\Gamma$. If we Legendre transform $W[J, \bar{\eta}, \eta]$ into $\Gamma[A, \bar{\psi}, \psi]$, we can replace the functional derivatives of $W$ with classical fields, and the sources with functional derivatives of $\Gamma$, i.e.

$$\frac{\delta W}{\delta J_\mu} = A_\mu, \quad \frac{\delta W}{\delta \eta} = \bar{\psi}, \quad \frac{\delta W}{\delta \bar{\eta}} = \psi$$

$$\frac{\delta \Gamma}{\delta A_\mu} = -J^\mu, \quad \frac{\delta \Gamma}{\delta \bar{\psi}} = -\eta, \quad \frac{\delta \Gamma}{\delta \psi} = -\bar{\eta}$$

with all the fields classical. This transforms Eq. (8.155) into

$$\frac{1}{\xi} \Box (\partial_\mu A^\mu(x)) - \partial_\mu \frac{\delta \Gamma}{\delta A_\mu(x)} + i\bar{\psi} \left( \frac{\delta \Gamma}{\delta \psi(x)} - \frac{\delta \Gamma}{\delta \bar{\psi}(x)} \right) = 0, \quad (8.161)$$

a master equation from which we can derive relations between different types of Green function. Differentiating with respect to $\psi(x_1)$ and $\bar{\psi}(x_2)$ and setting then the fields to zero gives us the most important one of the Ward-Takahashi identities, relating the 1PI 3-point function $\Gamma^{(3)}(x, x_1, x_2)$ to the 2-point function $\Gamma^{(2)}(x_1, x_2)$ of fermions,

$$\partial_\mu \Gamma^{(3)}_{\mu}(x, x_1, x_2) = i\bar{\psi}(\Gamma^{(2)}(x, x_1)\delta(x-x_2) - \Gamma^{(2)}(x_2, x)\delta(x-x_1)) \quad (8.162)$$

or, after Fourier transforming

$$k^\mu \Gamma^{(3)}_{\mu}(p, k, p+k) = \bar{\psi}S_F^{-1}(p+k) - \bar{\psi}eS_F^{-1}(p). \quad (8.163)$$

Taking the limit $k^\mu \to 0$, the usual Ward identity $\Gamma^{(3)}_{\mu}(p, 0, p) = e\partial_\mu S_F^{-1}(p)$ follows.

The Green functions in this equation are finite, renormalised quantities and thus $Z_1/Z_2$ has to be finite in any consistent renormalisation scheme too. In any scheme where we identify directly $e(0)$ with the measured electric charge, also the finite parts of $Z_1$ and $Z_2$ agree, i.e. the measured electric charge is universal.
8.6. Renormalisation and critical phenomena

Overview  The behaviour thermodynamical systems exhibit close to the critical points in their phase diagram is called “critical phenomena.” For a fixed number of particles, we can characterise thermodynamical systems using the free energy $F = U - TS$. Ehrenfest introduced the classification of phase transitions according to the order of the first discontinuous derivative of $F$ with respect to any thermodynamical variable $\phi$. Hence a phase transition where at least one derivative $\partial^n F/\partial \phi^n$ is discontinuous while all $\partial^{n-1} F/\partial \phi^{n-1}$ are continuous is called a $n$th order phase transition.

Critical phenomena are for a particle physicist interesting by at least three reasons:

- We can learn about symmetry breaking: We should look out for ideas how we can generate mass terms without violating gauge invariance. Systems like ferromagnets show that symmetries as rotation symmetry can be broken at low energies although the Hamiltonian governing the interactions is rotation symmetric. Another example is a plasma: Here the screening of electric charges modifies the Coulomb potential to a Yukawa potential; the photon has three massive degrees of freedom, still satisfying gauge invariance, $k_\mu \Pi^{\mu\nu}(\omega, k) = 0$, but with $\omega^2 - k^2 \neq 0$.

- Experimentally one finds that close a critical point, $T \to T_c$, the correlation length $\xi$ diverges, while otherwise correlations are exponentially suppressed. Thus the 2-point function of a certain order parameter $\phi$ scales as
  \[
  \langle \phi(x)\phi(0) \rangle \propto \exp(-|x|/\xi).
  \]
  Comparing this to the 2-point function of an Euclidean scalar field $\phi$,
  \[
  \langle \phi(x)\phi(0) \rangle \to \frac{4\pi}{|x|^2} \exp(-m|x|)
  \]
  in the limit $m|x| \gg 1$, we find the correspondence $\xi \propto m^{-1}$. Considering a statistical system on a lattice with spacing $a$, i.e. $m|x| = na/\xi$, we see that the continuum limit $a \to 0$ corresponds to $\xi \to \infty$ for finite $m$.
  Thus the correlation functions of a statistical system correspond for non-zero $a$ to bare Green functions and a finite value of the regulator of the corresponding quantum field theory. The connection to renormalised Green functions ($a \to 0$ or $\Lambda \to \infty$) is only possible when the statistical system is at a critical point.

- Near a critical point, $T \to T_c$, thermodynamical systems show a universal behaviour. More precisely, they fall in different universality classes which unify systems with very different microscopic behaviour. The various universality classes can be characterised by critical exponents, i.e. by the exponents $\gamma_i$ with which characteristic thermodynamical quantities $X_i$ diverge approaching $T_c$, i.e. $X_i = [b(T - T_c)]^{-\gamma_i}$.
  This phenomenon is similar to our realisation that e.g. two $\lambda \phi^4$ theories, one with $\lambda = 0.1$ and another one with $\lambda = 0.2$, are not fundamentally different but connected by a RGE transformation.

Landau’s mean field theory  Landau suggested that close to a second-order phase transition the free energy can be expanded as an even series in the order parameter. Considering e.g.
the magnetisation \(M\), we can write for zero external field \(H\) the free energy as

\[
F = A(T) + B(T)M^2 + C(T)M^4 + \ldots
\]  

(8.164)

We can find the possible value of the magnetisation \(M\) by solving

\[
0 = \frac{\partial F}{\partial M} = 2B(T)M + 4C(T)M^3.
\]  

(8.165)

The variable \(C(T)\) has to be positive in order that \(F\) is bounded from below. If also \(B(T)\) is positive, only the trivial solution \(M = 0\) exists. If however \(B(T)\) is negative, two solutions with non-zero magnetisation appear. Let us use a linear approximation, \(B(T) \approx b(T - T_c)\), and \(C(T) \approx c\) valid close to \(T_c\). Then

\[
M = \begin{cases} 
0 & \text{for } T > T_c, \\
\pm \left[ \frac{b}{4\pi} (T_c - T) \right]^{1/2} & \text{for } T < T_c.
\end{cases}
\]  

(8.166)

Note also that the ground-state breaks the \(M \to -M\) symmetry of the free energy for \(T < T_c\).

Representing the thermodynamical quantity \(M\) as integral of the local spin density,

\[
M = \int d^3 x \, s(x),
\]  

(8.167)

we can rewrite the free energy in a way resembling the Hamiltonian of a stationary scalar field,

\[
F = \int d^3 x \left[ (\nabla s)^2 + b(T - T_c)s^2 + cs^4 - H \cdot s \right].
\]  

(8.168)

Here, \((\nabla s)^2\) is the simplest ansatz leading to an alignment of spins in the continuous language. Minimising \(F\) will give us the ground-state of the system for a prescribed external field \(H(x)\) and temperature \(T\). For small \(s\), we can ignore the \(s^4\) term. The spin correlation function \(\langle s(x)s(0) \rangle\) is found as response to a delta function-like disturbance \(H_0 \delta(x)\) as

\[
\langle s(x)s(0) \rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{H_0 e^{i k \cdot x}}{k^2 + b(T - T_c)}
\]  

(8.169)

as it follows immediately by analogy with the Yukawa potential, \(m^2 \to b(T - T_c)\). Thus the correlation function is

\[
\langle s(x)s(0) \rangle = \frac{H_0}{4\pi r} e^{-r/\xi}
\]  

(8.170)

with

\[
\xi = \left[ b(T - T_c) \right]^{-1/2}.
\]  

(8.171)

Hence Landau’s theory reproduces the experimentally observed behaviour \(\xi \to \infty\) for \(T \to T_c\). Moreover, the theory predicts as critical exponent \(1/2\). Notice that the value of the exponent depends only on the polynomial assumed in the free energy, not on the underlying micro-physics. Thus another prediction of Landau’s theory is an universal behaviour of thermodynamical systems close to their critical points in the dependence on \(T - T_c\).

Experiments show that this prediction is too strong: Thermodynamical systems fall into different universality classes, and we should try to include some micro-physics into the description of critical phenomena.
8.6. Renormalisation and critical phenomena

Kadanoff’s block spin transformation Close to a critical point, collective effects play a decisive role even in case of short-range interactions. In d dimension, a particle is coupled by collective effects to \((\xi/a)^d\) particles and standard perturbative methods will certainly fail for \(\xi \to \infty\).

Kadanoff suggested to remove the short-wave length fluctuations by the following procedure:

Each step of a block spin transformation consists of i) dividing the lattice into cells of size \((2a)^d\), ii) assigning a common spin variable to the cell, iii) rescaling \(2a \to a\).

At each step, the number of strongly correlated spins is reduced. After \(n\) transformations, the correlation length decreases as \(\xi_n = \xi/(2^n)\). When the correlation length becomes of the order of the lattice spacing, collective effects play no role: All the physics can be read off from the Hamiltonian. If the procedure is not trivial, this implies that in each step the Hamiltonian changes. In particular, the coupling constant \(K\) is changed as

\[
K_2 = f(K), \quad K_3 = f(K_2) = f(f(K)), \ldots \tag{8.172}
\]

One-dimensional Ising model We illustrate the idea behind Kadanoff’s block spin transformation using the example of the one-dimensional Ising model. This model consists of spins with value \(s_i = \pm 1\) on a line with spacing \(a\), interacting via nearest neighbour interactions.

We consider only the piece of six spins shown in Fig. 8.6. The corresponding partition function is

\[
Z_6 = \sum_{s_{N-1},s_0,s_1,...,s_4} \exp \left[ K(s_{N-1}s_0 + s_0s_1 + \ldots s_3s_4) \right] \tag{8.173}
\]

\[
= \sum_{s_0',s_1',s_2'} \sum_{s_{N-1},s_1,s_3} \exp \left[ K(s_{N-1}s_0' + s_0's_1 + \ldots s_3s_4) \right]. \tag{8.174}
\]

The step \(a \rightarrow 2a\) requires to perform the sums over the unprimed spins. Expanding the exponentials using \((s_is_j)^{2n} = 1\) gives terms like

\[
\exp[K(s_0's_1)] = 1 + Ks_0's_1 + \frac{K^2}{2!} + \frac{K^3}{3!} s_0's_1 + \ldots \tag{8.175}
\]

\[
= \cosh(K) + s_0's_1 \sinh(K) \tag{8.176}
\]

\[
= \cosh(K)[1 + s_0's_1 \tanh(K)]. \tag{8.177}
\]

Figure 8.6.: One block spin transformation \(a \rightarrow 2a\) for an one-dimensional lattice model.
8. Renormalisation

The terms linear in $s_1$ cancel in the sum and we obtain

$$\sum_{s_1} \exp[K(s'_0 s_1)] \exp[K(s_1 s'_1)] = 2 \cosh^2(K)[1 + \tanh^2(K) s'_0 s'_1].$$  \hspace{1cm} (8.178)

Thus the summation over the unprimed spins changes the strength of the nearest neighbourhood interaction and generates additionally a new spin-independent interaction term. We try now to rewrite the last expression in a form similar to the original one,

$$2 \cosh^2(K)[1 + \tanh^2(K) s'_0 s'_1] = \exp[g(K) + K' s'_0 s'_1)].$$  \hspace{1cm} (8.179)

Using (8.177) to replace $\exp(K' s'_0 s'_1)$, we find

$$\tanh(K') = \tanh^2(K).$$ \hspace{1cm} (8.180)

This determines the function $g$ as

$$g(K) = \ln \left( \frac{2 \cosh^2 K}{\cosh K'} \right).$$ \hspace{1cm} (8.181)

The summation over the other spins $s_3, s_5, \ldots$ can be performed in the same way. Thus the partition function on a lattice of size $2a$ has the same nearest-neighbour interactions with a new coupling $K' \equiv K_1$ determined by (8.180). Iterating this procedure generates a renormalisation flow with

$$\tanh(K_n) = \tanh^{2^n}(K).$$  \hspace{1cm} (8.182)

**Fixed point behaviour** In general, we will not be able to calculate the transformation function $f(K)$. But even without the knowledge of $f(K)$, we can draw some important insight from general considerations. First, the exact RGE equation\footnote{A brief derivation for the curious is given in Section ??} is of the type of a heat or diffusion equation,

$$\partial_t X = \nabla^2 X,$$

on the space of functionals $X = \exp(-S)$. Its flow is therefore a gradient flow (“Fick’s law”) which has only two possible asymptotics: a runaway solution to infinity and the approach to a fixed point defined by $K_c = f(K_c)$.

With $\xi_{n+1} = \xi_n / 2$ and labeling the $n$ dependence implicitly vis $\xi_n = f(K_n)$ we can write

$$\xi(f(K)) = \frac{1}{2} \xi(K).$$ \hspace{1cm} (8.183)

At a fixed point $K_c = f(K_c)$ only two solutions exist,

$$\xi(K_c) = 0 \hspace{1cm} \text{and} \hspace{1cm} \xi(K_c) \to \infty.$$ \hspace{1cm} (8.184)

The second possibility corresponds to the approach of a critical point, allowing the limit $a \to 0$ and thus the continuum limit necessary for the transition to a QFT. This point is called a critical fixed point, while the fixed point with zero correlation length is called trivial.

We can now easily generalise our previous discussion of the fixed point behaviour for the beta function from one to $n$-dimensions. The general behaviour of the RGE flow can be
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Figure 8.7.: A two-dimensional illustration of the RGE flow: A, B and C are three fixed points on the critical surface $\xi = \infty$. The fixed points A and C have a stable direction along the critical surface, while B has two unstable directions. The trivial fixed point D is a stable fixed point, attracting all points starting not on the critical surface.

understood from the two-dimensional example in Fig. 8.7. The dashed lines show surfaces of constant correlation length, including a critical surface $\xi = \infty$. Also shown are three critical fixed points (A, B, C) and a trivial one (D). We remember that in each RGE step the correlation length decreases. Thus the trivial fixpoint is an attractor, i.e. inside a small enough neighbourhood all points will flow towards it. On the other hand, the critical line has at least one unstable direction, the one orthogonal to its surface: Even points infinitesimal close to the surface will flow away and eventually end in a trivial fixpoint. Moreover, we see that also inside the critical surface stable and unstable directions exist: The fixpoint B will attract all points in-between A and C (“its basin of attraction”).

We can identify universality classes of QFTs with stable critical fixpoints and their basin of attraction.

**Effective action, RGE flow and irrelevant operators** The RGE flow generates all kind of couplings compatible with the symmetries of the fundamental Hamiltonian. The task of understanding the RGE flow in an infinite-dimensional space of couplings is simplified by the following observation: Relevant and marginal interactions are typically already included in our starting Hamiltonian. For instance, in the case of the $\lambda \phi^4$ theory in $d = 4$, we include a constant term $\rho_\Lambda$ ($d = 0$, leading to a cosmological constant $\rho_\Lambda$), the mass term $m^2 \phi^2$ with $d = 2$, and the interaction $\lambda \phi^4$. Hence, the RGE flow will renormalise the values $\rho_\Lambda$, $m^2$ and $\lambda$ and introduce an infinite set of irrelevant operator $O_{d,i}$ with dimension $d \geq 6$.

It is now time to explain the reason why the non-renormalisable operators are called irrelevant in this context. Let us start from the Euclidean generating functional restricted to

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5If we do not, the normalisation constant of the path integral will do the job.
8. Renormalisation

wave-numbers below a cutoff, \( k \leq \Lambda \),

\[
Z[J] = \int \mathcal{D}\phi_\Lambda \exp \left( - \int d^4 x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right) \right),
\]

(8.185)

where \( \mathcal{D}\phi_\Lambda = \mathcal{D}\phi_{|k|\leq\Lambda} \). Now we want to integrate out the fields with wave-numbers between \( s\Lambda \leq k \leq \Lambda \) from the generating functional. We express \( \phi(x) \) as Fourier modes, obtaining e.g.

\[
S_{\text{int}} = \frac{1}{24} \int \frac{d^4 k_1}{(2\pi)^4} \cdots \frac{d^4 k_4}{(2\pi)^4} \delta(k_1 + \ldots + k_4) \phi(k_1) \cdots \phi(k_4).
\]

(8.186)

Then we introduce the field \( \tilde{\phi}(k) \) which coincides with the original field \( \phi \) in the range we want to integrate out and vanishes otherwise,

\[
\tilde{\phi}(k) = \begin{cases} 
\phi(k) & \text{for } s\Lambda \leq |k| \leq \Lambda, \\
0 & \text{otherwise}.
\end{cases}
\]

(8.187)

We set now \( \phi = \tilde{\phi} + \phi \) in Eq. (8.185) and call afterwards the integration variable \( \tilde{\phi} \) again \( \phi \). Then

\[
Z[J] = \int \mathcal{D}\phi_\Lambda e^{-S[\phi]} \mathcal{D}\tilde{\phi} \exp \left( - \int d^4 x \left( \frac{1}{2} \partial_\mu \phi + \partial_\mu \tilde{\phi} \right)^2 + \frac{1}{2} m^2 (\phi + \tilde{\phi})^2 + \frac{\lambda}{4!} (\phi + \tilde{\phi})^4 \right)
\]

\[
= \int \mathcal{D}\phi_\Lambda e^{-S[\phi]} e^{-S_{\text{int}}[\phi]} \mathcal{D}\tilde{\phi} \exp \left( - \int d^4 x \left( \frac{1}{2} \partial_\mu \tilde{\phi} \right)^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{\lambda}{6} \tilde{\phi}^3 + \frac{\lambda}{4} \tilde{\phi}^2 \tilde{\phi} + \frac{\lambda}{6} \phi \tilde{\phi}^3 + \frac{\lambda}{24} \phi^4 \right)
\]

\[
= \int \mathcal{D}\phi_\Lambda e^{-S_{\text{int}}[\phi]},
\]

(8.188)

where we used that the mixed quadratic terms vanish (being orthogonal). To proceed, we would have to expand the exponential and to evaluate it perturbatively. Apart from generating terms which renormalises the original parameters, we would obtain an infinite number of higher-dimension operators \( O_{d,i} \) with dimension \( d \geq 6 \). For instance \( O_{2n,1} = \phi^{2n} \), \( O_{2n,2} = \phi^{2n-1} \Box \phi \), etc. Thus the effective Lagrangian including fluctuations up to \( \Lambda_1 \equiv s\Lambda \) can be written as

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + \sum_{d \geq 6} \sum_i C_{d,i}(\Lambda_1) O_{d,i}.
\]

(8.189)

The coefficients \( C_{2n,1} \) are given by a loop integral over \( n \) propagators,

\[
C_{2n,1} \propto \lambda^n \int_{\Lambda_1} \frac{d^4 k}{(2\pi)^4} \left( \frac{1}{k^2 + m^2} \right)^n \propto \lambda^n \left( \frac{1}{\Lambda_1^{2n-4}} - \frac{1}{\Lambda_1^{2n-4}} \right).
\]

(8.190)

We see now that relevant operators are determined by the high-energy cutoff, marginal operators \( \propto \ln(\Lambda/\Lambda_1) \) are influenced in the same way by each decade in \( k \), while irrelevant operators are determined by the low-energy cutoff.

As a side remark, we note that both directions of the RGE flow—towards the UV or the IR—are useful discussing QFTs. The point of view of a RGE flow towards the IR is useful, if we want to connect a theory at high-scales to a simpler theory at lower energy scales. An example for this approach is chiral perturbation theory where one connects QCD to an effective theory of mesons and baryons at low energies. In the opposite view, we may look...
e.g. at the SM as an effective theory known to be valid up to scales around TeV and ask what happens if we increase the cutoff.

You may have noticed that we have rescaled the effective Lagrangian in Eq. (8.189) such that the kinetic term maintained its canonical normalisation. According to step iii) in the Kadanoff-Wilson prescription, we have to rescale \( Z \) after each step \( a \to f a \) (or \( \Lambda \to \Lambda / f \)). If we rescale distances by \( x \to x' = x / f \), the functional integral is again over modes \( \phi(x') \) with \( x > a \). Keeping the kinetic term invariant,

\[
\int d^4x (\partial_\mu \phi)^2 = \int d^4x' (\partial'_\mu \phi')^2 = \frac{1}{f^2} \int d^4x (\partial_\mu \phi')^2
\]

(8.191)

requires thus a rescaling of the field as \( \phi' = f \phi \). Let us consider now an irrelevant interaction, e.g. \( g_6 \phi^6 \). Then

\[
g_6 \int d^4x \phi^6 = \frac{g_6}{f^2} \int d^4x' \phi'^6
\]

(8.192)

shows that the new coupling \( g'_6 \) is rescaled as \( g'_6 = g_6 / f^2 \): As \( f \) grows and the cutoff scale \( \Lambda \) decreases, the value of an irrelevant coupling is driven to zero. Clearly, a relevant operator as the cosmological constant \( \rho \) or a mass term \( m^2 \phi^2 \) shows the opposite behaviour and grows. As result, irrelevant couplings are in our low-energy world suppressed and as first approximation a renormalisable theory emerges at low energies.

We can generalise now our earlier discussion of the two-dimensional RGE flow, Fig. 8.7. The RGE flow stops at fixed points on critical surfaces of low dimension. All initial values on the critical surface (or basin of attraction) flow to the critical fixed point. Directions perpendicular to the critical surface are controlled by the irrelevant interactions; flows beginning off the surface are driven to the trivial fixed point. On the way towards \( \xi \to \infty \) only the relevant and marginal interactions survive. Insisting not on \( \xi \to \infty \) and keeping a finite cutoff (somewhere between TeV and \( M_{\text{Pl}} \), depending on the limit of validity of the theory we assume) we can keep irrelevant interaction but may require some fine-tuning of their parameters.

**Summary of chapter**

Using a power counting argument for the asymptotic behaviour of the free Green functions, we singled out theories with dimensionless coupling constants: Such theories with marginal interactions are renormalisable, i.e. are theories with a finite number of primitive divergent diagrams. Consequently, the multiplicative renormalisation of the finite number of parameters contained in the classical (effective) Lagrangian is sufficient to obtain finite Green functions at any order perturbation theory.

The scale dependence of renormalised Green functions can be interpreted as a running of coupling constants and masses. The use of a running coupling sums up the leading logarithms of type \( \ln^n(\mu^2 / \mu_0^2) \), and a suitable choice of the renormalisation scale in a specific problems reduces the remaining scale dependence of any perturbative result.

Different interactions can be characterised by the asymptotic behavior of their coupling constants. Gauge theories with a sufficiently small number of fermions are the only renormalisable interactions which are asymptotically free, i.e. their running coupling constant goes to zero for \( \mu \to \infty \).

The non-perturbative approach of Wilson provides an argument why the SM as description of our low-energy world is renormalisable: Integrating out high-energy degrees of freedom,
irrelevant couplings are driven to zero and thus it is natural that a renormalisable theory emerges at low energies.

Further reading
Our discussion of the renormalisation of non-abelian gauge theories left out most details. For instance, we did not introduce the non-abelian analogue of the Ward-Takahshi identities which are easiest derived using the formalism of the BRST symmetry. I recommend those interested to fill the gaps to start with the books of Ramond and Pokorski. Banks gives a extremely concise and lucid discussion of renormalisation.

Problems

8.1 $g_s$-factor of gauge bosons.
Derive the $g_s$-factor of Yang-Mills gauge bosons from the non-abelian Maxwell equations.

8.2 Effective vertex for $\mu \rightarrow e\gamma$.
Derive the effective vertex $\Gamma^\mu$ for the transition $\mu \rightarrow e\gamma$ where all three particles are on-shell and the process violates parity. Use current conservation and that the photon is on-shell to show that $\Gamma^\mu = i q^\nu \sigma_{\mu\nu} [A(q^2) + B(q^2)\gamma^5]$, where $q^\nu$ is four-momentum of the photon.

8.3 Comparison of cutoff and DR.
Recalculate the three basic primitive diagrams of a scalar $\lambda\phi^4$ theory using as regularisation a cutoff $\Lambda$. Find the correspondence between the coefficients of poles in DR and divergent terms in $\Lambda$.

8.4 Primitive divergent diagrams of scalar QED.
Find the basic primitive diagrams of scalar QED
\[ \mathcal{L} = \frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{2}m^2\phi^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \]
and their superficial degree of divergence.

8.5 $\beta$ function of the $\lambda\phi^4$ theory.
The $\beta$ function determines the logarithmic change of the coupling constants. a) Show that the $\beta$ function can be written in DR as $\beta(\lambda) = -\varepsilon\lambda - \frac{\mu}{Z}\frac{dZ}{d\mu}\lambda$ with $\tilde{Z}^{-1} = Z^{-1}_\lambda Z^2_0$. b) Show that $Z^{-1}_\lambda = 1 - 3\lambda/(16\pi^2\varepsilon)$ in one loop approximation and find the $\beta$-function. c) Up to which order is the $\beta$ function scheme independent? d) Solve the differential equation for $\lambda(\mu)$.

8.6 Anomalous dimension.

8.7 General solution of RGE equation.
Find the general solution of the RGE equation (8.82) using the method of characteristics or the following analogue: Coleman suggested to consider the growth rate $g(x)$ of bacteria in an one-dimensional flow as analogue to the RGE.
a.) Show that the density $\rho(x,t)$ of bacteria satisfies
\[ \left( \frac{\partial}{\partial t} + v(x) \frac{\partial}{\partial x} - g(x) \right) \rho(x,t) = 0 \]
b.) The position of a fluid element is described by $\bar{x} = \bar{x}(x,t)$ with the initial condition $\bar{x}(x,0) = x$. Then $\bar{x}(x,t)$ satisfies
\[ \frac{d}{dt}\bar{x}(x,t) = v(x) \]
Show that for $\rho(x,0) = \rho_0(x)$ at later time $\rho(x,t)$ is given by
\[ \rho(x,t) = \rho_0(\bar{x}(x,t)) \exp \left( \int_0^t dt' g(\bar{x}(x,t')) \right) \]

8.8 Imaginary part of the photon polarisation tensor.
Derive the imaginary part of the photon polarisation tensor at one loop and show that it equals
the pair creation cross probability of a photon with virtuality $q^2$.

8.9 A toy model for the effective action approach.

Consider

$$Z = \int \text{d}x \text{d}y \exp\left[-(x^2 + y^2 + \lambda x^4 + \lambda x^2 y^2)\right]$$

as a toy model for the generating functional of two coupled scalar fields. Integrate out the field $y$, assume then that $\lambda$ is small: Expand first the result and then rewrite it as an exponential. Show that this process results in a.) a renormalisation of the mass term $x^2$, b.) a renormalisation of the coupling term $x^4$, and c.) the appearance of new (“irrelevant”) interactions $x^n$ with $n \geq 6$. 
9. Thermal field theory

9.1. Overview

Introduction  The Green functions we have considered so far were all defined as expectation value of products of fields in a pure state, the vacuum in the absence of real particles, $|0\rangle$. Out of these Green functions, we could build up our quantities of prime interest, decay or scattering amplitudes for $1\rightarrow n$ and $2\rightarrow n$ particles via the LSZ formalism. In this chapter, we discuss how Green functions should be calculated, if the vacuum is non-empty and described by a density matrix $\rho$. Examples for such systems are the early Universe or the dense, hot interior of a star. The simplest and at the same time most important cases of thermal systems are those in equilibrium.

In equilibrium statistical physics, the partition function is of central importance as all other thermodynamic quantities can be calculated from it. In the grand canonical ensemble (where both particles and energy may be exchanged between system and reservoir) it takes the form,

$$Z(V,T,\mu_1,\mu_2,\ldots) = \sum_n \langle n | e^{-\beta H - \mu_i N_i} | n \rangle = e^{-\beta \Omega}, \quad (9.1)$$

where $\beta \equiv 1/T$ is the inverse temperature of the system and $n$ denotes a complete set of quantum numbers. Finally, the Landau or grand canonical free energy

$$\Omega(U,S,N_i) = -T \ln Z = U - TS - \mu_i N_i = PV \quad (9.2)$$

connects the microscopic partition function to thermodynamics. We already saw that the partition function is closely related to the corresponding field theory in Euclidean space, while we can derive from $\Omega$ all relevant thermodynamical quantities: For instance, we obtain the pressure $P$ from $\Omega$ as $P = \partial \Omega/\partial V|_{T,\mu}$. In addition the expectation value of any observable, $O$, is given (setting in the following always $\mu_i = 0$) as

$$\langle O \rangle = Z^{-1} \text{Tr} \left[ O e^{-\beta H} \right]. \quad (9.3)$$

Calculational approaches  Two main approaches to calculations are used in thermal field theory:

- In the real-time formalism one applies the formula (9.3) valid for any observable directly to Green functions, that is

$$G(x_1,\ldots,x_n) = \langle T(\phi(x_1)\cdots\phi(x_n)) \rangle \quad (9.4)$$

$$= Z^{-1} \text{Tr} \left[ e^{-\beta H} T(\phi(x_1)\cdots\phi(x_n)) \right]. \quad (9.5)$$

The main advantage of this approach is that it can be extended to the non-equilibrium case. In particular, one can investigate the time evolution of a system towards thermal
equilibrium. As disadvantage, we note that the proper definition of the contours for the propagators becomes more complicated and one has to introduce therefore matrix-valued $2 \times 2$ propagators.

- In the imaginary time formalism, we rotate from Minkowski to Euclidean space, $t \rightarrow -i t$, so that the transition amplitude from an initial state, $|q(t_i)\rangle$, to a final state, $|q(t_f)\rangle$, is given by

$$
\langle q(t_f)| e^{-(t_f-t_i)H} | q(t_i)\rangle = \int_{q(t_i)}^{q(t_f)} Dq e^{-S},
$$

where $S$ is now the Euclidean action. If we set the evolution time, $t_f - t_i$, equal to the inverse temperature $\beta$ and integrate over all periodic paths $q(t_i) = q(t_f + \beta)$, we have

$$
\sum_q \langle q| e^{-\beta H} | q\rangle = \int_{q(t)}^{q(t+\beta)} Dq e^{-S} = Z.
$$

We now see that we have formally connected the path integral formulation of quantum mechanics (in Euclidean space) to the partition function of statistical mechanics.

In contrast to $T = 0$ field theory, Green functions and the partition function in the Euclidean are not not merely a mathematical tool but our main objects of interest. Since the Euclidean Green functions do depend on temperature instead of time, we are not able to describe time-dependent phenomena in this approach.

**Thermal Green functions** The trace in the partition function of statistical physics implies that we have to sum over configurations connecting the same physical state at $t$ and $t + \beta$. In the path integral corresponding to statistical mechanics, the periodicity condition $q(t) = q(t + \beta)$ for the real coordinate $q$ is clearly the only possible choice. In contrast, fields may only be observable through bilinear quantities, as e.g. $\bar{\psi}\Gamma\psi$ for a fermion field. This raises the question, if we should require periodic or anti-periodic boundary conditions for fermionic fields.

We consider first thermal Green functions $G_\beta$ for a free scalar field. From the Heisenberg equation for the field operator,

$$
\phi(t, x) = e^{iHt}\phi(0, x)e^{-iHt},
$$

we find

$$
G_\beta^+(t', x'; t, x) = \text{tr} \left[ e^{-\beta H}\phi(t', x')\phi(t, x) \right] / Z
$$

$$
= \text{tr} \left[ e^{-\beta H}\phi(t', x')e^{\beta H}e^{-\beta H}\phi(t, x) \right] / Z
$$

$$
= \text{tr} \left[ \phi(t' + i\beta, x')e^{-\beta H}\phi(t, x) \right] / Z
$$

$$
= \text{tr} \left[ e^{-\beta H}\phi(t, x)\phi(t' + i\beta, x') \right] / Z = G_\beta(t' + i\beta, x'; t, x).
$$

Hence for bosonic fields,

$$
G_\beta^+(t', x'; t, x) = G_\beta^+(t' + i\beta, x'; t, x),
$$

we set $G(t) = G^+\vartheta(t) + G^-\vartheta(-t)$ for bosonic and $S(t) = S^+\vartheta(t) - S^-\vartheta(-t)$ for fermionic propagators.
implying periodic boundary condition for the thermal propagators,

\[ G_\beta(t', x'; t, x) = G_\beta(t' + i\beta, x'; t, x). \]  

(9.11)

The derivation (9.9) goes through unchanged for fermionic fields,

\[ S_\beta^\pm(t', x'; t, x) = S_\beta^\pm(t' + i\beta, x'; t, x). \]  

(9.12)

But now the anti-commuting nature of fermionic fields—more specifically the minus sign in (5.71)—leads to anti-periodic boundary condition for their thermal propagators,

\[ S_\beta(t', x'; t, x) = -S_\beta(t' + i\beta, x'; t, x). \]  

(9.13)

Both periodicity conditions discretise the frequency spectrum of the wave-functions. Thus the Fourier transform of thermal fields contained in a box of size \( V \times \beta \) is given by

\[ \phi(t, x) = \frac{1}{\sqrt{\beta V}} \sum_{n=\infty}^{\infty} \sum_{p} \phi_{n, p} e^{-i(\omega_n t + p \cdot x)} \]  

(9.14)

with \( \omega_n = 2n\pi T \) for bosonic and \( \omega_n = (2n + 1)\pi T \) for fermionic fields, respectively, and \( n \in \mathbb{Z} \). The frequencies \( \omega_n \) are called Matsubara frequencies. Similarly, the Green functions for non-interacting fields are given in the limit \( V \to \infty \) by

\[ G_\beta(t, x) = \frac{1}{\beta} \sum_{n=\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} G_n(\omega_n, p) e^{-i(\omega_n t + p \cdot x)} \]  

(9.15)

with

\[ G_n(\omega_n, p) = \frac{1}{\omega_n^2 + p^2 + m^2}. \]  

(9.16)

Thermal Green functions and vertices come without imaginary units, because we have transformed the path integral to Euclidean time.

### 9.2. Scalar gas

We will illustrate the basics of thermal field theory considering the simplest example, a gas of scalar particles.

#### 9.2.1. Free scalar gas

The free energy density \( \mathcal{F} \) of a free scalar field obtained performing the path integral in the free partition function \( Z \) is given by

\[ \beta \mathcal{F} = \frac{1}{2} \sum_{n=\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left[ \omega_n^2 + k^2 + m^2 \right]. \]  

(9.17)

In order to evaluate the sum, it is sufficient to consider

\[ A = T \sum_n \ln \left[ -(i\omega_n)^2 + E_k^2 \right] \]  

(9.18)
where we set also $E_k^2 = k^2 + m^2$. We eliminate first the logarithm by differentiating $A$ w.r.t. $E_k$,
\[ \frac{dA}{dE_k} = -2TE_k \sum_n \frac{1}{(i\omega_n)^2 - E_k^2}. \]  
(9.19)

The function $dA/dE_k$ has in the complex $\omega$ plane poles along the imaginary axis at $\omega = i\omega_n = 2\pi n T i$ plus two poles on the real axis at $\omega = \pm E_k \mp i\varepsilon$, cf. Fig. 9.11. We convert the sum into a contour integral using Cauchy’s theorem in “reverse order” to evaluate the residua of the two enclosed poles at the orientation of the integration path. Now we can use Cauchy’s theorem in the “normal order” to evaluate the residua of the two enclosed poles at $\pm E_k \mp i\varepsilon$,
\[ \frac{dA}{dE_k} = E_k \sum_{\pm} \text{res}_{\pm E_k} \left\{ \frac{\coth(\beta\omega/2)}{\omega^2 - E_k^2} \right\} = \cosh(\beta E_k/2) = 1 + \frac{2}{e^{\beta E_k} - 1}. \]  
(9.21)

Integration gives
\[ \mathcal{F} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ E_k + 2T \ln(1 - e^{-\beta E_k}) + c \right]. \]  
(9.22)

The integration constant $c$ cancels against the normalization constant of the path integral; dropping also the $T = 0$ vacuum part and taking the high-temperature limit $T \gg m$ gives
\[ \mathcal{F} = T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta E_k}) \approx \frac{T^4}{2\pi^2} \int_0^\infty dx x^2 \ln \left(1 - \exp\{-\sqrt{x^2 + (\beta m)^2}\}\right) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24}. \]  
(9.23)

**Example:** Derive the high-temperature expansion $T \gg m$ of the free energy $\mathcal{F}$ given in Eq. 9.23. We perform a Taylor expansion of $\ln(1 - \exp\{-[x^2 + (\beta m)^2]^{1/2}\})$ around $\beta m = 0$,
\[ \int_0^\infty dx x^2 \ln(1 - e^{-x}) = -\sum_{n=1}^\infty \int_0^\infty dx x^2 e^{-nx} = -2 \sum_{n=1}^\infty \frac{1}{n^2} = -2\zeta(4) = -2 \frac{\pi^4}{90}, \]
In the first integral we expand the logarithm,
\[ \int_0^\infty dx x^2 \ln(1 - e^{-x}) = -\sum_{n=1}^\infty \int_0^\infty dx x^2 e^{-nx} = -2 \sum_{n=1}^\infty \frac{1}{n^4} = -2\zeta(4) = -2 \frac{\pi^4}{90}, \]
while we use in the second integral
\[ \frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^\infty e^{-nx}. \]

With the definition (9.1) for the Gamma function, the second integral results in $\Gamma(2)\zeta(2) = \pi^2/6$. 

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9. Thermal field theory

\[ \text{Im}(\omega) \quad \text{Re}(\omega) \]

\[ -\omega + i\varepsilon \]

\[ \omega = i\varepsilon \]

\[ -\omega + i\varepsilon \]

\[ \omega = i\varepsilon \]

Figure 9.1.: Poles and contours in the complex \( \omega \) plane used for the evaluation of the free energy \( F \).

9.2.2. Interacting scalar gas

As main application of the formalism of thermal field theory we will calculate the first quantum correction to the equation of state of a gas of scalar particles interacting with a \( \lambda \phi^4 \) interaction. This \( O(\lambda) \) correction is given by a two-loop vacuum diagram and serves us also as demonstration how mixed UV divergences are cancelled by counter terms found at lower order. After that we discuss the IR behaviour of the massless a \( \lambda \phi^4 \) theory. We will find that the plasma generates a thermal mass for the scalar particle, an effect we will try to imitate generating masses for the particles of the SM.

**Equation of state of a scalar gas** We aim now at the first-quantum correction to the equation of state of gas of massive scalar particles interacting with a \( \lambda \phi^4 \) interaction. The corresponding Hamiltonian is given by

\[
H = \int d^3x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]
\]

(9.24)

\[
= H_0 + H_{\text{int}},
\]

(9.25)

where \( H_0 = \sum_k \omega_k (1/2 + a_k^\dagger a_k) \). We expand also the Landau free energy in \( \lambda \),

\[
\Omega = \Omega_0 + 4 \varepsilon \Omega_1 \lambda^2 \Omega_2 + \ldots
\]

(9.26)

The interaction term produces to first order in \( \lambda \) the following vacuum diagram,

\[
\frac{\lambda}{4!} \langle \phi^4 \rangle = \frac{3\lambda}{4!} \langle \phi^2 \rangle \langle \phi^2 \rangle = \frac{\lambda}{8} \left( \Delta(t=0, x=0) \right)^2,
\]

where the factor three has come from the possible ways of joining the lines to form the two loops seen in the diagram. (Alternatively, we can use that \( \langle \phi^n \rangle \) is just the expectation value
of $\phi^n$ times a Gaussian.) We could now insert into the formula the Green function derived in
the imaginary time formalism. Instead, we apply a more instructive argument following the
steps in Eqs. (2.62-2.63): At $T = 0$, the vacuum is empty and the $a_k^\dagger a_k$ term in $H_0$ gives zero contribution,
\begin{equation}
\Delta(t = 0, x = 0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k}.
\end{equation}
(9.27)
For $T > 0$, the expectation value of the number operator $N_k = a_k^\dagger a_k$ is just the number distribution $n_k$ of $\phi$ particles,
\begin{equation}
\Delta(t = 0, x = 0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left(1 + \langle a_k^\dagger a_k \rangle \right)
\end{equation}
(9.28)
\begin{equation}
= \int \frac{d^3k}{(2\pi)^3} \frac{1 + 2n_k}{2\omega_k}.
\end{equation}
(9.29)
In thermal equilibrium the number density of a scalar field follows Bose-Einstein statistics, that is
\begin{equation}
n_k = \frac{1}{e^{\beta \omega_k} - 1}.
\end{equation}
(9.30)
More formally, this result can be derived directly from the periodicity condition of the Green
functions (problem 9.2). Note that we can view the propagator as the sum of a vacuum part
(“1/2”) and a thermal part (”$n_k$”). In the latter, the high energy modes are exponentially
suppressed and thus no UV divergences should appear in the temperature dependent parts
of physical observables. Thus our standard renormalisation program at $T = 0$ should apply
in the same way at $T > 0$.
Continuing the derivation of the first order correction to $\Omega$ we have
\begin{equation}
\frac{\lambda}{4!} \langle \phi^4 \rangle = \frac{\lambda}{8} \left[ \sum_k \frac{1 + 2n_k}{2\omega_k} \right]^2
\end{equation}
(9.31)
\begin{equation}
= \frac{\lambda}{8} \left[ \left( \sum_k \frac{1}{2\omega_k} \right)^2 + \left( \sum_k \frac{n_k}{\omega_k} \right)^2 + 2 \left( \sum_k \frac{1}{2\omega_k} \right) \left( \sum_k \frac{n_k}{\omega_k} \right) \right].
\end{equation}
(9.32)
The mixed term gives rise to a temperature dependent UV divergence, which from the previous
argument should not appear in our calculation. Thus we expect that this term will be cancelled
by an one-loop counter term.
We consider therefore the self-energy $\Sigma$, which is given by
\begin{equation}
\Sigma = \frac{\lambda}{2} \Delta(t = 0, x = 0) = \frac{\lambda}{2} \sum_k \frac{1 + 2n_k}{2\omega_k},
\end{equation}
(9.33)
which contains the usual $T = 0$ divergence coming from the unsuppressed sum over frequencies. We use physical perturbation theory, adding an additional interaction term of the form
$\delta m^2 \phi^2$ to the Lagrangian, where $\delta m^2$ is chosen such that $m$ corresponds to the physical mass. Thus $\delta m^2$ is determined by
\begin{equation}
\delta m^2 + \frac{\lambda}{2} \sum_k \frac{1}{2\omega_k} = 0.
\end{equation}
(9.34)
Because the product $\delta m^2 \Delta(0)$ is of $O(\lambda)$, we have to add the following vacuum diagram to $\Omega_1$, 

$$
\frac{1}{2} \delta m^2 \Delta(0,0) = \left( \sum_k \frac{\lambda}{2 \omega_k} \left( 1 + \frac{2 n_k}{\omega_k} \right) \right).
$$

Comparing this expression to the troublesome mixed term, we see that they agree but have the opposite sign. Thus the one-loop subdivergence cancels the temperature dependent UV divergence at two-loop in $\Omega_1$. As result, we obtain the consistent expression

$$
\frac{\lambda}{4!} \langle \phi^4 \rangle = \frac{\lambda}{8} \left( \sum_k \frac{n_k}{\omega_k} \right)^2 + \text{vac},
$$

which we can now calculate explicitly. For simplicity, we restrict ourselves to the high-temperature limit,

$$
\sum_k \frac{n_k}{\omega_k} = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\omega} e^{i \omega - 1} = \frac{T^2}{12},
$$

where we have used that $m/T \to 0$ at high $T$. So our final result for $\Omega_1$ is

$$
\Omega_1 = \frac{\lambda}{8} \left( \frac{1}{12} \right)^2 = \frac{\lambda}{1152} T^4,
$$

which we may compare to the non-interacting result, $\Omega_0 = \pi^2 T^4 / 90$: The ratio $\Omega_1/\Omega_0 \approx 10^{-3} \lambda$ seems to indicate a fast convergence of the perturbative expansion of the pressure for any reasonable value of the coupling.

We simply quote the result to three loops or second order in $\lambda$ from the literature,

$$
P = \frac{\pi^2 T^4}{9} \left[ - \frac{1}{10} \frac{1}{8 \pi^2} + \frac{1}{18} \ln \frac{\mu^2}{4\pi T} + \frac{31}{35} + C \left( \frac{\lambda}{16 \pi^2} \right)^2 \right].
$$

The parameter $\mu$ in Eq. (9.39) is not the chemical potential but as usually the renormalisation scale. As the pressure is a physical quantity, it should not depend upon such a parameter, but the truncation used in the expansion causes this $\mu$ dependence, which then leads to a $\mu$ dependence at $O(\lambda^3)$. Examining this $\mu$ dependence of $P$ we have

$$
\frac{dP}{d\mu} = \frac{\pi^2 T^4}{9} \left[ - \frac{1}{10} \frac{1}{8 \pi^2} \frac{d\lambda}{d\mu} + \frac{3}{8} \left( \frac{(\lambda/16 \pi^2)}{\beta=3\lambda^2/16\pi^2} \right)^2 + O(\lambda^3) \right] = O(\lambda^3).
$$

Thus the dependence on the renormalisation scale $\mu$ is of higher order in $\lambda$ than the order of perturbation theory we are working with. Still we can try to minimise the remaining dependence by a suitable choice of $\mu$: Clearly, a sensible choice in this case is $\mu = 4\pi T$, for which the logarithm vanishes. Still, any other choice is mathematically as correct as this one. Instead of being worried about this dependence on the renormalisation scale, we may take advantage of it as follows: Varying the renormalisation scale $\mu$ in a “reasonable range”, say between $\mu = 2\pi T$ and $\mu = 8\pi T$, we obtain an error estimate for the missing higher-order corrections.

Finally, we note that $\mu \propto T$ implies that a QCD plasma becomes in the large temperature limit an asymptotically free gas of quarks and gluons, while at $T = O(\Lambda_{\text{QCD}})$ a phase transition to colourless mesons and baryons happens.
9.2. Scalar gas

Figure 9.2.: Left: A second order correction to the mass; Right: Ring diagram, each external bubble corresponds to an insertion of $m_D^2$.

**IR behaviour** We have found a fast convergence of the perturbative expansion from the results for the pressure of a scalar gas. For applications particularly interesting is the case of a massless particle and we consider now as a toy-model for QCD a $\lambda \phi^4$ theory with $m = 0$.

Looking back at our previous result for the self-energy, Eq. (9.33), setting $m \to 0$ and dropping the vacuum term, we have

$$\Sigma = \frac{\lambda}{2} \Delta(t = 0, x = 0) \to \frac{\lambda}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{n_k}{\omega_k} = \frac{\lambda T^2}{2} \frac{2}{12} . \quad (9.41)$$

Thus the thermal part of the self-energy induces at first order in $\lambda$ a thermal or Debye mass,

$$m_D^2 = \frac{\lambda T^2}{2} \frac{2}{12} . \quad (9.42)$$

Switching to the covariant form of the thermal propagator, Eq. (9.15), the contribution shown in the left panel of figure 9.2 at second order is

$$\Sigma_2 = -\frac{\lambda}{2} T \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{m_D^2}{(\omega_n^2 + k^2)^2} . \quad (9.43)$$

While in the terms with $n \neq 0$ the $\omega_n$ act as a IR cutoff, we see that for $n = 0$ and thus $\omega_0 = 0$ the integral is proportional to $\int dk/k^2$ and thus IR divergent. If we go to higher terms in the expansion and add additional loops to the ‘primary loop’ as shown in the right panel of figure 9.2 then the IR divergence only increases in power.

The solution to this problem is to account for the thermal mass of the scalar particle properly. If we use an effective propagator which includes the Debye mass of the particle,

$$\frac{1}{\omega_n^2 + k^2} \to \frac{1}{\omega_n^2 + k^2 + m_D^2} , \quad (9.44)$$

then the $n = 0$ term of Eq. (9.43) with $\omega_0 = 0$ is given by

$$\Sigma_{2,n=0} = -\frac{\lambda}{2} T \int \frac{d^3 k}{(2\pi)^3} \frac{m_D^2}{(k^2 + m_D^2)^2} = -\frac{\lambda^2}{4} \left( \frac{T^2}{12} \right) \left( \frac{T}{8\pi m_D} \right) . \quad (9.45)$$
and thus finite.

Since the Debye mass scales as $m_D \propto \lambda^{1/2}$, the contribution of $\Sigma_{2,n=0}$ in the perturbative expansion is not of order $\lambda^2$ but of order $\lambda^{3/2}$. Thus we obtained a term which is non-analytic in the coupling—which can not happen, if we sum a finite number of terms. As explanation, we have to look at the expansion of the effective propagator for small $m^D$ (restricting ourselves again to the $n = 0$ term),

$$\frac{1}{k^2 + m^2_D} = \frac{1}{k^2} - \frac{m^2_D}{k^4} + \frac{m^4_D}{k^6} + \ldots$$  \hspace{1cm} (9.46)

Here we can view e.g. the $m^4_D/k^6$ term as a ring diagram with three massless propagators $k^{-2}$ and two factors $m^2_D$ produced by self-energy insertions. Thus including the thermal mass corresponds to summing up the infinite sum of diagrams shown in the right panel of figure 9.2.

We can formalise the inclusion of the Debye mass in the originally massless scalar theory as follows: We reorganise perturbation theory by adding a mass term to the free Lagrangian and subtracting it from the interaction term,

$$L_0 = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$ \hspace{1cm} (9.47)

$$L_{\text{int}} = -\frac{\lambda}{4!} \phi^4 + \frac{1}{2} m^2 \phi^2.$$ \hspace{1cm} (9.48)

Here we may set $m^2 = m^2_D$ or keep it as a free parameter to be determined by, for example, that the free energy is independent of this parameter

$$\frac{dF}{dm^2} = 0.$$ \hspace{1cm} (9.49)

This reformulation of the perturbative expansion in thermal field theories is called screened or optimised perturbation theory.

**Symmetry restoration at high temperature** Let us reconsider the Hamiltonian. We have chosen the sign of $\lambda$ such that $H$ is bounded from below, and $m^2 > 0$ ensures that the minimal energy is obtained for $\langle \phi \rangle = 0$. If we add in this case the thermal mass,

$$V(\phi, T) = \frac{1}{2} \left( m^2 + \frac{\lambda}{24} T^2 \right) \phi^2 + \frac{\lambda}{4!} \phi^4,$$ \hspace{1cm} (9.50)

we increase simply the effective mass of the $\phi$ particle. Something more drastic happens, if we switch the sign of the vacuum mass term and set $m^2 < 0$. Then for $T = 0$ the minimal energy is obtained for $\phi_0 = \pm \sqrt{-6m^2/\lambda}$. Thus we expect that the vacuum is non-empty and filled with a non-zero, classical field $\phi$, having either the value $+\phi_0$ or $-\phi_0$. Picking out one of the two breaks the $\phi \to -\phi$ symmetry of the Hamiltonian. Increasing the temperature, the positive thermal mass grows. Above some critical temperature, the effective mass becomes therefore again positive and the minimal energy configuration becomes $\phi_0 = 0$. This is shown graphically in figure 9.3.

This behaviour resembles the one we know from a ferromagnet: Below some critical temperature, spontaneous magnetisation breaks rotation invariance, which is above $T_c$ restored.
9.A. Appendix: Equilibrium statistical physics in a nutshell

The distribution function $f(p)$ of a free gas of fermions or bosons in kinetic equilibrium are

$$f(p) = \frac{1}{\exp[\beta(E - \mu)] \pm 1} \tag{9.51}$$

where $\beta = 1/T$ the inverse temperature, $E = \sqrt{m^2 + p^2}$, and $\pm 1$ refers to fermions and -1 to bosons, respectively. As we will see later, photons as massless particles stay also in an expanding universe in equilibrium and may serve therefore as a thermal bath for other particles. A species $X$ stays in kinetic equilibrium, e.g., if in the reaction $X + \gamma \rightarrow X + \gamma$ the energy exchange with photons is fast enough.

The chemical potential $\mu$ is the average energy needed, if an additional particle is added, $dU = \sum_i \mu dN_i$. If $\mu$ is zero, If the species $X$ is also in chemical equilibrium with other species, e.g. via the reaction $X + \bar{X} \leftrightarrow \gamma + \gamma$ with photons, then their chemical potentials are related by $\mu_X + \mu_{\bar{X}} = 2\mu_{\gamma} = 0$.

The number density $n$, energy density $\rho$ and pressure $P$ of a species $X$ follows as

$$n = \frac{g}{(2\pi)^3} \int d^3 p f(p) \tag{9.52}$$

$$\rho = \frac{g}{(2\pi)^3} \int d^3 p E f(p) \tag{9.53}$$

$$P = \frac{g}{(2\pi)^3} \int d^3 p \frac{p^2}{3E} f(p) \tag{9.54}$$

The factor $g$ takes into account the internal degrees of freedom like spin or colour. Thus for a photon, a massless spin-1 particle $g = 2$, for an electron $g = 4$, etc.
9. Thermal field theory

Derivation of the pressure integral Eq. (9.54)

a. Intuitive for a classical gas: old ex.

b. Comparing the 1. law of thermodynamics, \( dU = TdS - PdV \), with the total differential \( dU = (\partial U/\partial S)_V dS + (\partial U/\partial V)_S dV \) gives \( P = -V (\partial E/\partial V)_f \).

We write \( \partial E/\partial V = (\partial E/\partial p)(\partial p/\partial L)(\partial L/\partial V) \). To evaluate this we note that \( \partial E/\partial p = p/E \), that from \( V = L^3 \) it follows \( \partial L/\partial V = 1/3L^2 \) and that finally the quantisation conditions of free particles, \( p_k = 2\pi k/L \) implies \( \partial p/\partial L = -p/L \). Combined this gives \( \partial E/\partial V = -p^2/(3EV) \).

In the non-relativistic limit \( T \ll m \), \( e^{\beta(m-\mu)} \gg 1 \) and thus differences between bosons and fermions disappear,

\[
\begin{align*}
n &= \frac{g}{2\pi^2} e^{-\beta(m-\mu)} \int_0^\infty dp \, p^2 e^{-\frac{p^2}{2m}} = g \left( \frac{mT}{2\pi} \right)^{3/2} \exp[-\beta(m-\mu)] \\
\rho &= mn \\
P = nT \ll \rho
\end{align*}
\]

These expressions correspond to the classical Maxwell-Boltzmann statistic\(^2\). The number of non-relativistic particles is exponentially suppressed, if their chemical potential is small. Since the number of protons per photons is indeed very small in the universe (cf. Ex. 4), and therefore also the number of electron (the universe should be neutral), the chemical potential \( \mu \) can be neglected in cosmology at least for protons and electron.

In the relativistic limit \( T \gg m \) with \( T \gg \mu \) all properties of a gas are determined by its temperature \( T \),

\[
\begin{align*}
n &= \frac{gT^3}{2\pi^2} \int_0^\infty dx \, x^2 e^{-x^2} = \frac{\zeta(3)}{\pi^2} gT^3 \\
\rho &= \frac{gT^4}{2\pi^2} \int_0^\infty dx \, x^3 e^{-x^2} = \frac{\pi^2}{30} gT^4 \\
p &= \rho/3
\end{align*}
\]

where for bosons \( \varepsilon_1 = \varepsilon_2 = 1 \) and for fermions \( \varepsilon_1 = 3/4 \) and \( \varepsilon_2 = 7/8 \), respectively.

Since the energy density and the pressure of non-relativistic species is exponentially suppressed, the total energy density and the pressure of all species present in the universe can be well-approximated including only the relativistic ones,

\[
\begin{align*}
\rho_{\text{rad}} &= \frac{\pi^2}{30} g_* T^4 \\
p_{\text{rad}} &= \rho_{\text{rad}}/3 = \frac{\pi^2}{90} g_* T^4
\end{align*}
\]

where

\[
g_* = \sum_{\text{bosons}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left( \frac{T_i}{T} \right)^4 .
\]

\(^2\)Integrals of the type \( \int_0^\infty dx x^n e^{-ax^2} \) can be reduced to a Gaussian integral by differentiating with respect to the parameter \( a \).
Entropy

Rewriting the first law of thermodynamics, \( dU = TdS - PdV \), as
\[
dS = \frac{dU}{T} + \frac{P}{T} dV = \frac{d(V\rho)}{T} + \frac{p}{T} dV = \frac{V}{T} \frac{dp}{dT} dT + \frac{\rho + P}{T} dV
\]
(9.64)
and comparing this expression with the total differential \( dS(T,V) \), one derives
\[
\frac{\partial S}{\partial V}T = \frac{\rho + P}{T}.
\]
(9.65)
Since the RHS is independent of \( V \) for constant \( T \), we can integrate and obtain
\[
S = \frac{\rho + P}{T} V + f(T).
\]
(9.66)
The integration constant \( f(T) \) has to vanish to ensure that \( S \) is an extensive variable, \( S \propto V \).
The total entropy density \( s \equiv S/V \) of the universe can again approximated by the relativistic species,
\[
s = \frac{2\pi^2}{45} g_* S T^3
\]
(9.67)
where now
\[
g_* = \sum_{\text{bosons}} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left( \frac{T_i}{T} \right)^3
\]
(9.68)
The entropy \( S \) is an important quantity because it is conserved during the evolution of the universe. Conservation of \( S \) implies that \( S \propto g_* S R^3 T = \text{const.} \) and thus the temperature of the Universe evolves as
\[
T \propto g_*^{-1/3} R^{-1}.
\]
(9.69)
When \( g_* \) is constant, the temperature \( T \propto 1/R \). Consider now the case that a particle species, e.g. electrons, becomes non-relativistic at \( T \sim m_e \). Then the particles annihilate, \( e^+ + e^- \rightarrow \gamma \gamma \), and its entropy is transferred to photons. Formally, \( g_* \) decreases and therefore the temperature decreases for a short period less slowly than \( T \propto 1/R \).

Since \( s \propto R^{-3} \) and also the net number of particles with a conserved charge, e.g. \( n_B \equiv n_B - n_B \propto R^{-3} \) if baryon number \( B \) is conserved, the ratio \( n_B/s = \text{const.} \)

Summary of chapter

Thermal Green functions are (anti-) periodic functions in imaginary time, leading to discrete energies with \( \omega_n = 2n\pi T \) for bosonic and \( \omega_n = (2n+1)\pi T \) for fermionic fields, respectively. No new UV divergences appear for \( T > 0 \), since the thermal distribution function vanish exponentially for \( E/T \rightarrow \infty \). In a plasma, even massless particles can aquire a temperature dependent (Debye) mass and symmetries of the Lagrangian may be hidden at low temperatures.

Further reading

Blaizot’s lecture notes [Bla11] are a useful starting point before one turns to text books dedicated to thermal field theory as [KG11].
9.1 Free energy of a free scalar field.
Calculate the free energy of a free scalar field using the Matsubara/imaginary time formalism: Express first the free energy density as

\[ \mathcal{F} = \frac{1}{2} T \sum_n \int \frac{d^3p}{(2\pi)^3} \ln(\omega_n^2 + p^2 + m^2). \]

Use then the residue theorem and that \( \coth(\beta \omega/2) \) has poles at the right position to convert the sum into a contour integral. Finally, discard the \( T = 0 \) part and calculate the integral in the high-temperature limit \( T/m \gg 1 \).

9.2 Bose-Einstein distribution.
Derive (9.30) from (9.11).
10. Symmetries and symmetry breaking

We have seen in the last chapter that the discrete $Z_2$ symmetry of our standard $\lambda \phi^4$ Lagrangian could be hidden at low temperatures, if we choose a negative mass term in the zero temperature Lagrangian. Although such a choice seems at first sight unnatural, we will investigate this case in the following in more detail. Our main motivation is the expectation that hiding a symmetry by choosing a non-invariant ground-state retains the “good” properties of the symmetric Lagrangian. Coupling then such a scalar theory to a gauge theory, we hope to break gauge invariance in a “gentle” way which allows e.g. gauge boson masses without spoiling the renormalisability of the unbroken theory.

As additional motivation we remind that couplings and masses are not constants but depend on the scale considered. Thus it might be that the parameters determining the Lagrangian of the Standard Model at low energies originate from a more complete theory at high scales, where the mass parameter $\mu^2$ is originally still positive. In such a scenario, $\mu^2(Q^2)$ may become negative only after running it down to the electroweak scale $Q = m_Z$.

10.1. Symmetry breaking and Goldstone’s theorem

Let us start classifying the possible destinies of a symmetry:

- Symmetries may be exact. In the case of local gauge symmetries as U(1) or SU(3) for the electromagnetic and strong interactions, we expect that this holds even in theories beyond the SM. In contrast, there is no good reason why global symmetries of the SM as $B-L$ should be respected by higher-dimensional operators originating from a more complete theory at high scales.

- A classical symmetry may be broken by quantum effects. As a result, the corresponding Noether currents are non-zero and the Ward identities of the theory are violated. If the anomalous symmetry is a local gauge symmetry, the theory becomes thereby non-renormalisable. Moreover, we would expect e.g. in case of QED that the universality of the electric charge does not hold exactly.

- The symmetry is explicitly broken by some “small” term in the Lagrangian. An example for such a case is isospin which is broken by the mass difference of the $u$ and $d$ quarks.

- The Lagrangian contains an exact symmetry but the ground-state is not symmetric under the symmetry. In field theory, the ground-state corresponds to the mass spectrum of particles. As a result, the symmetry of the Lagrangian is not visible in the spectrum of physical particles. If the ground-state breaks the original symmetry because one or several scalar fields acquire a non-zero vacuum expectation value, one calls this spontaneous symmetry breaking (SSB). As the symmetry is not really broken on the Lagrangian level, a perhaps more appropriate name would be “hidden symmetry.”
In this and the following chapter, we discuss the case of spontaneous symmetry breaking, first in general and then applied to the electroweak sector of the SM. Since the breaking of an internal symmetry should leave Poincaré symmetry intact, we can give only scalar quantities \( \phi \) a non-zero vacuum expectation value \( \langle \phi \rangle = 0 \). This excludes non-zero vacuum expectation values (vev) for tensor fields, which would single out a specific direction. On the other hand, we can construct scalars as \( \langle \phi \rangle = \langle \bar{\psi} \psi \rangle \neq 0 \) out of the product of multiple fields with spin. In the following, we will always treat \( \phi \) as an elementary field, but we should keep in mind the possibility that \( \phi \) is a composite object, e.g. a condensate of fermion fields, \( \langle \bar{\psi} \psi \rangle \), similar to the case of superconductivity.

### Spontaneous breaking of discrete symmetries

We will first consider the simplest example of a theory with a broken symmetry: A single scalar field with a discrete reflection symmetry. Consider the familiar \( \lambda \phi^4 \) Lagrangian, but with a negative mass term which we include into the potential \( V(\phi) \),

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4} \phi^4 = \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi). \tag{10.1}
\]

The Lagrangian is invariant under the symmetry operation \( \phi \rightarrow -\phi \) for both signs of \( \mu^2 \).

The field configuration with the smallest energy is a constant field \( \phi_0 \), chosen to minimise the potential

\[
V(\phi) = -\frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4, \tag{10.2}
\]

which has the two minima

\[
\phi_0 = \pm v = \sqrt{\mu^2/\lambda}. \tag{10.3}
\]

In quantum mechanics, we learn that the wave-function of the ground state for the potential \( V(x) = -\frac{1}{2} \mu^2 x^2 + \lambda x^4 \) will be a symmetric state, \( \psi(x) = \psi(-x) \), since the particle can tunnel through the potential barrier. In field theory, such tunnelling can happen in principle too. However, the tunnelling probability is proportional to the volume \( L^3 \), and vanishes in the limit \( L \rightarrow \infty \): In order to transform \( \phi(x) = -v \) into \( \phi(x) = +v \) we have to switch an infinite number of oscillators, which clearly costs an infinite amount of energy.

Thus in quantum field theory, the system has to choose between the two vacua \( \pm v \) and the symmetry of the Lagrangian is broken in the ground state. Had we used our usual Lagrangian with a positive mass term, the vacuum expectation value of the field would have been zero, and the ground state would respect the symmetry.

Quantising the theory \( \mathcal{L} \) with the negative mass around the usual vacuum, \( |0\rangle \) with \( \langle 0 | \phi | 0 \rangle = \phi_c = 0 \), we find modes behaving as

\[
\phi_k \propto \exp(-i\omega t) = \exp(-i\sqrt{-\mu^2 + |k|^2} \ t), \tag{10.4}
\]

which can grow exponentially for \( |k|^2 < \mu^2 \). More generally, exponentially growing modes exists, if the potential is concave at the position of \( \phi_c \), i.e. for

\[
m^2_{\text{eff}}(\phi_c) = V''(\phi_c) = -\mu^2 + 3\lambda \phi_c^2 < 0 \tag{10.5}
\]

or \( |\phi_c| < \sqrt{\mu^2/(3\lambda)} \).

Clearly, the problem is that we should, as always, expand the field around the ground-state \( v \). This requires that we shift the field as

\[
\phi(x) = v + \xi(x), \tag{10.6}
\]
splitting it into a classical part \( \langle \phi \rangle = v \) and quantum fluctuations \( \xi(x) \) on top of it. Then we express the Lagrangian as function of the field \( \xi \),

\[
\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}(\partial_\mu \xi)^2 - \frac{1}{2}(2\mu^2)\xi^2 - \mu\sqrt{\lambda}\xi^3 - \frac{\lambda}{4}\xi^4. \tag{10.7}
\]

In the new variable \( \xi \), the Lagrangian describes a scalar field with positive mass \( m_\xi = \sqrt{2}\mu > 0 \). The original symmetry is no longer apparent: Since we had to select one out of the two possible ground states, a term \( \xi^3 \) appeared and the \( \phi \to -\phi \) symmetry is broken. The new cubic interaction term rises now the question, if our scalar \( \phi \to -\phi \) theory becomes non-renormalisable after SSB: As we have no corresponding counter-term at our disposal, the renormalisation of \( \mu \) and \( \lambda \) has to cure also the divergences of the interaction.

Finally, we note that the contribution \( \mu^4/(4\lambda) \) to the energy density of the vacuum is in contrast to the vacuum loop diagrams generated by \( Z[0] \) classical and finite. Thus it is unlikely that we can use the excuse that “quantum gravity” will solve this problem. Moreover, we see later that symmetries will be restored at high temperatures or at early times in the evolution of the Universe. Even if we take the freedom to shift the vacuum energy density, we have either before or after SSB an unacceptable large contribution to the cosmological constant (problem 10.1).

**Spontaneous breaking of continuous symmetries** Our main aims to understand the SSB of the electroweak gauge symmetry. As next step we look therefore at a system with a global continuous symmetry. We discussed already in section 3.1 the case of \( N \) real scalar fields described by the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left[ (\partial_\mu \phi)^2 + \mu^2 \phi^2 \right] - \frac{\lambda}{4} (\phi^2)^2. \tag{10.8}
\]

Since \( \phi = \{\phi_1, \ldots, \phi_N\} \) transforms as a vector under rotations in field space,

\[
\phi_i \to R_{ij} \phi_j \tag{10.9}
\]

with \( R_{ij} \in O(n) \), the Lagrangian is clearly invariant under orthogonal transformations.

Before we consider the general case of arbitrary \( N \), we look at the case \( N = 2 \) for which the potential is shown in Fig. 10.1 Without loss of generality, we choose the vacuum pointing in the direction of \( \phi_1 \): Thus \( v = \langle \phi_1 \rangle = \sqrt{\mu^2/\lambda} \) and \( \langle \phi_2 \rangle = 0 \). Shifting the field as in the discrete case gives

\[
\mathcal{L} = \frac{1}{2} \mu^4 + \frac{1}{2}(\partial_\mu \xi)^2 - \frac{1}{2}(2\mu^2)\xi^2 + \mathcal{L}_{\text{int}}, \tag{10.10}
\]

i.e. the two degrees of freedom of the field \( \phi \) split after SSB into one massive and one massless mode.

Since the mass matrix consists of the coefficients of the terms quadratic in the fields, the general procedure for the determination of physical masses is the following: Determine first the minimum of the potential \( V(\phi) \). Expand then the potential up to quadratic terms,

\[
V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)_i(\phi - \phi_0)_j \partial^2 V_{\phi_i \phi_j} + \cdots \tag{10.11}
\]
The term of second derivatives is a symmetric matrix with elements $M_{ij} \geq 0$, because we evaluate it by assumption at the minimum of $V$. Diagonalising $M_{ij}$ gives as eigenvalues the squared masses of the fields. If the potential has $n > 0$ flat directions, the vacuum is degenerated and $n$ massless modes appear. The eigenvectors of the mass matrix $M_{ij}$ are called the mass eigenstates or physical states. Propagators and Green functions describe the evolution of fields with definite masses and should be therefore build up on these states.

Looking at Fig. 10.1 suggests to use polar instead of Cartesian coordinates in field space. In this way, the rotation symmetry of the potential and the periodicity of the flat direction is reflected in the variables describing the scalar fields. Introducing first the complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, the Lagrangian becomes

$$L = \partial_\mu \phi^\dagger \partial^\mu \phi + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2. \quad (10.12)$$

Next we set

$$\phi(x) = \rho(x)e^{i\vartheta(x)} \quad (10.13)$$

and use $\partial_\mu \phi = [\partial_\mu \rho + i\rho \partial_\mu \vartheta]e^{i\vartheta}$ to express the Lagrangian in the new variables,

$$L = (\partial_\mu \rho)^2 + \rho^2 (\partial_\mu \vartheta)^2 + \mu^2 \rho^2 - \lambda \rho^4. \quad (10.14)$$

Shifting finally again the fields as $\rho = v + \xi$ with $v = \sqrt{\mu^2/2\lambda}$, we find

$$L = \frac{\mu^4}{4\lambda} + \frac{\mu^2}{2\lambda} (\partial_\mu \vartheta)^2 + (\partial_\mu \xi)^2 - 2\mu^2 \xi^2 - 2\mu \sqrt{2\lambda} \xi^3 - \lambda \xi^4 + \left[ \sqrt{\frac{2\mu^2}{\lambda}} \xi + \xi^2 \right] (\partial_\mu \vartheta)^2. \quad (10.15)$$

The phase $\vartheta$ which parametrises the flat direction of the potential $V(\vartheta, \xi)$ remained massless. This mode is called Goldstone (or Nambu-Goldstone) boson and has derivative couplings to the massive field $\xi$, given by the last term in Eq. (10.15). This is a general result, implying that static Goldstone bosons do not interact. Another general property of Goldstone bosons is that they carry the quantum number of the corresponding symmetry generator. Thus they are scalar or pseudo-scalar particles. The only exception is the case when a supersymmetry which has fermionic generators is spontaneously broken.

Let us now discuss briefly the case of general $N$ for the Lagrangian (10.8). The lowest energy configuration is again a constant field. The potential is minimised for any set of fields $\phi_0$ that satisfies

$$\phi_0^2 = \frac{\mu}{\lambda}. \quad (10.16)$$

This equation only determines the length of the vector, but not its direction. It is convenient to choose a vacuum such that $\phi_0$ points along one of the components of the field vector. Aligning $\phi_0$ with its $N$th component,

$$\phi_0 = \left(0, \ldots, 0, \sqrt{\frac{\mu}{\lambda}}\right), \quad (10.17)$$

we now follow the same procedure as in the previous example. First we define a new set of fields, with the $N$th field expanded around the vacuum

$$\phi(x) = (\phi^k(x), v + \xi(x)), \quad (10.18)$$
10.1. Symmetry breaking and Goldstone’s theorem

where \( k \) now runs from 1 to \( N - 1 \). Then we insert this, and the value \( v = \sqrt{\mu^2/\lambda} \) for the vacuum expectation value into the Lagrangian, and obtain

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi^k)^2 + \frac{1}{2} (\partial_\mu \xi)^2 - \frac{1}{2} (2\mu^2)\xi^2 + \frac{1}{4} \frac{\mu^4}{\lambda} - \sqrt{\lambda} \mu \xi^3 - \sqrt{\lambda} \mu \xi (\phi^k)^2 - \frac{\lambda}{2} \xi^2 (\phi^k)^2 - \frac{\lambda}{4} [(\phi^k)^2]^2 - \frac{\lambda}{4} \xi^4.
\]

This Lagrangian describes \( N - 1 \) massless fields and a single massive field \( \xi \), with cubic and quartic interactions. The \( O(N) \) symmetry is no longer apparent, leaving as symmetry group the subgroup \( O(N - 1) \), which rotates the \( \phi^k \) fields among themselves. This rotation describes movements along directions where the potential has a vanishing second derivative, while the massive field corresponds to oscillations in the radial direction of \( V \). This can be visualised for \( N = 2 \), where we get the "Mexican hat" potential shown in figure 10.1.

**Goldstone’s theorem** The observation that massless particles appear in theories with spontaneously broken continuous symmetries is a general result, known as Goldstone’s theorem. The first example for such particles was suggested by Nambu in 1960: He showed that a massless quasi-particle appears in a magnetised solid, because the magnetic field breaks rotation invariance. Goldstone applied soon after that this idea to relativistic QFTs and showed that massless scalar elementary particles appear in theories with SSB. Since no massless scalar particles are known to exist, this theorem appeared to be a dead end for the application of SSB to particle physics. So our task is two-fold: First we should derive Goldstone’s theorem and then we should find out how we can bypass the theorem applying it to our case of interest, gauge theories.

The theorem is obvious at the classical level: Consider a Lagrangian with a symmetry \( G \) and a vacuum state invariant under a subgroup \( H \) of \( G \). For instance, choosing a Lagrangian invariant under \( G = O(3) \) and picking out a vacuum along \( \phi_3 \), the subgroup \( H = O(2) \) of rotation around \( \phi_3 \) keeps the vacuum invariant. Let us denote with \( U(g) \) a representation of \( G \) acting on the fields \( \phi \) and with \( U(h) \) a representation of \( H \), respectively. Since we
10. Symmetries and symmetry breaking

Consider constant fields, derivative terms in the fields vanish and the potential $V$ alone has to be symmetric under $G$, i.e.

$$V(U(g)\phi) = V(\phi). \quad (10.20)$$

Moreover, we know that the vacuum is kept invariant for all $h$, $\phi'_0 = U(h)\phi_0$, but changes for some $g$, $\phi'_0 \neq U(g)\phi_0$. Using the invariance of the potential and expanding $V(U(g)\phi_0)$ for an infinitesimal group transformation gives

$$V(\phi_0) = V(U(g)\phi_0) = V(\phi_0) + \frac{1}{2} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \bigg|_0 \delta \phi_i \delta \phi_j + \ldots, \quad (10.21)$$

where $\delta \phi_i$ denotes the resulting variation of the field. Equation (10.21) implies that

$$M_{ij} \delta \phi_i \delta \phi_j = 0. \quad (10.22)$$

The variation $\delta \phi_i$ depends on whether the transformation belong to $U(h)$ or not: In the former case, the vacuum $\phi_0$ is unchanged, $\delta \phi_i = 0$ and (10.22) is automatically satisfied. If on the other hand $g$ does not belong to $H$, i.e. is a member of the left coset $G/H$, then $\delta \phi_i \neq 0$, implying that the mass matrix $M_{ij}$ has a zero eigenvalue. It is now clear that the number of massless particles is simply determined by the dimensions of the two groups $G$ and $H$: The number of Goldstone bosons is equal to the number of symmetries spontaneously broken, or to the dimension of the left coset $G/H$.

Quantum case The previous discussion was based on the classical potential. Thus we should address the question if this picture survives quantum corrections.

Noether’s theorem tells us that every continuous symmetry has associated to its generators $g_i$ conserved charges $Q_i$. On the quantum level this means the operators $Q_i$ commute with the Hamiltonian, $[H, Q_i] = 0$. Subtracting the cosmological constant, we have $H |0\rangle = 0$. If the vacuum is invariant under the symmetry $Q$, then $\exp(\imath \vartheta Q) |0\rangle = |0\rangle$. For the infinitesimal form of the symmetry transformation, $\exp(\imath \vartheta Q) \approx 1 + \imath \vartheta Q$, and we conclude that the charge annihilates the vacuum,

$$Q |0\rangle = 0. \quad (10.23)$$

Or, in simpler words, the vacuum has the charge 0.

Now we came to the case we are interested in, namely that the symmetry is spontaneously broken and thus $Q |0\rangle \neq 0$. We first determine the energy of the state $Q |0\rangle$. From

$$HQ |0\rangle = (HQ - \underbrace{QH |0\rangle}_{H|0\rangle = 0}) = [H, Q] |0\rangle = 0, \quad (10.24)$$

we see that at least another state $Q |0\rangle$ exists which has as the vacuum $|0\rangle$ zero energy.

We represent the charge operator as the volume integral of the time-like component of the corresponding current operator,

$$Q = \int \mathrm{d}^3 x J^0(t, x). \quad (10.25)$$

The state

$$|s\rangle = \int \mathrm{d}^3 x e^{\imath k \cdot x} J^0(t, x) |0\rangle \rightarrow Q |0\rangle \quad \text{for} \quad k \rightarrow 0 \quad (10.26)$$

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10.2. Renormalisation of theories with SSB

When we went through the SSB of the scalar field, we saw that new $\phi^3$ interactions were introduced. The question then arises, are new renormalisation constants needed when a symmetry is spontaneously broken? This would make these theories non-renormalisable.

We can address this question in two ways. One possibility is to repeat our analysis of the renormalisability of the scalar theory in section 8.3.2, but now for the broken case with $\mu^2 < 0$. Then we would find that the $\phi^3$ term becomes finite, renormalising fields, mass and coupling as in the unbroken case. This is not unexpected, because shifting the field $\phi \to \phi = \phi - v$, which is an integration variable in the generating functional, should not affect physics. On the other hand, such a shift reshuffles the splitting $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ in our standard perturbative expansion in the coupling constant.

To avoid this problem, we analyse SSB in the following in a different way: We develop as a new tool the loop expansion which is based on the effective action formalism. Additionally of being not affected by a shift of the fields, this formalism allows us to calculate the potential including all quantum corrections in the limit of constant fields.

First we recall the definition of the classical field,

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = \frac{1}{Z} \int \mathcal{D} \phi \phi(x) \exp\{i[S + \int d^4x' J(x') \phi(x')])\}$$

(10.28)

and of the effective action

$$\Gamma[\phi_c] = W[J] - \int d^4x' J(x') \phi_c(x') \equiv W[J] - \langle J\phi \rangle,$$

(10.29)

which lead to the converse relation for $J$,

$$J(x) = -\frac{\delta \Gamma[\phi]}{\delta \phi(x)}.$$  

(10.30)

**Effective potential** In general we will not be able to solve the effective action. Studying SSB, we can however make use of a considerable simplification: The fields we are interested in are constant, and it should be therefore useful to perform a gradient expansion of the effective action $\Gamma[\phi]$,

$$\Gamma[\phi] = \int d^4x \left[-V_{\text{eff}}(\phi) + \frac{1}{2} F_2(\phi)(\partial_\mu \phi)^2 + \ldots\right].$$

(10.31)

We will suppress the subscript $c$ on the classical field from now on and use brackets $\langle J\phi \rangle$ to indicate integration.
Here, we introduced also the effective potential $V_{\text{eff}}(\phi)$ as the zeroth order term of the expansion in $(\partial_\mu \phi)^2$, i.e. the only term surviving for constant fields.

If we now choose the source $J(x)$ to be constant, the field $\phi(x)$ has to be uniform too, $\phi(x) = \phi$, by translation invariance. Together this implies that

$$\Gamma = -\Omega V_{\text{eff}},$$

(10.32a)

$$-J = \delta \Gamma[\phi] = -\Omega \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi},$$

(10.32b)

where $\Omega$ is the space-time volume. Hence, as announced, we only have to calculate the effective potential, not the full effective action. In the absence of external sources, $J = 0$, Eq. (10.32b) simplifies to $V^\prime_{\text{eff}}(\phi) = 0$. Thus this is the quantum version of our old approach where we minimised the classical potential $V(\phi)$ in order to find the vacuum expectation value of $\phi$. Note that Eq. (10.32b) contains all quantum corrections to the classical potential, the only approximation made so far neglecting gradients of the classical field.

In order to proceed, we use that we know the classical potential and we assume that quantum fluctuations are small. Then we can perform a saddle-point expansion around the classical solution $\phi_0$, given by the solution to

$$\delta \left\{ S[\phi] + \langle J \phi \rangle \right\} \bigg|_{\phi_0} = 0$$

(10.33)

or,

$$\Box \phi_0 + V^\prime(\phi_0) = J(x).$$

(10.34)

We write the field as $\phi = \phi_0 + \tilde{\phi}$, i.e. as a classical solution with quantum fluctuations on top. Then we can approximate the path integral by

$$Z = \exp\left\{ \frac{i}{\hbar} W \right\} \approx \exp\left\{ \frac{i}{\hbar} \left[ S[\phi_0] + \langle J \phi_0 \rangle \right] \right\} \int D\tilde{\phi} \exp\left\{ \frac{i}{\hbar} \int d^4x \frac{1}{2} \left[ (\partial_\mu \tilde{\phi})^2 - V^\prime\prime(\phi_0) \tilde{\phi}^2 \right] \right\},$$

(10.35)

where $V^\prime\prime$ is the second derivative of the potential term of the theory. Planck’s constant $\hbar$ has been restored to indicate that what we are doing here is an expansion in $\hbar$, or a loop expansion, which we will show later. The functional integral over $\tilde{\phi}$ is quadratic and can be formally solved directly, it is equal to det($\Box + V^\prime\prime(\phi_0)$)$^{-1/2}$. Using the identity $\ln \det A = \text{tr} \ln A$, we find

$$W = S[\phi_0] + \langle J \phi_0 \rangle + \frac{i\hbar}{2} \text{tr} \ln[\Box + V^\prime\prime(\phi_0)] + O(\hbar^2).$$

(10.36)

To evaluate the operator trace, we write out the definition and insert two complete sets of plane waves,

$$\text{tr} \ln[\Box + V^\prime\prime] = \int d^4x \langle x | \ln[\Box + V^\prime\prime] | x \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \langle x | k \rangle \langle k | \ln[\Box + V^\prime\prime] | q \rangle \langle q | x \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \ln[-k^2 + V^\prime\prime].$$

(10.37)
Performing the Legendre transform and putting everything together, we obtain for the effective potential including the first quantum corrections

$$V_{\text{eff}}(\phi_0) = V(\phi_0) - \frac{i\hbar}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left[ k^2 - V''(\phi_0) \right] + O(\hbar^2).$$  \hspace{1cm} (10.38)

As an example we can use the $\lambda\phi^4$ theory, with

$$V''(\phi_0) = \mu^2 + \frac{1}{2} \lambda \phi_0^2,$$  \hspace{1cm} (10.39)

where we see that $V''(\phi_0)$ can be interpreted as an effective mass, consisting of the $\mu^2$ and the contribution $\frac{1}{2} \lambda \phi_0^2$ due to the constant background field $\phi_0$. The total effective potential at order $O(\hbar^2)$ consists of the classical potential $V(\phi)$, i.e. the classical energy density of a scalar field with vacuum expectation value $\phi$, while the first quantum correction is given by the zero-points energies of a scalar particle with effective mass $V''(\phi_0)$.

Not surprisingly, the effective potential is divergent and we have to introduce counter-terms that eliminate the divergent parts. Our effective potential is then

$$V_{\text{eff}}(\phi_0) = V(\phi) + \frac{\hbar}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left( \frac{k^2 + V''(\phi_0)}{k_E^2} \right) + B\phi_0^2 + C\phi_4^4 + O(\hbar^2).$$  \hspace{1cm} (10.40)

Here, we Wick rotated the integral to Euclidean space and subtracted an infinite constant in order to make the logarithm dimensionless. (Equivalently we could have added an additional constant counter-term $\Lambda$ renormalising the vacuum energy density.) The integral can be solved in different regularisation schemes. Here we will expand the logarithm,

$$\ln \left( 1 + \frac{V''}{k_E^2} \right) = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{V''}{k_E^2} \right)^n,$$  \hspace{1cm} (10.41)

and cutoff the integral at some large momenta $\Lambda$. The first two terms of the sum will depend on the cutoff, being proportional to $\Lambda^2$ and $\ln(\Lambda^2/V'')$, respectively. Performing the integral and neglecting terms that vanish for large $\Lambda$, we obtain

$$V_{\text{eff}}(\phi_0) = V(\phi_0) + \frac{\Lambda^2}{32\pi^2} V''(\phi_0) + \frac{V''(\phi_0)^2}{64\pi^2} \ln \left( \frac{V''(\phi_0)}{\Lambda^2} \right).$$  \hspace{1cm} (10.42)

Now we see that if we start out with a massless $\lambda\phi^4$, our cutoff-dependent terms are

$$V'' = \frac{1}{2} \lambda \phi_0^2,$$  \hspace{1cm} and $$(V'')^2 = \frac{\lambda^2}{4} \phi_0^4,$$  \hspace{1cm} (10.43)

which both can be absorbed into the counter-terms $B$ and $C$ by imposing appropriate renormalisation conditions.

Let us stress the important point in this result: The renormalisation of the $\lambda\phi^4$ theory using the effective potential approach is not affected at all by a shift of the field: We are free to use both signs of $\mu^2$ and any value of the classical field $\phi_0$ in Eq. (10.42). Independently of the sign of $\mu^2$, we need only symmetric counter-terms, as a cubic term does not appear at all.

---

2 Integrating $i\Delta_P(0)$ w.r.t. $m^2$ reproduces the one-loop term.
10. Symmetries and symmetry breaking

Figure 10.2.: Perturbative expansion of the one-loop effective potential $V^{(1)}_{\text{eff}}$ for the $\lambda \phi^4$ theory; all external legs have zero momentum.

We can rephrase this point as follows: If we renormalise before we shift the fields, we know that we obtain finite renormalised Green functions. But shifting the fields does not change the total Lagrangian. Thus the effective action and the effective potential are unchanged too. Consequently the theory has to stay renormalisable after SSB.

Let us now discuss what happens with a non-renormalisable theory in the effective potential approach. Including e.g. a $\phi^6$ term leads to $(V'')^2 \propto \phi^8$ which requires an additional counter-term $D \phi^8$, generating in turn even higher order terms and so forth. Thus in this case an infinite number of counter-terms is needed for the calculation of $V^{(1)}_{\text{eff}}$. How does this finding go together with our statement that non-renormalisable theories are predictive below a certain cutoff scale $\Lambda$? The reason for this apparent contradiction becomes clear, if we look again at the series expansion of the logarithm in the one loop contribution $V^{(1)}_{\text{eff}}$.

\[
V^{(1)}_{\text{eff}} = i \sum_{n=1}^{\infty} \int \frac{d^4k}{(2\pi)^4} \frac{1}{2n} \left[ \frac{V''(\phi_0)}{k^2} \right]^n.
\] (10.44)

This contribution is an infinite sum of single loops with progressively more pairs of external legs with zero-momentum attached, see Fig. [10.2] for the case of $V''(\phi_0) = \frac{1}{2} \lambda \phi_0^2$. (We added the factor $i$, because we returned to Minkowski space; the symmetry factor $2n$ appearing automatically in this approach accounts for the symmetry of a graph with $n$ vertices under rotations and reflection.) As we saw, the superficial degree of divergence increases with the number of external particles for a $\lambda \phi^6$ theory and $n > 4$. Hence every single diagram in the infinite sum contained in $V^{(1)}_{\text{eff}}$ diverges and requires a counter-term of higher order in $\phi^n$. Therefore the effective potential approach is not useful for non-renormalisable theories.

Expansion in $\hbar$ as a loop expansion To show that the expansion in $\hbar$ is really a loop expansion, we introduce artificially a parameter $a$ into our Lagrangian so that

\[
\mathcal{L}(\phi, \partial_\mu \phi, a) = a^{-1} \mathcal{L}(\phi, \partial_\mu \phi).
\] (10.45)

Now we want to determine the power $P$ of $a$ in an arbitrary 1PI Feynman graph, $a^P$: A propagator is the inverse of the quadratic form in $\mathcal{L}$ and contributes thus a positive power $a$, while each vertex $\propto \mathcal{L}_{\text{int}}$ adds a factor $a^{-1}$. The number of loops in an 1PI diagram is given by $L = I - V + 1$, cf. Eq. (8.42), where $I$ is the number of internal lines and $V$ is the number of vertices. Putting this together we see that

\[
P = I - V = L - 1
\] (10.46)
and it is clear that the power of \( a \) gives us the number of loops.

We should stress that using a loop expansion does not imply a semi-classical limit, \( S \gg \hbar \): Our fictitious parameter \( a \) is not small; in fact, it is one. The loop expansion is not affected by a shift in the fields, since \( a \) multiplies the whole Lagrangian. Thus this procedure is particularly useful discussing the renormalisation of theories with SSB.

**Effective action as generating functional for 1PI Green functions** We have now all the necessary tools in order to show that the tree-level graphs generated by the effective action \( \Gamma[\phi] \) correspond to the complete scattering amplitudes of the corresponding action \( S[\phi] \). We compare our familiar generating functional

\[
Z[J] = \int D\phi \exp\{iS + \langle J\phi \rangle\} = e^{iW[J]},
\]

with the functional \( V_a[J] \) of a fictitious field theory whose action \( S \) is the effective action \( \Gamma[\phi] \) of the theory \( (10.47) \) we are interested in,

\[
V_a[J] = \int D\phi \exp \left\{ \frac{i}{a}(\Gamma[\phi] + \langle J\phi \rangle) \right\} = e^{iu_a[J]}.
\]

Additionally, we introduced the parameter \( a \) with the same purpose as in \( (10.45) \): In the limit \( a \to 0 \), we can perform a saddle-point expansion and the path integral is dominated by the classical path. From \( (10.35) \), we find thus

\[
\lim_{a \to 0} aU_a[J] = \Gamma[\phi] + \langle J\phi \rangle = W[J],
\]

where we used the definition of the effective action, Eq. \( (10.29) \), in the last step. The RHS is the sum of all connected Green functions of our original theory. The LHS is the classical limit of the fictitious theory \( V_a[J] \), i.e. it is the sum of all connected tree graphs of this theory. Equation \( (8.147) \) shows the vertices of this theory are given by \( \Gamma^{(n)}(x_1, \ldots, x_n) \), i.e. the 1PI Green functions of our original theory. Thus we can represent the connected graphs of \( W \) as tree graphs whose effective vertices are the sum of all 1PI graphs with the appropriate number of external lines.

**Another proof of the Goldstone theorem** With the help of the effective potential we can give another simple proof of the Goldstone theorem. We know that the zero of the inverse propagator determines the mass of a particle. From Eq. \( (8.148) \), the exact inverse propagator in momentum space for a set of scalar fields is given by

\[
\Delta_{ij}^{-1}(p^2) = \int d^4 x e^{ip(x-x')} \frac{\delta^2 \Gamma}{\delta \phi_i(x) \delta \phi_j(x')}.
\]

Massless particles correspond to zero eigenvalues of this matrix equation for \( p^2 = m^2 \). If we set \( p = 0 \), the fields are constant. But differentiating the effective action w.r.t. to constant fields is equivalent to differentiating simply the effective potential,

\[
\frac{\partial^2 V_{\text{eff}}}{\partial \phi_i(x) \partial \phi_j(x')} = 0.
\]

Thus our previous analysis of Goldstone’s theorem using the classical potential holds also in the quantum case, if we simply replace the classical by the effective potential.
10. Symmetries and symmetry breaking

**Coleman-Weinberg Problem** Sidney Coleman and Erick Weinberg \([CW73]\) used this formalism to investigate if quantum fluctuations could trigger SSB in an initially massless theory. Rewriting the effective potential a bit we have

\[
V_{\text{eff}}(\phi) = \left[ \frac{\Lambda^2}{64\pi^2} \lambda + B \right] \phi^2 + \left[ \frac{\lambda}{4!} + \frac{\lambda^2}{(16\pi)^2} \ln \frac{\phi^2}{\Lambda^2} + C \right] \phi^4. \tag{10.52}
\]

Now we impose the renormalising conditions, first

\[
\frac{d^2V_{\text{eff}}}{d\phi^2} \bigg|_{\phi=0} = 0, \tag{10.53}
\]

which implies that

\[
B = -\frac{\lambda \Lambda^2}{64\pi^2}. \tag{10.54}
\]

When renormalising the coupling constant, we have to pick a different point than \(\phi = 0\), due to the logarithm being ill-defined there. This means that we have to introduce a scale \(\mu\). Taking the fourth derivative and ignoring terms that are independent of \(\phi\), we find

\[
\frac{d^4V_{\text{eff}}}{d\phi^4} \bigg|_{\phi=\mu} = \lambda = 24 \frac{\lambda^2}{(16\pi)^2} \ln \frac{\mu^2}{\Lambda^2}. \tag{10.55}
\]

We can convince ourselves that this expression gives the correct beta function,

\[
\beta(\mu) = \mu \frac{\partial \lambda}{\partial \mu} = 3 \frac{\lambda}{16\pi^2} \lambda^2 + \mathcal{O}(\lambda^3). \tag{10.56}
\]

Using the complete expression for Eq. (10.55), we can determine \(C\) and obtain for the renormalised effective potential (problem 10.6)

\[
V_{\text{eff}}(\phi) = \frac{\lambda(\mu)}{4!} \phi^4 + \frac{\lambda^2(\mu)}{(16\pi)^2} \ln \frac{\phi^2}{\mu^2} + \mathcal{O}(\lambda^3). \tag{10.57}
\]

This potential has two minima outside of the origin, so it seems that SSB does indeed happen. These minima lie however outside the expected range of validity of the one loop approximation: Rewriting the potential as \(V_{\text{eff}}(\phi) = \lambda \phi^4 / 4! (1 + a \lambda \ln(\phi^2 / \mu^2) + \ldots)\) suggest that we can trust the one-loop approximation only as long as \((3/32\pi^2)\lambda \ln(\phi^2 / \mu^2) \ll 1\).

10.3. Abelian Higgs model

After we have shown that the renormalisability is not affected by SSB, we now try to apply this idea to a the case of a gauge symmetry. First of all, because we aim to explain the masses of the \(W\) and \(Z\) bosons as consequence of SSB. Secondly, we saw that SSB of global symmetries leads to massless scalars which are however not observed. As SSB cannot change the number of physical degrees of freedom, we hope that each of the two diseases is the cure of the other: The Goldstone bosons which would remain massless in a global symmetry disappear becoming the required additional longitudinal degrees of freedom of massive gauge bosons in case of the SSB of a gauged symmetry.
The Abelian Higgs model, which is the simplest example for this mechanism, is obtained by gauging a complex scalar field theory. Introducing in the Lagrangian \( \text{(10.12)} \) the covariant derivative
\[
\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \tag{10.58}
\]
and adding the free Lagrangian of an U(1) gauge field gives
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi) \dagger (D^\mu \phi) + \mu^2 \phi \dagger \phi - \lambda (\phi \dagger \phi)^2. \tag{10.59}
\]
The symmetry breaking and Higgs mechanism is best discussed changing to polar coordinates in field-space, \( \phi = \rho \exp\{i\vartheta\} \). Then we insert
\[
D_\mu \phi = [\partial_\mu \rho + i \rho (\partial_\mu \vartheta + eA_\mu)] e^{i\vartheta} \tag{10.60}
\]
to the Lagrangian, obtaining
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \rho^2 (\partial_\mu \vartheta + eA_\mu)^2 + (\partial_\mu \rho)^2 + \mu^2 \rho^2 - \lambda \rho^4. \tag{10.61}
\]
The only difference to the ungauged model is the appearance of the gauge field in the prospective mass term \( \rho^2 (\partial_\mu \vartheta + eA_\mu)^2 \). This allows us to eliminate the angular mode \( \vartheta \) which shows up nowhere else by performing a gauge transformation on the field \( A_\mu \): The action of a U(1) gauge transformation \( A_\mu \to A_\mu' = A_\mu - \partial_\mu \Lambda \) on the original field \( \phi \) is just a phase shift, hence \( \rho \) is unchanged and \( \vartheta \) is shifted by a constant, \( \vartheta \to \vartheta' = \vartheta + e\Lambda \). This means that if we consider the gauge invariant combination
\[
B_\mu = A_\mu + \frac{1}{e} \partial_\mu \vartheta \tag{10.62}
\]
as new variable, we eliminate \( \vartheta \) completely, as \( F_{\mu\nu}(A_\mu) = F_{\mu\nu}(B_\mu) \) is gauge invariant,
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e^2 \rho^2 (B_\mu)^2 + (\partial_\mu \rho)^2 + \mu^2 \rho^2 - \lambda \rho^4. \tag{10.63}
\]
It is now evident that the Goldstone mode \( \vartheta \) has disappeared, while the new gauge invariant field \( B_\mu \) obtained a mass and an interaction term \( e \rho \). Thus the U(1) symmetry is hidden using the massless \( A_\mu \) field as variable, while the field \( B_\mu \) produces a gauge invariant mass term in \( \text{(10.63)} \).

Eliminating the field \( \rho \) in favour of fluctuations \( \chi \) around the vacuum \( v = \sqrt{\mu^2/\lambda} \), i.e. shifting as usually the field as
\[
\rho = \frac{1}{\sqrt{2}} (v + \chi), \tag{10.64}
\]
we find after some algebra as new Lagrangian
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 (B_\mu)^2 + e^2 v \chi (B_\mu)^2 + \frac{1}{2} e^2 \chi^2 (B_\mu)^2 \\
+ \frac{\mu^4}{4\lambda} + \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2} (\sqrt{2}m)^2 \chi^2 - \sqrt{\lambda} \mu \chi^3 - \frac{\lambda}{4} \chi^4. \tag{10.65}
\]
As in the ungauged model we obtain a \( \chi^3 \) self-interaction and a contribution to the vacuum energy density. But the gauge field \( B_\mu \) acquired the mass \( M = ev \), therefore having now three
spin degrees of freedom. The additional longitudinal one has been delivered by the Goldstone boson which in turn disappeared: The gauge field has eaten the Goldstone boson, so to speak.

We also see that the number of degrees of freedom before SSB (2 + 2) matches the number afterwards (3 + 1). The phenomenon that breaking spontaneously a gauge symmetry does not lead to massless Goldstone bosons because they become the longitudinal degree of freedom of massive gauge bosons is called the Higgs effect.

The gauge transformation we used to eliminate the $\vartheta$ field corresponds to the Higgs model in the unitary gauge, where only physical particles appear in the Lagrangian. The massive gauge boson is described by the Procca Lagrangian and we know that the resulting propagator becomes constant for large momenta. Hence, this gauge is convenient for illustrating the concept of the Higgs mechanism, but not suited for loop calculations.

A different way to consider the model is to keep the Cartesian fields $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. Then the Lagrangian is

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \left[ (\partial_\mu \phi_1 - eA_\mu \phi_2)^2 + (\partial_\mu \phi_2 + eA_\mu \phi_1)^2 \right] + \frac{\mu^2}{2} (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2. \quad (10.66)
$$

Performing the shift due to the SSB, $\phi_1 = v + \tilde{\phi}_1$ and $\phi_2 = \tilde{\phi}_2$, the Lagrangian becomes

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu^2 - eA^\mu \partial_\mu \tilde{\phi}_2 \\
+ \frac{1}{2} \left[ (\partial_\mu \tilde{\phi}_1)^2 - 2\mu^2 \tilde{\phi}_1^2 \right] + \frac{1}{2} (\partial_\mu \tilde{\phi}_2)^2 + \ldots, \quad (10.67)
$$

where we have omitted interaction and vacuum terms not relevant to the discussion. As we see, the Goldstone boson $\tilde{\phi}_2$ does not disappear and it couples to the gauge field $A_\mu$. On the other hand, the mass spectrum of the physical particles is the same as in the unitary gauge. The degrees of freedom before and after breaking the symmetry do not match, hence there is an unphysical degree of freedom in the theory, namely that corresponding to $\tilde{\phi}_2$.

**Gauge fixing and gauge boson propagator** In order to make the generating functional $Z[J^\mu, J, J^*]$ of the abelian Higgs model well-defined, we have to remove the gauge freedom of the classical Lagrangian. Using the Faddeev-Popov trick to achieve this implies to add a gauge-fixing and a Faddeev-Popov ghost term to the classical Lagrangian,

$$
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} = \mathcal{L}_{\text{cl}} - \frac{1}{2\xi} G^2 - \bar{c} \frac{\partial G}{\partial \vartheta} c. \quad (10.68)
$$

Here $G = 0$ is a suitable gauge condition, $\vartheta$ are the generators of the gauge symmetry and $c, \bar{c}$ are Grassmannian ghost fields.

In the unbroken abelian case we used as gauge condition $G = \partial_\mu A^\mu$. With the gauge transformation $A^\mu \to A^\mu - \vartheta^\mu \vartheta$ the ghost term becomes simply $\mathcal{L}_{\text{FP}} = \bar{c}(-\Box)c$. Thus the ghost fields completely decouple from any physical particles, and the ghost term can be absorbed in the normalisation.

In the present case of a theory with SSB, we want to use the Faddeev-Popov term to cancel the mixed $A^\mu \partial_\mu \phi_2$ term. Therefore we include the Goldstone boson $\phi_2$ in the gauge condition,

$$
G = \partial_\mu A^\mu - \xi e \phi_2 = 0. \quad (10.69)
$$
From

\[ \phi_2 = \frac{\partial_\mu A^\mu}{\xi e v}, \]  

we see that the unitary gauge corresponds to \( \xi \to \infty \). We calculate first \( G^2 \),

\[ G^2 = (\partial_\mu A^\mu)(\partial_\nu A^\nu) - 2\xi ev \phi_2 \partial_\mu A^\mu + \xi^2 e^2 v^2 \phi_2^2, \]  

integrate partially the cross term and insert the result into \( \mathcal{L}_{\text{gf}} \),

\[ \mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} G^2 = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - ev A_\mu \partial^\mu \phi_2 - \frac{1}{2} \xi (ev)^2 \phi_2^2. \]  

Now we see that the second term cancels the unwanted mixed term in \( \mathcal{L}_{\text{cl}} \), while a \( \xi \) dependent mass term \( \xi M^2 \) for \( \phi_2 \) appeared.

If we write out the terms in \( \mathcal{L}_{\text{eff}} \) quadratic in \( A_\mu \) and \( \phi_2 \),

\[ \mathcal{L}_{\text{eff},2} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu^2 + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} \xi M^2 \phi_2^2, \]  

we can find the boson propagator. Using the antisymmetry of \( F_{\mu\nu} \) and a partial integration, we transform \( F^2 \) into standard form,

\[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \left( \partial_\mu A^\nu \partial^\mu A_\nu - \partial_\nu A_\mu \partial^\mu A_\nu \right) \]
\[ = \frac{1}{2} \left( A_\nu \partial_\mu \partial^\mu A_\nu - A_\mu \partial^\mu \partial_\nu A_\mu \right) \]
\[ = \frac{1}{2} A_\mu \left( g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right) A_\nu. \]

The part of the Lagrangian quadratic in \( A_\mu \) then reads

\[ \mathcal{L}_A = \frac{1}{2} A_\mu \left[ g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right] A_\nu + \frac{1}{2} A_\mu g^{\mu\nu} M^2 A_\nu + \frac{1}{2\xi} A_\mu \partial^\mu \partial^\nu A_\nu \]
\[ = \frac{1}{2} A_\mu \left[ g^{\mu\nu} (\Box + M^2) - (1 - \xi^{-1}) \partial^\mu \partial^\nu \right] A_\nu. \]

To find the propagator we want to invert the term in the bracket, denote this term by \( P^{\mu\nu}(k) \). If we go to momentum space, then

\[ P^{\mu\nu} = -(k^2 - M^2)g^{\mu\nu} + (1 - \xi^{-1}) k^\mu k^\nu. \]  

This can be split into a transverse and a longitudinal part by factoring out terms proportional to \( P_T^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \),

\[ P^{\mu\nu} = -(k^2 - M^2) \left( P_T^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} \right) + (1 - \xi^{-1}) k^\mu k^\nu \]
\[ = -(k^2 - M^2) P_T^{\mu\nu} - \left( \frac{k^2 - M^2}{k^2} - 1 + \xi^{-1} \right) k^\mu k^\nu \]
\[ = -(k^2 - M^2) P_T^{\mu\nu} - \xi^{-1} (k^2 - \xi M^2) P_L^{\mu\nu}, \]  

(10.74)
10. Symmetries and symmetry breaking

where the longitudinal part is $P^\mu_\nu L = \frac{k^\mu k^\nu}{k^2}$. Since $P^\mu_\nu T$ and $P^\mu_\nu L$ as projection operators are orthogonal to each other, we can invert the two parts separately and obtain

$$iD^\mu_\nu (k^2) = \frac{-iP^\mu_\nu T}{k^2 - M^2 + i\varepsilon} + \frac{-i\xi P^\mu_\nu L}{k^2 - \xi M^2 + i\varepsilon}$$

$$= \frac{-i}{k^2 - M^2 + i\varepsilon} \left( g^\mu_\nu - (1 - \xi) \frac{k^\mu k^\nu}{k^2 - \xi M^2 + i\varepsilon} \right). \quad (10.76)$$

As we see, the transverse part propagates with mass $M^2$, while the longitudinal part propagates with mass $\xi M^2$. The limit $\xi \to \infty$ corresponds again to the unitary gauge and $\xi = 1$ corresponds to the easier Feynman-'t Hooft gauge. For finite $\xi$ we see that the propagator is proportional to $k^{-2}$ and no problems arise in loop calculations, as it did in the unitary gauge.

The Goldstone boson $\phi_2$ has the usual propagator of a scalar particle, however with gauge-dependent mass $\xi M^2$.

**Ghosts** Using the Faddeev-Popov ansatz for the gauge introduces ghosts field through the term

$$\mathcal{L}_{FP} = -\bar{c} \frac{\delta G}{\delta \bar{c}} c \quad (10.77)$$

into the Lagrangian. To calculate $\delta G/\delta \bar{c}$, we have to find out how the gauge fixing condition $G$ changes under an infinitesimal gauge transformation. Looking first at the change of the complex field,

$$\phi \to \tilde{\phi} = \phi + i e \bar{c} \phi = \phi + i e \bar{c} \frac{1}{\sqrt{2}} (v + \phi_1 + i \phi_2), \quad (10.78)$$

we see that the fields $\phi_1$ and $\phi_2$ are mixed under the gauge transformation.

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \bar{c} \quad (10.79)$$

$$\phi_1 \to \tilde{\phi}_1 = \phi_1 - e \bar{c} \phi_2 \quad (10.80)$$

$$\phi_2 \to \tilde{\phi}_2 = \phi_2 + e \bar{c} (v + \phi_1). \quad (10.81)$$

Inserting this into the gauge fixing condition $(10.69)$ and differentiating with respect to the generator, we obtain

$$\frac{\delta G}{\delta \bar{c}} = \delta \left( \partial_\mu \tilde{A}^\mu + \xi e v \tilde{\phi}_2 \right) = \square + \xi e^2 v (v + \phi_1). \quad (10.82)$$

Thus after spontaneous symmetry breaking the ghost particles receive a $\xi$-dependent mass and interact with the Higgs field $\phi_1$. To see this explicitly we insert $\delta G/\delta \bar{c}$ into the ghost Lagrangian,

$$\mathcal{L}_{FP} = -\bar{c} \left[ \square + \xi e^2 v (v + \phi_1) \right] c = (\partial^\mu c)(\partial_\mu c) - \xi M^2 \bar{c} c - \xi e^2 v \phi_1 \bar{c} c. \quad (10.83)$$

The second term corresponds to the mass $\xi e^2 v^2 = \xi M^2$ for the ghost field, while the third one describes the ghost-ghost-Higgs interaction.

To sum this up, we have the following propagators in the $R_\xi$ gauge, where we follow common practise and denote with $h$ the physical Higgs boson and with $\phi$ the Goldstone boson:
10.3. Abelian Higgs model

Gauge boson $A_\mu$ with mass $M_A = ev$

$$
\mu \sim \nu \quad \Rightarrow \quad [g_{\mu \nu} + (1 - \xi) \frac{k_\mu k_\nu}{k^2 - \xi M^2 + i \epsilon}]
$$

Higgs boson $h$ with mass squared $M^2_h = 2\mu^2$

$$
\sim \quad \Rightarrow \quad \frac{i}{k^2 - M^2 + i \epsilon}
$$

Goldstone boson $\phi$ with mass squared $\xi M^2_A$

$$
\sim \quad \Rightarrow \quad \frac{i}{k^2 - \xi M^2_A + i \epsilon}
$$

Ghost $c$ with mass squared $\xi M^2_A$

$$
\sim \quad \Rightarrow \quad \frac{i}{k^2 - \xi M^2_A + i \epsilon}
$$

Before we finish this chapter, we should answer why the Goldstone theorem does not apply to the case of the Higgs model. The characteristic property of gauge theories that no manifestly covariant gauge exists which eliminates all gauge freedom is also responsible for the failure of the Goldstone theorem: In the first version of our proof, we may either choose a gauge as the Coulomb gauge. Then only physical degrees of freedom of the photon propagate, but the potential $A^0(x)$ drops only as $1/|x|$ and the charge $Q$ defined in (10.25) becomes ill-defined. Alternatively, we can use a covariant gauge as the Lorentz gauge. Then the charge is well-defined, but unphysical scalar and longitudinal photons exist. The Goldstone theorem does apply, but the massless Goldstone bosons do not couple to physical modes.

In the second version of our proof, the effective potential for the scalar and for the gauge sector do not decouple and mix by the same reason after SSB. This invalidates our analysis including only scalar fields.

Summary of chapter Examining spontaneous symmetry breaking of internal symmetries, we found three qualitatively different types of behaviours: For a broken global continuous symmetry, Goldstone’s theorem predicts the existence of massless scalars. In the case of broken approximate symmetries, this can explain the existence of light scalar particles—an example are pions. The case of broken global continuous symmetry which are exact seems to be not realised in nature, since no massless scalar particles are observed. If we gauge the broken symmetry, the would-be massless Goldstone bosons become the longitudinal degrees of freedom required for massive spin-1 bosons. Finally, neither Noether’s nor Goldstone’s theorems apply to the case of discrete symmetries; therefore the breaking of discrete symmetries does not change the mass spectrum of the theory.
We developed the effective potential as a tool to study the renormalisability of spontaneously broken theories. This approach allows the calculation of all quantum corrections to the classical potential in the limit of constant fields and is invariant under a shift of fields. Thereby we could establish that renormalisability is not affected by SSB.

Further reading

Our discussion of the effective potential is based on the 1966 Erice lecture “Secret Symmetry” of S. Coleman [Col88].

Problems

10.1 Contribution to the vacuum energy density from SSB.
Calculate the difference in the vacuum energy density before and after SSB in the SM using \( v = 256 \text{ GeV} \) and \( m_h^2 = 2 \mu^2 = (125)^2 \text{ GeV}^2 \). Compare this to the observed value of the cosmological constant.

10.2 Scalar Lagrangian after SSB.
Derive Eq. (10.10) and write down the explicit form of \( L_{\text{int}} \).

10.3 Quantum corrections to \( \langle \phi \rangle \).
We implicitly assumed that quantum corrections are small enough to that the field stays at the chosen classical minimum. Calculate \( \langle \phi(0)^2 \rangle \) for \( d \) space-time dimensions and show that this assumption is violated for \( d \leq 2 \).

10.4 Instability of \( \langle \phi \rangle \).
Calculate the imaginary part of the self-energy for a scalar field with the Lagrangian (10.1), i.e. with a negative squared mass \( \mu^2 < 0 \). Discuss the physical interpretation.

10.5 Goldstone mode as zero mode.
Show that the state \( |s \rangle \) defined in Eq. (10.26) has zero energy for \( k \to 0 \).

10.6 Coleman-Weinberg problem.
Derive Eq. (10.57), find the minima of the potential and discuss the validity of the one-loop approximation.

10.7 Effective potential at finite temperature.
Find the effective potential at finite temperature \( T > 0 \) for a scalar with mass \( m \) and a \( \lambda \phi^4 \) self-interaction.
11. GSW model of electroweak interactions

Fermi introduced a current-current interaction between four fermions as explanation for the nuclear beta-decay. After the discovery of parity violation in weak interactions, it was realised that the weak currents have a $V - A$ form, $L_{\text{Fermi}} = G_F / \sqrt{2} J^\mu(x) J_\mu(x)$ with e.g. $J_\mu(x) = \bar{\psi} e \gamma_\mu (1 - \gamma^5) \psi + h.c$ for the leptonic current. Using this Lagrangian, all experimental data about weak interactions known until 1973 could be explained. Being a dimension six interaction, the Fermi theory belongs however according to our power counting analysis to the class of non-renormalisable theories. Attempts to develop a more complete theory of weak interactions started therefore already in the late 1950ies. A first step towards a renormalisable gauge theory for weak interactions was the introduction of “intermediate vector bosons” $W^\pm$ with mass $m_W$, leading to an interaction $J^\mu W^\pm_\mu$ of similar type as in QED and to a dimensionless coupling constant $g \propto G_F / m_W^2$.

Using these new vector bosons, we can write the charged current interaction in a more economical way introducing doublets of left-handed fermions, $(\nu_e e)_L$ and $(u d)_L$.

Associating then the bosons $W^\pm$ with the ladder operators $\tau_\pm = (\tau_1 \pm i \tau_2) / \sqrt{2}$ (we denote the Pauli matrices in this context with $\tau_i$), the charged current interaction becomes

$$L_{\text{Fermi}} = \frac{G_F}{2\sqrt{2}} (\bar{\nu}_e - e)_L (\tau^+ W^+_\mu + \tau^- W^-_\mu) \gamma^\mu (\nu_e - e)_L.$$ 

The doublet structure of fermions in the Fermi interaction suggests to use $SU(2)$ as gauge group for weak interactions. The gauge group is often called weak isospin and denoted by $SU(2)_L$ in order to stress that only left-handed fermions participate in this gauge interaction, while right-handed fermions transform as singlets. If it would be possible to identify $\tau_3$ with the photon, a unification of the weak and electromagnetic forces would have been achieved. There are three major obstacles defeating this attempt:

- The fermion mass term $m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$ is not gauge invariant.
- The generators of $SU(2)$ are traceless, which means that the multiplets must have zero net charge.
- The currents generated by $\tau_\pm$ should have a vector minus axial vector ($V - A$) structure, while the electromagnetic current has to be a pure vector current.

The last argument seems to be impossible to overcome using a single gauge group. Glashow was the first to realise that Nature may not always choose the most economical solution: He suggested that the gauge group of weak and electromagnetic interactions is the product of two groups, $SU(2)_L \otimes U(1)_Y$, where the $Y$ stands for hypercharge. Salam and Weinberg added to this model the idea of SSB, using the Higgs effect to break $SU(2)_L \otimes U(1)_Y$ down to $U(1)_{\text{em}}$. In this form, the Glashow-Salam-Weinberg (GSW) model of electroweak interactions survived all experimental tests until present.
11. GSW model of electroweak interactions

11.1. Gauge sector

A $SU(2) \otimes U(1)$ gauge theory contains four gauge bosons, of which only one should remain massless. If we use as scalar field to break the gauge symmetry a complex $SU(2)$ doublet, then we add four real degrees of freedom. Three of them will become the longitudinal degrees of freedom for the three massive gauge bosons, so that just one physical Higgs field remains in this most economical version of electroweak symmetry breaking.

We choose the complex scalar $SU(2)$ doublet as

$$\Phi = \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{array} \right). \quad (11.1)$$

The corresponding scalar Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi) \cdot \partial^\mu \Phi + \mu^2 \Phi \cdot \Phi - \lambda (\Phi \cdot \Phi)^2 \quad (11.2)$$

is invariant under both $SU(2)$ and $U(1)$ transformations of $\Phi$,

$$\Phi \rightarrow \exp \left( \frac{i\alpha \cdot \tau}{2} \right) \Phi,$$

$$\Phi \rightarrow \exp \{i\vartheta \} \Phi.$$

We avoid an electrically charged vacuum by choosing the vacuum expectation value in the $\phi^0$ direction,

$$\langle 0 | \Phi | 0 \rangle = \left( \begin{array}{c} 0 \\ \sqrt{2} \end{array} \right). \quad (11.3)$$

Electroweak symmetry breaking should leave $U_{em}(1)$ invariant. With this choice of the vev, we see that the combination $1 + \tau_3$ keeps the ground state invariant,

$$\delta \Phi = i\varepsilon (1 + \tau_3) \Phi = i\varepsilon \left( \begin{array}{cc} 2 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ \sqrt{2} \end{array} \right) = 0. \quad (11.4)$$

Since the hypercharge generator is the identity in the weak isospin space, we should associate therefore the combination $Y + \tau_3$ with the electric charge, $Q \propto Y + \tau_3$.

Now we gauge the model, introducing covariant derivatives,

$$\partial^\mu \Phi \rightarrow D^\mu \Phi = \left( \partial^\mu + \frac{ig}{2} \tau \cdot W^\mu + \frac{ig'}{2} B^\mu \right) \Phi, \quad (11.5)$$

where we suppressed a unit matrix in isospin space in front of $\partial^\mu$ and $B^\mu$. The field-strengths

$$F^\mu_{\phantom{\mu}a} = \partial^\mu W^\mu_a - \partial^0 W^\mu_a - g\varepsilon_{abc} W^\mu_b W^\mu_c \quad (11.6)$$

$$G^\mu_{\phantom{\mu}a} = \partial^\mu B^\mu_a - \partial^\nu B^\mu_a \quad (11.7)$$

correspond to the three $SU(2)_L$ gauge fields $W$ and to the $U_Y(1)$ field $B^\mu$. The couplings of the two groups are $g$ and $g'$, respectively, and the structure constants of $SU(2)$ are the completely antisymmetric tensor $\varepsilon_{abc}$. The Lagrangian describing the Higgs-gauge sector is then

$$\mathcal{L} = -\frac{1}{4} F^2 - \frac{1}{4} G^2 + (D_\mu \Phi) \cdot (D^\mu \Phi) - V(\Phi). \quad (11.8)$$
11.1. Gauge sector

We now spontaneously break this theory using the unitary gauge, since we are mainly interested in the mass spectrum of physical particles. We separate $\phi^0$ into the vev $v$ and fluctuations $h(x)$ as

$$\Phi = \left( \frac{1}{\sqrt{2}} (0, v + h(x)) \right) = \frac{v + h}{\sqrt{2}} \chi.$$  \hspace{1cm} (11.9)

Inserting the part containing $v$ into Eq. (11.8) gives as mass terms for the gauge bosons

$$L_m = \frac{v^2}{2} \frac{g}{\sqrt{2}} \left( \frac{g}{2} \tau \cdot W^\mu + g' B^\mu \right) \left( \frac{g}{2} \tau \cdot W^\mu + g' B^\mu \right) \chi.$$  \hspace{1cm} (11.10)

Multiplying out the brackets and using $$(\tau \cdot W)^2 = W^2$$ and $\chi^\dagger \tau \cdot W^\mu \chi = -W^\mu_3$, we find

$$L_m = \frac{v^2}{2} \left[ \frac{g^2}{4} (W_1^2 + W_2^2 + W_3^2) + \frac{g'^2}{4} B^2 - \frac{gg'}{2} W_3^\mu B_\mu \right].$$  \hspace{1cm} (11.11)

Apparently the neutral fields $W^\mu_3$ and $B^\mu$ are connected, and therefore we rewrite the mass term as

$$L_m = \frac{g^2}{8} v^2 (W_1^2 + W_2^2) + \frac{g'^2}{8} (gW_3^\mu - g'B^\mu)^2.$$  \hspace{1cm} (11.12)

The two fields in the first bracket are the gauge bosons we know from the charged current interactions in the Fermi theory,

$$W^\pm = \frac{1}{\sqrt{2}} (W_1 \mp iW_2).$$  \hspace{1cm} (11.13)

The second bracket corresponds to a new neutral massive gauge boson called $Z$ boson, which is a mixture of the $B$ and the $W_3$ fields,

$$Z^\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left( gW_3^\mu - g'B^\mu \right) = \cos \theta_W W_3^\mu - \sin \theta_W B^\mu.$$  \hspace{1cm} (11.14)

The mixing of the $B$ and the $W_3$ field is parametrised by $\theta_W$ called Weinberg angle: For $\theta_W = 0$, the mixing disappears and hypercharge and electric charge become identical.

The combination of $W_3$ and $B$ orthogonal to $Z$ does not show up in the mass Lagrangian, and should therefore correspond to the massless photon,

$$A^\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left( g'W_3^\mu + gB^\mu \right) = \sin \theta_W W_3^\mu + \cos \theta_W B^\mu.$$  \hspace{1cm} (11.15)

We can now write $L_m$ in terms of physical fields,

$$L_m = \frac{1}{2} m_W W^\mu_1 W^- \mu + \frac{1}{2} m_Z^2 Z_\mu Z^\mu,$$  \hspace{1cm} (11.16)

reading off the gauge boson masses, $m_W = gv/2$, $m_Z = (g^2 + g'^2)^{1/2} v/2 = m_W/(\cos \theta_W)$ and $m_A = 0$, as function of the still unknown values of the coupling constants $g, g'$ and the vev $v$. Thus in the GSW theory, the mass ratio of the gauge bosons is fixed at tree level by the Weinberg angle, $m_W/m_Z = \cos \theta_W$.  

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11. GSW model of electroweak interactions

**Couplings** The new coupling constants \(g\) and \(g'\) should be related to the electromagnetic coupling \(e\) and the Weinberg angle \(\vartheta_W\). We can find the connection between these parameters, if we replace the original fields \(W^\mu_3\) and \(B^\mu\) with the physical fields \(Z^\mu\) and \(A^\mu\) in the covariant derivative (11.5) by inverting Eqs. (11.14) and (11.15),

\[
gW^\mu_3 \tau_3 + g' B^\mu = g(\cos \vartheta_W Z^\mu + \sin \vartheta_W A^\mu) \tau_3 + g'(- \sin \vartheta_W Z^\mu + \cos \vartheta_W A^\mu) \tau_3
\]

Using then \(\tan \vartheta_W = g'/g\), we obtain

\[
gW^\mu_3 \tau_3 + g' B^\mu = \frac{1}{2} g \sin \vartheta_W (\tau_3 + 1) A^\mu + \frac{g}{2 \cos \vartheta_W} (\tau_3 - \sin^2 \vartheta_W (\tau_3 + 1)) Z^\mu.
\]

If we assign weak isospin 1/2 to a lepton doublet, then the electric charge of the charged lepton is

\[
e = g \sin \vartheta_W = g' \cos \vartheta_W,
\]

while the neutrino stays as desired neutral. Remarkably, the electric coupling is smaller than the weak one: Weak interactions are weak, because the mass of \(m_W\) implies a short range, not because the coupling is small.

We can use the remaining piece of the covariant derivative to derive the connection between the Fermi constant \(G_F\) and the vev \(v\) (problem 11.1), finding

\[
G_F = \frac{g^2}{8m_W^2} = \frac{1}{2v^2}
\]

or \(v \approx 246\) GeV. Thus a measurement of the gauge boson masses determines the gauge couplings \(g\) and \(g'\), which in turn fix \(\vartheta_W\).

11.2. Fermions

**Fermion masses** Our remaining task discussing the electroweak mass spectrum at the tree-level is to generate masses for the fermions. The fermion mass term \(m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)\) becomes gauge invariant, if we can promote the parameter \(m\) into a SU(2) doublet. This is easiest accomplished introducing a Yukawa coupling \(y_f = m/v\) between the lepton doublet \(L\), singlet \(e_R\) and the scalar doublet \(\Phi\),

\[
\mathcal{L}_Y = -y_f \left( \bar{L} \Phi e_R + \bar{e}_R \Phi^* L \right).
\]

This term generates fermion masses as well as Yukawa interactions between fermions and the Higgs: Inserting the vacuum expectation value of \(\Phi\), the mass term is

\[
\mathcal{L}_m = -\frac{y_f v}{\sqrt{2}} \left[ \left( \begin{array}{c} \bar{\nu}_e \\ \bar{e} \end{array} \right)_L \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L \right] = -m_f \bar{e} e,
\]

i.e. it is a normal Dirac mass term.

In this way we can generate masses for the down-like fermions with \(\tau_3 = -1/2\) like the electron. In order to generate masses for the other half, we must use the charge conjugated Higgs doublet, \(\bar{\tau}_2 \phi^*\).
To see why this works, we look at the action of an $SU(2)_L$ transformation on the quark and anti-quark doublets. The quarks transform as

$$ q \rightarrow q' = Uq = \exp \left( \frac{i\alpha \cdot \tau}{2} \right) q \quad \text{with} \quad q = \begin{pmatrix} u \\ d \end{pmatrix}, $$

while the charge conjugated version of this relation is

$$ q^* \rightarrow q^* = U^*q^* = \exp \left\{ -\frac{i\alpha \cdot \tau^*}{2} \right\} \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}. $$

Now the bar denotes antiparticles. The complex conjugated Pauli matrices are

$$ \tau_1^* = \tau_1, \quad \tau_2^* = -\tau_2, \quad \tau_3^* = \tau_3. $$

This means that $\{ \tau_1, -\tau_2, \tau_3 \}$ satisfy the same Lie algebra as the original matrices, and is an appropriate basis for the antiquarks. This representation is called $2^*$, while the original one is called the $2$ representation. These two representations must be unitary equivalent, that is for some unitary matrix $V$ we have

$$ V \exp \left( -\frac{i\alpha \cdot \tau^*}{2} \right) V^{-1} = \exp \left( \frac{i\alpha \cdot \tau}{2} \right). $$

For infinitesimal $|\alpha|$, this equation becomes

$$ V(-\tau^*)V^{-1} = \tau $$

implying $V = i\tau_2$. We then have

$$ V \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}. $$

This doublet transforms as the $2$ representation. This means that we can use the charge conjugated field $i\tau_2 \phi^*$ to generate Dirac masses for up-like quarks and, if desired, for neutrinos.

As the Yukawa couplings $y_f$ are not predicted, the fermion masses are arbitrary in the GSW theory. Theoretical prejudice suggests as “natural” values of the couplings around $O(0.1)$, as in the case of the gauge couplings $g$ and $g'$. Even excluding neutrinos, there is instead a large hierarchy in the values of the Yukawa couplings, ranging from $y_e = \sqrt{2}m_e/v \sim 10^{-5}$ for the electron to $y_t = \sqrt{2}m_t/v \sim 1$ for the top quark.

**Flavour mixing** Accounting for the second and third generation of leptons and quarks, the coupling $y_f$ introduced in Eq. (11.21), and hence also the mass $m_f$, become arbitrary $3 \times 3$ matrices, $y$ and $m$. In particular, there is no reason for $y$ to be either diagonal or hermitian. It can still be diagonalized by a biunitary transformation, i.e. by

$$ S^\dagger mT = m_D, $$

where $S$ and $T$ are two different unitary matrices and $m_D$ is diagonal and positive. This means that the weak eigenstates $\psi = \{ \psi_e, \psi_\mu, \psi_\tau \}$, which are diagonal in the interaction basis but have indefinite masses, can be transformed into mass eigenstates $\psi'$ as

$$ \bar{\psi}_L m \psi_R = \bar{\psi}_L S S^\dagger m T T^\dagger \psi_R = \bar{\psi}_L m_D \psi'_R. $$
These new eigenstates will of course no longer be diagonal in the interaction basis, which will lead to flavour mixing.

If we look at the weak current $J^\mu$, and insert the mass eigenstates,

$$J^\mu = \bar{\nu}_L \gamma^\mu e_L = \bar{\nu}'_L \gamma^\mu S^\dagger L S e_L,$$

we see that only the product $S^\dagger L S e \equiv U$ will be observable. This means that we have some freedom of choice, and it is convention to choose mixing only for the neutrinos, $S_L = 1$ and $U = S_\nu$. In the same way, we shift the mixing in the quark sector completely to the down-like quarks. One denotes the mass eigenstates as $(d, s, b)$ and the weak eigenstates as $(d', s', b')$, respectively, while $(u, c, t)$ are unmixed by definition. The two matrices $U$ which arise due to this phenomenon are known as the MNS-matrix in the neutrino case and the CKM-matrix in the quark case.

Note that the mixing matrices cancel in the case of the neutral current: Because the neutral currents couples e.g. neutrinos to neutrinos, products like $S^\dagger L S^\nu e \equiv U$ appear which are equal to unity due to unitarity.

**Conservation laws**

### 11.3. Higgs sector

All particles of the SM except the Higgs particle have been firmly established. The search for the Higgs has been the major goal of the Large Hadron Collider at CERN which reported in 2012 the discovery of scalar resonance with mass $\sim 125$ GeV. At the time of writing, the production and decay modes of this new scalar are compatible with that of a SM Higgs particle.

Since the Higgs mass $m^2_H = 2\lambda v^2$ depends on the unknown self-coupling $\lambda$, it is a priori a free parameter. In this section, we will sketch how one can find theoretical bounds for $\lambda$ and thus the Higgs mass. Vice versa, assuming that the scalar resonance is indeed the SM Higgs, we can use its mass 125 GeV to investigate until which energy scale the SM is valid.

**Lower bound for $m_H$**

Calculating the beta function of the Higgs self-coupling, we have to add to the $\lambda (\phi^\dagger \phi)^2$ term we examined already the effects from all other massive SM particles. In practise, it is often sufficient to include only the massive gauge bosons, $W^\pm$ and $Z$, and the top quark as the only fermion with a Yukawa coupling of order one. Then one finds for the beta function (note the different normalization of $V(\Phi)$)

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2} \left\{ 12\lambda^2 + 12\lambda y_t^2 - 12y_t^4 - \frac{3}{2} \lambda (3g^2 + g'^2) + \frac{3}{16} \left[ 2g^4 + (g^2 + g'^2)^2 \right] \right\}$$

with $t = \ln (\mu^2/\mu_0^2)$. Note the signs of the different terms: the gauge boson loops and the scalar self-energy contribute positively, while the fermion loop has a negative sign. The coupling $\lambda$ must be positive for the vacuum state to be stable. We are looking first for a lower bound on $m_h = \sqrt{2\lambda v}$. For small values of $\lambda$, we neglect all terms of $O(\lambda)$,

$$\frac{d\lambda}{dt} \approx \frac{1}{16\pi^2} \left[ -12y_t^4 + \frac{3}{16} (2g^4 + (g^2 + g'^2)^2) \right].$$
In general, we should solve the matrix of RGEs for $\lambda$, $y_t$, $g$ and $g'$ simultaneously, which is only numerically possible. For simplicity, we neglect therefore the scale dependence of the RHS and obtain

$$\lambda(\Lambda) - \lambda(v) = \frac{1}{16\pi^2} \left[ -12y_t^4 + \frac{3}{16}(2g^4 + (g^2 + g'^2)^2) \right] \ln \left( \frac{\Lambda^2}{v^2} \right).$$

(11.33)

Imposing $\lambda(\Lambda) > 0$, we get a lower bound for $m_H^2$,

$$m_H^2 > \frac{v^2}{8\pi^2} \left[ -12y_t^4 + \frac{3}{16}(2g^4 + (g^2 + g'^2)^2) \right] \ln \left( \frac{\Lambda^2}{v^2} \right).$$

(11.34)

**Upper bound for $m_H$** An upper bound for $m_H$ corresponds to the case of large $\lambda$. Therefore we need to keep now only the self-coupling of the Higgs in the beta-function,

$$\beta(\lambda) = \frac{d\lambda}{dt} = \frac{3\lambda^2}{4\pi^2},$$

(11.35)

leading to a Landau pole at

$$\mu_L = \mu_0 \exp \left\{ \frac{4\pi^2}{3\lambda_0} \right\}.$$  

(11.36)

We need $\lambda$ to be small enough to do perturbation theory. Including a cutoff, $\Lambda$, $\mu_L > \Lambda$, and asking for $\lambda < \infty$, we obtain the inequality

$$\frac{3\lambda(\mu_0)}{4\pi^2} \ln \left( \frac{\Lambda^2}{\mu_0^2} \right) < 1.$$  

(11.37)

Using $m_H^2 = 2\lambda(v)v^2$ and $v = \mu_0$, we find as upper bound for $m_H$ as function of the cutoff scale,

$$m_H^2 < \frac{8\pi^2 v^2}{3 \ln(\Lambda^2/v^2)}.$$  

(11.38)

Combining the two bounds, it is clear that the possible range of Higgs masses shrinks for increasing $\Lambda$, and disappears for a large but finite value of the cutoff. This is a sign that the electroweak theory is a trivial theory: We can keep an interacting theory only for a finite value of the cutoff, while in the limit $\Lambda \to \infty$ only the trivial solution $\lambda = 0$ is possible.

Now we turn this argument around: If the scalar particle with mass 125 GeV measured at the LHC is indeed the SM Higgs, at which scale $\Lambda$ new physics has latest to appear, changing the SM RGE flow? In figure 11.1, the bounds obtained using the full RGE of the SM are shown as green band (lower limit) and black lines (upper limit for $\lambda < \pi$ and $\lambda < 2\pi$): The green band suggests $\Lambda < 10^8$ GeV, implying that the SM cannot be a complete theory valid up to the Planck scale. We will come back to this plot, discssuing the other, less restrictive versions of the lower bound later.

**Perturbative unitarity** In the SM, all particle masses are given by the product of the vev $v$ and some coupling constant. Particle masses much larger than the electroweak scale require therefore large coupling constants, leading to the breakdown of perturbation theory. More precisely, a properly normalised amplitude $A$ has to be bounded, $|A| \leq 1$, reflecting the unitarity of the S-matrix.
11. GSW model of electroweak interactions

Figure 11.1.: The scale $\Lambda$ at which the two-loop RGEs drive the quartic SM Higgs coupling non-perturbative (upper curves), and the scale $\Lambda$ at which the RGEs create an instability in the electroweak vacuum (lower curves). The absolute vacuum stability bound is displayed by the light shaded [green] band, while the less restrictive finite-temperature and zero-temperature metastability bounds are medium [blue] and dark shaded [red], respectively. Figure taken from [?].

To see how too heavy particles within the SM violate perturbative unitarity we consider the case of $2 \to 2$ elastic scattering. The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |M|^2. \quad (11.39)$$

Using a partial wave decomposition we can express the Feynman amplitude for the scattering of spinless particles as

$$M(s, \vartheta) = 16\pi \sum_{l=0}^{\infty} (2l+1) P_l(\cos \vartheta) T_l, \quad (11.40)$$

where $T_l$ is the $l$th partial wave and $P_l$ are the Legendre polynomials. Inserting the partial wave expansion into (11.39) and integrating gives for the total cross section

$$\sigma = \frac{8\pi}{s} \sum_{l=0}^{\infty} (2l+1)(2l'+1) T_l T_l^* \int_{-1}^{1} d(\cos \vartheta) P_l(\cos \vartheta) P_{l'}(\cos \vartheta) \frac{\lambda}{(\omega+n)^{l+l'}}$$

$$= \frac{16\pi}{s} \sum_{l=0}^{\infty} (2l+1)|T_l|^2, \quad (11.41)$$

where we have used that the $P_l$’s are orthogonal polynomials. The optical theorem gives with $P_l(1) = 1$ on the other hand

$$\sigma = \frac{\Im [M(\vartheta = 0)]}{2p_{\text{cms}}\sqrt{s}} = \frac{8\pi}{p_{\text{cms}} \sqrt{s}} \sum_{l=0}^{\infty} (2l+1) \Im(T_l). \quad (11.42)$$
Comparing the two expressions for the total cross section $\sigma$ in the ultrarelativistic limit, using $s \sim p_{\text{cms}}^2/4$, immediately yields the unitarity requirement

$$|T_l|^2 = \Im(T_l),$$

which in turn implies $|T_l| \leq 1$. Thus the partial amplitude $T_l$ lies within a circle of radius one centered at $(0, i/2)$ in the complex plane. For the real part of the amplitude, relevant for elastic scattering, the stronger bound

$$|\Re(T_l)| < \frac{1}{2}$$

applies.

The simplest possible application is the elastic scattering of Higgs bosons: In lowest order perturbation theory, the Feynman amplitude of this process is simply $iA(hh \rightarrow hh) = 6 i \lambda$. As the amplitude is independent of the scattering angle, it corresponds to pure s-wave scattering, $l = 0$. Thus

$$T_0 = \frac{A}{16 \pi} = \frac{3 \lambda}{8 \pi} < \frac{1}{2}$$

or $m_h = \sqrt{2\lambda} v \lesssim 800$ GeV. For larger values of $m_h$, the non-sensical result $T_l > 1/2$ indicates that perturbation theory fails. Similar findings are obtained, if one calculates the scattering of longitudinal gauge bosons as function of $m_h$. These considerations lead to the belief that one would either find the Higgs boson at a multi-TeV collider as the LHC or discover a new strongly interacting sector coupled to the gauge bosons.

**Comparison with electroweak corrections** The fermion masses are widely spread in the SM,

$$m_\nu, \ldots, m_b < m_W, m_Z < m_t$$

Therefore it is often useful to integrate out heavy degrees of freedom. The decoupling theorem derived by Appelquist and Corrasonz states that heavy degrees of freedom decouple at low energies,

$$\mathcal{L}(\Phi, g, \phi, g') \xrightarrow{\text{intgr. out } \Phi} \mathcal{L}_{\text{eff}}(\phi, Z g') + \mathcal{O}\left(\frac{E}{m_\Phi}\right),$$

if $\mathcal{L}$ and $\mathcal{L}_{\text{eff}}$ are renormalizable. As an example for this behavior, we have seen that the effect of heavy particles in the anomalous magnetic moment of leptons is suppressed.

In the case of the electroweak sector, decoupling does not happen in several phenomenologically important cases: For example sending $m_t \rightarrow \infty$, but keeping $m_b$ finite breaks SU(2) invariance. This makes $\mathcal{L}_{\text{eff}}$ non-renormalizable, and therefore decoupling does not apply. Similarly, $m_h \rightarrow \infty$ results in an higgless effective theory which is not renormalisable.

**Example:** Higgs production via gluon fusion.

The Higgs couples to the massless gluons only at the one-loop level via a quark loop. Apart from being the most important production process of Higgs bosons in a proton-proton collider like LHC, this process illustrates also the non-decoupling of heavy fermions.

We calculate the Feynman amplitude for the process using dimensional regularization in $n = 4 - 2\epsilon$ dimensions: Although we know that we will obtain a finite result, individual terms in intermediate

\footnote{In this case, a symmetry factor $1/S = 1/2!$ has to be included both in (11.39) and (11.41) which therefore drops out from the unitarity bound (11.43).}

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steps are divergent (cf. Eq. (11.51)).

For a fermion of mass \( m_f \) in the loop, the Feynman rules give

\[
\mathcal{A} = -(-i g_s)^2 \text{tr}(T_A T_B) \left( \frac{-im_f}{v} \right) \int \frac{d^n k}{(2\pi)^n} \frac{(i)^3 N^{\mu\nu}}{D} \varepsilon_\mu(p)\varepsilon_\nu(q) \tag{11.46}
\]

The denominator is \( D = (k^2 - m_f^2)(k+p)^2 - m_f^2[(k-q)^2 - m_f^2] \). We use first the usual Feynman parameterization to combine the denominators,

\[
\frac{1}{ABC} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{dy}{[Ax + By + C(1-x-y)]^3}. \tag{11.47}
\]

We find

\[
\frac{1}{D} \to 2 \int dx dy \frac{1}{[k^2 - m_f^2 + xym_f^2]^3}. \tag{11.48}
\]

We evaluate the numerator assuming transverse gluons, i.e. dropping terms proportional to \( p_\mu \) or \( q_\nu \), obtaining

\[
N^{\mu\nu} = 4m \left[ \eta^{\mu\nu} \left( m_f^2 - k^2 - \frac{m_h^2}{2} \right) + 4k^\mu k^\nu + p^\mu q^\nu \right]. \tag{11.49}
\]

Now we shift momenta in the integration, drop terms linear in \( k' \) from the numerator and use the relation

\[
\int d^n k' \frac{k'^\mu k'^\nu}{(k'^2 - C)^m} = \frac{1}{n} \eta^{\mu\nu} \int d^n k' \frac{k'^2}{(k'^2 - C)^m}. \tag{11.50}
\]

The integrals can be done using the standard formulas of dimensional regularization,

\[
\int \frac{d^n k'}{(2\pi)^n} \frac{k'^2}{(k'^2 - C)^3} = \frac{i}{32\pi^2} (4\pi)^\varepsilon \Gamma(1+\varepsilon) (2 - \varepsilon) C^{-\varepsilon}, \tag{11.51}
\]

\[
\int \frac{d^n k'}{(2\pi)^n} \frac{1}{(k'^2 - C)^3} = -\frac{i}{32\pi^2} (4\pi)^\varepsilon \Gamma(1+\varepsilon) C^{1-\varepsilon}. \tag{11.52}
\]

The amplitude is

\[
\mathcal{A}(gg \to h) = -\frac{\alpha_s m_f^2}{2\pi v} \delta_{AB} \left( \eta^{\mu\nu} \frac{m_h^2}{2} - p^\nu q^\mu \right) \int dx dy \left( \frac{1 - 4xy}{m_f^2 - xym_h^2} \right) \varepsilon_\mu(p)\varepsilon_\nu(q). \tag{11.53}
\]

Evaluating the two integrals over the Feynman parameters in the limit \( m \gg m_h \), we find

\[
\mathcal{A}(gg \to h) = -\frac{\alpha_s m_f^2}{6\pi v} \delta_{AB} \left( \eta^{\mu\nu} \frac{m_h^2}{2} - p^\nu q^\mu \right) \varepsilon_\mu(p)\varepsilon_\nu(q). \tag{11.54}
\]

Thus gluon fusion \( gg \to h \) is independent of the mass of the heavy fermion in the loop in the limit \( m \gg m_h \). Thus we can use the decay width \( \Gamma(h \to gg) \) as a counter for all fermions with an arbitrary high mass \( m \)—as long as their mass is generated by the SM Higgs effect. In this case, the suppression of heavy particles in loops is compensated by their increasing Yukawa couplings \( y_f \propto m_f \) to the Higgs \( h \).

**Electroweak radiative corrections** In our theory so far we have introduced all in all 24 (26) parameters (problem 11.2). In the Higgs and gauge sector, we can predict with only four parameters \((v, \lambda, g, g')\) the masses of four particles as well as the electroweak interactions of all fermions. Much less satisfactory is the description of the fermion sector, which requires as experimental input eight (or ten, if neutrinos are Majorana particles) elements of the mixing
11.3. Higgs sector

matrices. Additionally, the twelve fermion masses have to be known in order to determine the fermion-Higgs interactions.

It is convenient to choose $\alpha_{\text{em}}$, $G_F$, $m_Z$, which are experimentally most precisely known, to be the parameters for the gauge sector instead of $g$, $g'$, $v$. This set of parameters may be defined on-shell. As a typical example for the structure of the electroweak radiative corrections, we write down the expression for $m_W$ with this set of parameters,

$$m^2_W = \frac{\pi \alpha_{\text{em}} G_F}{\sqrt{2} \sin^2(\vartheta_W)(1 - \Delta r)}.$$  \hfill (11.55)

The quantity $\Delta r$ is the one-loop correction to the $W$ mass, and its dominant contributions are

$$\Delta r = \Delta r_0 - \frac{1 - \sin^2(\vartheta_W)}{\sin^2(\vartheta_W)} \Delta \rho + \Delta r_{\text{rem}}$$  \hfill (11.56)

with

$$\Delta r_0 = 1 - \frac{\alpha_{\text{em}}(0)}{\alpha_{\text{em}}(m_Z)^2}$$  \hfill (11.57)

$$\Delta \rho = \frac{3G_F (m_t^2 - m_b^2)}{8\pi^2 \sqrt{2}}$$  \hfill (11.58)

$$\Delta r_{\text{rem}} = \frac{\sqrt{2} G_F m_W^2}{16\pi^2} \left[ \frac{11}{3} \ln \left( \frac{m_h^2}{m_W^2} \right) - \frac{5}{6} \right].$$  \hfill (11.59)

The first term $\Delta r_0$ accounts simply for the running of $\alpha_{\text{em}}$ from $q^2 = m_Z^2$ to the electroweak scale $q^2 = m_Z^2$. The second contribution, $\Delta \rho$, depends quadratically on the mass difference between members of the same fermion doublet. We recognize such a behavior again as an example for non-decoupling due to the breaking of SU(2) in the limit $m_t/m_b \to \infty$. Finally, the reminder is dominated by the effects of the Higgs, but is only logarithmically dependent on $m_h$. Thus even a precise determination of the electroweak parameters leaves a relatively large range of Higgs masses open.

**Goldstone boson equivalence theorem**  One might expect that in processes where the energy is much higher than the electroweak scale, $s \gg m_W^2$, the unbroken theory becomes a valid description. In particular, we should be able to replace a massive gauge boson by a massless transverse boson plus a massive Goldstone boson. More precisely, the equivalence theorem states that amplitudes involving longitudinal gauge bosons are equal to those of the unphysical Goldstone bosons up to corrections of $\mathcal{O}(m_W/E)$,

$$\mathcal{A}(W^+_L, W^-_L, Z_L) = \mathcal{A}(\phi^+, \phi^-, \phi_3) + \mathcal{O}(m_W/E).$$  \hfill (11.60)

An example for this correspondence which allows to simplify calculations in the high-energy limit is discussed in problem 11.5. The Higgs-Goldstone Lagrangian is

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} h)^2 + \frac{1}{2} (\partial_{\mu} w^+ \partial^{\mu} w^- + \frac{1}{2} (\partial_{\mu} z)^2 - \lambda v h^2$$

$$- \lambda v (2 w^+ w^- + z^2 + h^2) - \frac{1}{4} \lambda (2 w^+ w^- + z^2 + h^2)^2.$$  \hfill (11.61)

(11.62)
11. Summary of chapter

The electroweak sector of the SM is described by a $SU_L(2) \otimes U_Y(1)$ gauge symmetry which is broken spontaneously to $U_{em}(1)$. Three out of the four degrees of the complex scalar $SU(2)$ doublet are pseudo Goldstone bosons ($w^\pm, z$) which become the longitudinal degrees of freedom of the three massive gauge bosons, $W^\pm$ and $Z$. The remaining scalar degree of freedom becomes a physical Higgs boson.

The SM does not explains why SSB happens. Also the Yukawa sector and the replication of three families remain mysteries within this theory.

Further reading

An excellent source especially for phenomenological aspects of the SM is [DGH94]. The Feynman rules for the electroweak model in $R_\xi$ gauge are derived e.g. in Romão and Silva (2012); this reference provides also conversion rules between various sign conventions commonly used discussing gauge theories.

Problems

11.1 Fermi constant $G_F$.
Find the connection between the charged current in the Fermi and the GSW theory, and determine thereby the numerical value of $v$.

11.2 Mixing matrices.
Derive the number of free parameters in the quark and neutrino mixing matrices as the sum of real mixing angles $\theta_i$ and phases $\phi_i$ for $n$ generation of Dirac and Majorana fermions, respectively. Take into account that not all phases of a fermion field need to be observable.

11.3 Feynman rules for Higgs-gauge sector.
Derive the Feynman rules for the vertices between the scalar and the vector sector.

11.4 Higgs decay $h \rightarrow \gamma \gamma$.
Calculate the matrix element for the process $h \rightarrow \gamma \gamma$ following the example $h \rightarrow gg$.

11.5 Higgs decay $h \rightarrow W^+W^-$. 
Calculate the matrix element and the decay width for Higgs decay into transverse and longitudinal W-bosons, $h \rightarrow W_T^+W_T^-$ and $h \rightarrow W_L^+W_L^-$. Consider then the decay into Goldstone bosons, $h \rightarrow w^+w^-$, and show that $\mathcal{M}(h \rightarrow W_T^+W_T^-) = \mathcal{M}(h \rightarrow w^+w^-) + \mathcal{O}(m_W/m_h)$.

11.6 Scattering of longitudinal gauge bosons.
Consider the scattering of longitudinal gauge bosons, $W_L^+W_L^- \rightarrow W_L^+W_L^-$, which can be found to $\mathcal{O}(M_W^2/s)$ from the Goldstone boson scattering. Find the s-wave amplitude in the limit $m_h \ll s$ and $m_h \gg s$. Apply the unitarity condition (11.44) and derive the limits where perturbative unitarity breaks down and the Higgs-gauge sector of the SM becomes a strongly interacting theory.

11.7 Goldstone boson fermion couplings.
Derive the Feynman rules for the vertices of fermions with charged or neutral Goldstone bosons $G$ in $R_\xi$ gauge. Use

$$\phi(x) = \left( \frac{1}{\sqrt{2}} \begin{pmatrix} G^+(x) \\ v + h(x) + iG^0(x) \end{pmatrix} \right)$$

(11.63)

to read off the vertices of Goldstone bosons with fermions (assuming no mixing/one generation).
12. Phase transitions and topological defects

We introduced at the end of chapter 9 the idea that spontaneously broken symmetries of field theories could be restored at high temperatures. In this case the hot, early Universe would be in a symmetric phase, followed by transitions to phases with more and more broken symmetries. Phase transitions in the early Universe can lead to observable consequences for mainly two reasons: First, the state of the Universe deviates from thermodynamical equilibrium during a first-order phase transition. This is an important pre-requisite that processes like baryogenesis which require out-of-equilibrium conditions can work successfully. Second, phase transitions lead generically to the formation of topological defects. These defects are zero-, one- or two-dimensional extended solutions of the classical equation of motions which contain in their core the unbroken vacuum $\langle \phi \rangle = 0$. Depending on the symmetry breaking scale $\langle \phi \rangle = 0$ and their dimensionality, they lead to (un-) desirable cosmological consequences and can thus be used to constrain particle physics models beyond the SM.

12.1. Phase transitions in the abelian Higgs model

Effective potential at $T > 0$ We can include quantum and thermal fluctuations by studying the temperature-dependent effective potential at the one-loop level. We obtain the latter replacing vacuum expectation values with thermal averages in the definition of classical fields, the effective action and effective potential at non-zero temperatures. In particular, we can transform the $T = 0$ effective potential \[ V_{\text{eff}}^{(1)}(\phi) = V(\phi) + \frac{\hbar}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left[ k^2 + V''(\phi) \right] + O(\hbar^2), \] (12.1) into the temperature-dependent effective potential replacing the integration over the continuous energy $k_0$ by a summation over discrete Matsubara frequencies $\omega_n$,

$$\beta V_{\text{eff}}(\phi) = \beta V(\phi) + \frac{1}{2} \sum_n \int \frac{d^3 k}{(2\pi)^3} \ln \left[ \omega_n^2 + k^2 + V''(\phi) \right] + O(\hbar^2).$$ (12.2)

The sum over $n$ is performed in the same way as in the calculation of the free energy $\mathcal{F}$ of a free scalar gas in section 9.2.1.

$$\beta V_{\text{eff}}^{(1)}(\phi) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left[ \beta E_k + 2 \ln(1 - e^{-\beta E_k}) \right],$$ (12.3)

where we replaced also $V''(\phi) = m^2$. Next we split the one-loop contribution $\beta V_{\text{eff}}^{(1)}(\phi)$ into the $T = 0$ vacuum part $V_{\text{eff}}^{(1)}(\phi, 0)$ and a temperature dependent thermal part $V_{\text{eff}}^{(1)}(\phi, T)$,

$$V_{\text{eff}}^{(1)}(\phi) = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left[ k^2 + m^2 \right] + T \int \frac{d^3 k}{(2\pi)^3} \ln \left( 1 - e^{-\beta E_k} \right),$$ (12.4)
12. Phase transitions and topological defects

where we dropped a $\phi$-independent constant.

In order to simplify the discussion we consider separately the two limiting cases that quantum or thermal fluctuations dominate. In the latter case, $V^{(1)}_{\text{eff}}(\phi, 0) \ll V^{(1)}_{\text{eff}}(\phi, T)$, we evaluate $V^{(1)}_{\text{eff}}(\phi, T)$ in the high-temperature limit (compare to the derivation of (9.23)),

$$V^{(1)}_{\text{eff}}(\phi, T) = T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta E_k}) = \frac{T^4}{2\pi^2} \int_0^\infty dx x^2 \ln(1 - e^{-\sqrt{x^2 + a^2}}) = -\frac{\pi^2 T^4}{90} + \frac{a^2 T^4}{12} + \ldots .$$

(12.5)

with $a = \beta V''(\phi)$. For our standard choice $V''(\phi) = \mu^2 + \frac{1}{2} \lambda \phi^2$, the effective potential becomes

$$V(\phi) + V^{(1)}_{\text{eff}}(\phi, T) = -\frac{\pi^2 T^4}{90} - \frac{1}{2} \mu^2 \phi^2 + \frac{1}{24} \lambda \phi^2 T^2 + \frac{\lambda}{12} \phi^4 .$$

(12.6)

The terms quadratic in the field $\phi$ agree with the thermal or Debye mass $m_D^2 = \frac{\lambda T^2}{2\pi^2}$ we found in (9.42). The minimum of $V_{\text{eff}}(\phi)$ moves smoothly away from $\phi = 0$ below the critical temperature, no barrier is formed which implies a second order phase transition.

An example for a first order phase transition is given by the abelian Higgs model. In the limit $e^4 \gg \lambda$, the $T = 0$ part of the radiative corrections due to the vector boson is important. Neglecting the $T > 0$ correction, the effective potential is

$$V^{(1)}_{\text{eff}}(\phi, T) = -\frac{\mu^2 \phi^2}{2} (1 - \xi) + \frac{\lambda \phi^4}{4!} \left(1 - \frac{3}{2} \xi\right) + \frac{\xi \omega^4}{4} \left[\ln \left(\frac{\phi^2}{\mu^2}\right)\right] ,$$

(12.7)

where $\xi = \frac{3e^4}{16\pi^2 \lambda}$ parametrises the relative size of the loop corrections due to scalar and gauge bosons. For $\xi \geq 2$, a barrier between the true and the false vacuum is formed and a first-order phase transition results.

12.2. Decay of the false vacuum

When the Universe goes through a first order phase transition cooling down, the minimum at $\phi = 0$ in the potential of a set of scalar fields may change from being the global to be a (false) local minimum. Since there is a barrier between the two minima, the field $\phi$ has to tunnel from the false to the true minimum. The true vacuum nucleates at a localized position in space-time, when a quantum (or thermal) fluctuation tunnels through (or above) the barrier. It forms an extending bubble which contains the energetically favored ground-state. In the case of thermal fluctuations, this phenomenon is known to everybody from boiling water.

The equivalent problem in quantum mechanics is the tunneling through the barrier in a double-well potential $V(x)$ depicted in the left panel of Fig. 12.1. The tunneling probability $P$ can be calculated using the WKB method as

$$P \sim \exp \left(-i \int_a^b dx \sqrt{2m[V(x) - E]}\right) ,$$

(12.8)

where $a$ and $b$ are the beginning and end point of the tunneling trajectory. In order to translate this prescription to a field theory, we should rewrite the tunneling probability $P$ first as an Euclidean path integral. The exponent is the integral over the (imaginary) momentum of the particle, which we can express as

$$i \int dx p = i \int dt p \dot{x} = i \int dt (E + L) = \int dt E (-L_E) .$$

(12.9)
In the last step we assumed that the energy of the particle is normalized to zero and changed to Euclidean time \( t_E = it \). Now we can rewrite the tunneling probability as an Euclidean path integral,

\[
P = \int Dx \exp(-S_E[x])
\]

(12.10)

with

\[
S_E[x] = \int dx \left[ \frac{1}{2} \frac{d^2 x}{d t_E^2} + V(x) \right].
\]

(12.11)

Since the potential \( V \) changes sign performing a Wick rotation, a classically forbidden path in Minkowski space corresponds to an allowed Euclidean path shown in the right panel of Fig. 12.1. Thus finding the tunneling probability between two vacua amounts to finding solutions in the Euclidean theory which connect the two vacua and minimize the action.

**Figure 12.1.** Tunneling through a double-hill potential in Minkowski space corresponds to the classically allowed path from one maxima to another one in Euclidean space.

**Bounce solution** We now turn to the case we are interested in, namely the tunneling of a scalar field \( \phi \) sitting in the false, metastable vacuum \( a \) into the true vacuum \( b \), cf. Fig. 12.2. In Euclidean space, we should find solutions of the classical action starting from \( b \) which bounce at the classical turning point \( a \) back to \( b \). These solutions are called “bounce” or instantons, since the tunneling happens instantaneously in Minkowski space. Moreover, these solutions should have a finite action such that they give a non-zero contribution to the semi-classical limit of the path integral. In principle, we should find all solution with a finite action and then integrate their contribution. Since the tunneling probability is however exponentially suppressed, the integral is dominated by the solution which minimizes the action.

Solutions which are symmetric under rotations minimize the gradient energy. We use therefore an \( O(4) \) symmetric ansatz, where the real scalar field \( \phi \) is only a function of the radial coordinate \( r^2 = x^2 + t_E^2 \). Then the Euclidean action for a real scalar field becomes

\[
S = 2\pi^2 \int_0^\infty dr \left[ \frac{1}{2} \left( \frac{d\phi}{dr} \right)^2 + V(\phi) \right]
\]

(12.12)

and the field equation simplifies to (problem 12.2)

\[
\frac{d^2 \phi}{dr^2} + \frac{3}{r} \frac{d\phi}{dr} - \frac{dV}{d\phi} = 0.
\]

(12.13)
12. Phase transitions and topological defects

\[ V(\phi) \]

\[ \phi_b \quad \phi_a \]

Figure 12.2.: Left: Transition from the false vacuum \((a)\) to the true one \((b)\) requires in Minkowski space tunneling through a barrier. Right: In Euclidean space, a classically allowed path starts from \(b\) and bounces at \(a\) back to \(b\); the zero point of the potential is chosen to agree with \(a\).

The boundary conditions \(\phi \to 0\) for \(r \to \infty\) and \(d\phi/dr|_{r=0} = 0\) ensure a regular solution at \(r = 0\) and a finite action. Viewing \((12.13)\) as a mechanical problem, it describes the motion of a particle in a potential with the friction term \(\frac{3}{r} \frac{d\phi}{dr}\). Thus the two parts \(a \to b\) and \(b \to a\) of the bounce are not equivalent: We should choose the starting position \(\phi(t = -\infty)\) uphill of \(b\) such that at \(t = +\infty\) the field comes to rest at \(a\).

We can proceed analytically, if we consider the limiting case that either the potential difference between the true and the false vacuum is much smaller or much larger than the potential barrier which separates them. We will consider here the first case, which corresponds to the so called thin-wall approximation. This allows us to approximate the potential as

\[ V(\phi) = V_0(\phi) + \mathcal{O}(\varepsilon), \tag{12.14} \]

where \(V_0(\phi)\) is the symmetric potential and the difference \(\varepsilon = V(\phi_+) - V(\phi_-)\) between the false and the true minima is a small parameter. For small \(\varepsilon\), the volume energy \(\propto \varepsilon r^4\) of a bubble dominates the surface energy \(\propto r^3\) only, if the bubble is sufficiently large. In this case, the thickness of the bubble wall, i.e. the region where \(d\phi/dr\) deviates significantly from zero, is also much smaller than the bubble size. Thus we can neglect the \(\frac{3}{r} \frac{d\phi}{dr}\) term in \((12.13)\), and the field equation simplifies to the one of an one-dimensional problem,

\[ \phi'' \equiv \frac{d^2\phi}{dr^2} = \frac{dV_0}{d\phi}. \tag{12.15} \]

Using the chain rule, \(dV_0/d\phi = V_0' / \phi'\) and \((\phi')^2 = 2\phi'' \phi'\), we find after one \(x\) integration

\[ \frac{1}{2} \phi'^2 - V_0 = c. \tag{12.16} \]

We determine the integration constant by asking that the contribution to the action \(S\) goes to zero for \(r \to \infty\). With \(\phi'(\infty) = 0\), this gives \(c = -V(\phi_+)\). Separating variables in \((12.16)\) leads to

\[ r = \int \frac{d\phi}{\sqrt{2[V_0(\phi) - V(\phi_+)])}}. \tag{12.17} \]
12.2. Decay of the false vacuum

To gain more insight, we consider now a specific potential. We choose our favorite potential,

\[ V_0(\phi) = \frac{\lambda}{4} (\phi^2 - \eta^2)^2. \]  

(12.18)

Compared to chapter 10, we shifted the potential such that the two minima at \( \phi = \pm \eta = \pm \sqrt{\mu^2/\lambda} \) have as required the value zero, \( V_0(\phi_{\pm}) = 0 \). Inserting this potential into (12.17), we find

\[ r = \mp \frac{\sqrt{2}}{\eta \sqrt{\lambda}} \arctanh(\phi/\eta) + r_0. \]  

(12.19)

The integration constant \( r_0 \) determines the position of the bubble wall. Inverting (12.19) gives as field profile

\[ \phi_{\pm}(r) = \mp \eta \tanh \left( \frac{\eta \sqrt{\lambda}}{\sqrt{2}} (r - r_0) \right). \]  

(12.20)

The solution \( \phi_{\pm} \) interpolate between the two vacua \( \pm \eta \) of the symmetric potential \( V_0(\phi) \). The thickness of the bubble wall is determined by the argument of the tanh and is given by \( \delta \sim \sqrt{2}/(\eta \sqrt{\lambda}) \).

Knowing the solution \( \phi(r) \), we can calculate the action. For \( r \gg r_0 \), the field is constant \( \phi \approx \eta \) and \( V(\phi_{\mp}) = 0 \). Therefore, the contribution \( S_> \) from this region to the action is zero. For \( r \ll r_0 \), the field \( \phi \approx -\eta \) is again constant, but the potential contributes \( V(\phi_-) \approx -\varepsilon \) or

\[ \frac{1}{2} (\phi')^2 + \frac{dV}{dr} \approx -\varepsilon \]

and thus

\[ S_< = -\frac{1}{2} \pi^2 \varepsilon r_0^4. \]  

(12.21)

Finally, the contribution from the bubble wall is with (12.16)

\[ S = 2\pi^2 r_0^3 \int dr \left[ \frac{1}{2} (\phi')^2 + V(\phi) \right] = \int dx 2V(\phi) = \]  

(12.22)

\[ = 2\pi^2 r_0^3 \int_{\phi_1}^{\phi_2} d\phi \sqrt{2V(\phi)} = 2\pi^2 r_0^3 I, \]  

(12.23)

with \( \phi_{1/2} \approx \phi(r_0 \mp \xi \delta) \) and \( \xi = \mathcal{O}(1) \). The quantity

\[ I = \int_{\phi_1}^{\phi_2} d\phi \sqrt{2V(\phi)} \]  

(12.24)

has the interpretation of the surface tension of the bubble. Adding the two contributions, the total action follows as

\[ S = -\frac{1}{2} \pi^2 \varepsilon r_0^4 + 2\pi^2 r_0^3 I. \]  

(12.25)

We find the solution which gives the largest tunneling probability minimizing the action \( S \) w.r.t. \( r_0 \),

\[ r_0 = \frac{3I}{\varepsilon}. \]  

(12.26)
For our choice \((12.17)\) for the potential, the surface tension is \(I = 2\mu^3/(3\lambda)\) and the action
\[
S = \frac{27\pi^2 I^4}{2\varepsilon^3} = \frac{16\pi^2 \mu^{12}}{3\lambda^4 \varepsilon^3}.
\]
(12.27)

For \(\varepsilon \to 0\), the action becomes infinite. Equivalently, the transformation from one to another ground-state \(\pm \eta\) of the symmetric potential \(V_0(\phi)\), costs an infinite amount of energy, as we have argued in chapter 10.

The tunneling probability per time and volume follows then as
\[
p = A \int \mathcal{D}\phi \exp(-S[\phi]),
\]
where the pre-factor \(A\) is determined by \(\text{Det}(\Box - V''(\phi))^{-1/2}\) and its zero-modes \([CC77]\). In practice, the result is rather insensitive to the exact value of \(A\), and setting \(A \sim V''(\phi_0)\) is sufficient for simple estimates.

We can now justify the assumptions made: Our starting assumption that \(\varepsilon\) is small implies that \(r_0\) is large, \(r_0 \propto 1/\varepsilon\). Moreover, a small \(\varepsilon\) implies that the bubble of true vacuum is separated by a thin wall from the metastable vacuum, \(\delta \sim (\varepsilon r_0)^{-1/3}\). Therefore, this approximation is called the thin-wall approximation.

In order to describe the time evolution of a bubble, we have to continue analytically the O(4) symmetric solution \(r_0^2 = x^2 + t_0^2\) back to Minkowski space. From \(r_0^2 = x^2 - t^2\), we see that the bubble extends close to the speed of light for \(t \gg r_0\). The resulting O(3,1) symmetry means that all observers\(^1\) in inertial systems will measure the same expansion law.

**Scalar instantons** For the case of a massless \(\lambda\phi^4\) theory we can find a class of exact solutions for the bounce \((12.13)\), which are often called scalar instantons. For a massless particle, the classical solution should fall off as a power-law. Then dimensional analysis tells us that \(\phi(r) \sim 1/r\) for \(r \to \infty\). Since the massless theory is scale-invariant, the solutions should be parametrised by an arbitrary parameter \(\rho\) characterising their size. This suggests to insert as ansatz \(\phi(r) \propto \rho/(r^2 + \rho^2)\) into \((12.13)\) what results in (problem 12.3)
\[
\phi(r) = \left(\frac{8}{\lambda}\right)^{1/2} \frac{\rho}{r^2 + \rho^2}.
\]
(12.29)

Thus the action of the bounce becomes
\[
S = \frac{27\pi^2 I^4}{2\varepsilon^3} = \frac{8\pi^2}{3\lambda}.
\]
(12.30)

Although the SM vacuum is metastable, the probability that inside the past-light cone of the observed universe a tunneling happened is practically zero.

**Example:** Application to the metastable SM vacuum:
We use \(V = \lambda_{\text{eff}} \phi^4/4\) as crude approximation for the scalar potential of the SM, where \(\lambda_{\text{eff}}\) is the running coupling at the scale \(\mu = \phi\). We can check using the thin-wall approximation that the exact result \((12.30)\) makes sense: The surface tension is \(I \sim |\lambda_{\text{eff}}|/2\phi^3/3\) and \(\varepsilon \sim |\lambda_{\text{eff}}|\phi^4/4\). Thus the action of the bounce becomes
\[
S = \frac{27\pi^2 I^4}{2\varepsilon^3} = \frac{8\pi^2}{3|\lambda_{\text{eff}}|}.
\]

---

\(^1\)The facts that the bubble extends with \(v \sim 1\) and that the bubble is thin imply (un?)-fortunately that observers will be dissolved without having the time to notice.
12.3. Topological defects

agreeing with (12.30). An analysis of the RGE of the SM gives for the effective Higgs boson coupling at the Planck scale $\lambda_{\text{eff}}(M_{\text{Pl}}) \approx -0.01$. For the estimate of the probability $P$ that a bubble of the true vacuum has nucleated in the past-light cone of an observer, we can set $V T \sim t_0^4$ with $t_0$ as the present age of the universe. Setting the pre-factor $A$ by dimensional reasons equal to $A \sim \phi^4 \sim M_{\text{Pl}}^4$, the tunneling probability is
\[
P \sim (t_0 M_{\text{Pl}})^4 \exp\{-8\pi^2 / (3|\lambda_{\text{eff}}|)\} \sim 10^{-11000}.
\]

Despite of the enormous volume factor, $(t_0 M_{\text{Pl}})^4 \sim (8 \times 10^{60})^4$, the tunneling probability is exceedingly small.

12.3. Topological defects

A rather generic consequence of phase transitions is the formation of extended solutions to the field equations which are stable by virtue of a topological quantum number. The latter arises if the space of possible field configurations is given a non-trivial topological structure by the condition that a functional $F[\phi]$ of the fields $\phi$ as the potential energy or the Euclidean action is finite.

Two field configurations $\phi_1$ and $\phi_2$ are topologically equivalent, if they can be transformed continuously into each other keeping the functional $F[\phi]$ finite during the transformation. Thus topologically equivalent field configurations form equivalence classes which can characterized by a topological quantum number and are separated by an infinite energy barrier. Within each equivalence class, the minimal energy solution will be stable. The spatially uniform ground-state we have assumed up to now as our vacuum in perturbative calculations is thus disconnected from other stable solutions which cannot spread out to become spatially uniform due to their non-trivial topology.

As a system crosses its critical temperature cooling down, it performs a transition from its symmetric state to a state with broken symmetry. While the correlation length $\xi$ becomes formally infinite at the phase transition, the order parameter in the broken phase can take the same value only in a finite volume, restricted by the finite propagation speed of the relevant waves. In the case of the expanding Universe, we expect the formation of order one topological defect per causally connected region.

Domain walls are two-dimensional topological defects which separate three-dimensional volumes containing a different vacua. Examples are ferromagnets where domains of uniform magnetization exist. Depending on the gauge group and the pattern of symmetry breaking, topological defects with one and zero dimensions are also possible: In the first case, an one-dimensional line or string contains a vacua different from the surrounding, while it is the second case a point-like object called monopole.

We will proceed as in chapters 10 and 11, moving from a Higgs models with a single scalar field and increasing then their number. At the same, the dimensionality of the topological defects formed will decrease. As start, we will consider however the Sine-Gordon model which is defined in 1+1 space-time dimensions.

Sine-Gordon solitons We have chosen in general as potential $V(\phi)$ a polynomial in the field $\phi$. In the case of SSB, the periodicity of the angular variable in $\phi = \eta e^{i \vartheta}$ leads however to a periodicity of the potential. An example for a Lagrangian with a periodic potential is the
Sine-Gordon model defined by

\[ \mathcal{L} = \frac{1}{2}(\partial_{\mu} \phi)^2 - V(\phi) = \frac{1}{2}(\partial_{\mu} \phi)^2 - \frac{a}{b^2}[1 - \cos(b\phi)] \]  

(12.32)

with \( \mu = 0, 1 \) and two real parameters \( a \) and \( b \). Expanding the potential \( V(\phi) \) for small \( \phi \),

\[ \mathcal{L} = \frac{1}{2}(\partial_{\mu} \phi)^2 - \frac{1}{2}a\phi^2 - \frac{ab^2}{4!}\phi^4 + \ldots \]  

(12.33)

and comparing to our usual \( \lambda\phi^4 \) potential, we see that they agree for small \( \phi \) if we identify\(^2\)

\( a = m^2 \) and \( b^2 = \lambda \). The potential of the Sine-Gordon model is periodic, and has zeros for

\( \phi = 2\pi n/\sqrt{\lambda} \).

From the Lagrangian \( \mathcal{L} \), the wave equation

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{a}{b}\sin(b\phi) = 0 \]  

(12.34)

follows. It admits static and traveling wave solutions, \( \phi(x,t) = f(\pm(x - vt)) \equiv f(\pm \xi) \). You can check that

\[ f(\xi) = \frac{4}{b}\arctan[\exp(\pm \gamma \sqrt{a/b}\xi)] \]  

(12.35)

is a solution with \( \gamma = (1 - v^2)^{-1/2} \) as usual Lorentz factor, problem \( 15.1 \). The solution with the plus sign is called a kink, the one with minus an anti-kink. The kink interpolates between the \( n = 0 \) ground-state at \( x \rightarrow -\infty \) and \( n = 1 \) \( (\phi = 2\pi/\sqrt{\lambda}) \) at \( x \rightarrow \infty \). The extension \( \ell \) of the soliton is determined by the argument of the arctan and is given by \( \ell \sim \sqrt{b/(\gamma \sqrt{a})} = \lambda/(\gamma m) \).

Figure 12.3.: Left: Kink solution \( \phi(\xi) \) for \( a = b = 1 \); Right: its derivative \( d\phi(\xi)/d\xi \).

The energy (or mass) of a static solution interpolating between \( n = 0 \) and \( n = 1 \) is

\[ E = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + 2V(\phi) \right] = \int_{-\infty}^{\infty} dx \ 2V(\phi) = \int_0^{2\pi/b} d\phi \frac{\partial \phi}{\partial x} V(\phi) = \int_0^{2\pi/b} d\phi \sqrt{2V(\phi)} = \frac{\sqrt{2a}}{b^2} \int_0^{2\pi} d\vartheta [1 - \cos(\vartheta)]^{1/2} = \frac{8m}{\lambda} \]  

(12.36)

(12.37)

\(^2\)Recall that a scalar field has dimension \( [\phi] = (n - 2)/2 \) in \( n \) space-time dimensions. Thus the argument of \( \cos(b\phi) \) is dimensionless while the \( \phi^4 \) interaction requires a \( m^2 \) pre-factor.
Here we used in the second step the requirement that $E < \infty$, i.e. Eq. (12.16) with $c = 0$. More generally, a static solution connecting $n_1$ and $n_2$ has the energy density $E/A = 8|n_1 - n_2|m/\lambda$. Thus the energy spectrum of these solutions is discrete and inverse proportionally to the coupling constant. The latter property implies that these solutions are not accessible in a perturbative calculation which is a power series in the coupling constant $\lambda$. In the weak coupling regime, the mass $8m/\lambda$ of the soliton is much larger than the mass $m$ of the elementary field $\phi$, while in the strong coupling limit the soliton is the lightest excitation.

We can imagine the Sine-Gordon model as a chain of arrows, which the force $F = -\partial_x V$ tries to orient e.g. downwards. Then a kink is a solution which points from $x = -\infty \ldots -\ell$ downwards, turning at $\xi \sim 0$ by $2\pi$, and pointing at $x \gtrsim \ell$ again downwards. To untwist the chain, we would have to turn the arrows on one of the two side of $\xi$ which would cost an infinite amount of energy. Therefore the winding number $n = n_1 - n_2$ is a conserved quantum number and the solutions are stable. We can understand why the winding number is called a topological quantum numbers as follows: We can map $\mathbb{R}$ on a compact interval by identify the points $x = -\infty$ and $x = \infty$. Then the kink becomes a Möbius band, which cannot be smoothly transformed into on a circle $S^1$.

A conserved quantum number implies the existence of a conserved current: The condition $E < \infty$ requires that the field approaches for $x \pm \infty$ one of the vacuum states, and thus

$$\phi(\infty) - \phi(-\infty) = \frac{2\pi}{\sqrt{\lambda}} \nu. \quad (12.38)$$

We can rewrite this is an integral,

$$\int_{-\infty}^{\infty} dx \partial_x \phi = \frac{2\pi}{\sqrt{\lambda}} \nu. \quad (12.39)$$

Since the solution is two-dimensional, we can set as current

$$j^\mu = \frac{\sqrt{\lambda}}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \phi. \quad (12.40)$$

The antisymmetry of $\varepsilon^{\mu\nu}$ ensures then that $\partial_\mu j^\mu = 0$. The conserved charge follows with $\varepsilon^{01} = 1$ as

$$Q = \int dx j^0 = \frac{\sqrt{\lambda}}{2\pi} \int dx \partial_x \phi = \frac{\sqrt{\lambda}}{2\pi} [\phi(\infty) - \phi(-\infty)] = n_1 - n_2 , \quad (12.41)$$

i.e. equals the winding number $\nu$. In contrast to the conserved charges we have encountered up to now, the current is not the Noether current of a global symmetry and we did not have to use the equation of motions in its derivation. Instead, the charge has a topological origin.

Solitons maintain their shape, although they are the solution of a non-linear equation. The distinctive property of the Sine-Gordon solitons in 1+1 space-time dimensions is that this holds also for the asymptotic regions of a scattering event: In particular, two solitons emerge from a collision unchanged except possibly for a phase shift, although the superposition principle is not valid.

**Domain walls** We now move to topological defects in 3+1 dimensions considering two-dimensional topological defects called domain walls. Assuming that the domain wall is uniform in the $y - z$ plane, its action is

$$S[\phi] = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} (\phi')^2 + V(\phi) \right] . \quad (12.42)$$
Instead of integrating the equation of motions, as in Eqs. \(12.15\) to \(12.17\), we will use now an argument due to Bogomolnyi: He suggested to rewrite the action “completing the square” as

\[
S[\phi] = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \phi' \mp \sqrt{2V(\phi)} \right]^2 \pm \int_{\phi(-\infty)}^{\phi(\infty)} d\tilde{\phi} \sqrt{2V(\tilde{\phi})}.
\]

The second integral depends only on the boundary values of the field at infinity. For field configurations which connect the same ground-state at \(x = -\infty\) and \(+\infty\), this boundary term vanishes, and the minimum \(S[\phi] = 0\) of the action is attained for constant fields. If the field configurations connect however different ground-states, then we can associate the second integral to a non-zero topological charge. The winding number in the Sine-Gordon case is a specific example for such a topological charge.

A lower bound for the action for fields with non-zero topological charge is

\[
S[\phi] \geq \int_{-\eta}^{\eta} d\tilde{\phi} \sqrt{2V(\tilde{\phi})}
\]

which is attained when the first integral vanishes. In this case, we arrive separating again variables at

\[
x = \pm \int \frac{d\phi}{\sqrt{2V(\phi)}}.
\]

To proceed, we have to choose a definite potential. For the choice \(12.18\), we come back to the solution

\[
\phi_{\pm}(x) = \mp \eta \tanh \left( \frac{\eta \sqrt{\lambda}}{\sqrt{2}} (x - x_0) \right)
\]

we have used in the calculation of the bounce. The solution \(\phi_{\mp}\) interpolates between the two vacua \(\pm \eta\) of the symmetric potential \(V_0(\phi)\) and contains at \(x = x_0\) a two-dimensional plane with the unbroken vacuum \(\phi = 0\).

More generally, two-dimensional surfaces of finite size which separate domains with opposite values of \(\eta\) are formed during a phase transition. They are called domain walls; their cosmological implications are discussed later on.

**Global cosmic strings** We continue our way through possible topological defects looking at the consequences of a global continuous symmetry. Thus we replace the single real field by a set of two real or one complex field \(\phi = \phi_1 + i\phi_2\) with potential

\[
V(\phi) = \frac{\lambda}{4} \left( \phi^\dagger \phi - \eta^2 \right)^2.
\]

We use cylinder coordinates \(\rho, \xi, z\), and search for static solutions using as ansatz

\[
\phi = \eta e^{i\eta \theta} f(\rho).
\]

The phase \(e^{i\eta \theta}\) can be chosen arbitrarily at \(\rho = 0\), unless the field is zero at the origin. In order to ensure a single-valued field, we have therefore to impose the boundary condition \(f(\rho) \to 0\) for \(\rho \to 0\). In the opposite limit, \(\rho \to \infty\), the field should approach one of its minima,

\[
\phi \to \eta e^{i\eta \theta} \quad \text{and} \quad f(\rho) \to 1,
\]
in order to minimize the energy.

For the ansatz (12.48), the field equations are separable and we find from $\Delta \phi = \lambda (\phi^2 - \eta^2)\phi$ as differential equation for $f$

$$\frac{d^2 f}{d\xi^2} + \frac{1}{\xi} \frac{df}{d\xi} - \frac{n^2}{\xi^2} f = \xi(f^2 - 1)f. \quad (12.50)$$

Here we have introduced also the new dimensionless variable $\xi = \lambda^{1/2} \eta\rho$. This differential equation together with the two boundary conditions for $\rho \to 0$ and $\rho \to \infty$ has to be solved numerically.

We can however estimate the extension of a cosmic string by dimensional arguments. The scale of the problem is set by $\rho/\xi = \lambda^{1/2} \eta^{-1}$. Thus the core containing the unbroken vacuum $\phi = 0$ has the radius $O(\lambda^{1/2} \eta^{-1})$, while for larger distances $\rho \gg \lambda^{1/2} \eta^{-1}$ the field approaches the broken phase $|\phi| = \eta$.

The extended solution should have again a finite energy $E$ per length $L$,

$$\frac{E}{L} = \int_0^\infty d\rho \int_0^{2\pi} d\vartheta \left[ |\nabla \phi|^2 + V(\phi) \right] \quad (12.51)$$

$$= \int_0^\infty d\rho \int_0^{2\pi} d\vartheta \left[ \partial_\rho \phi \partial_\rho \phi + \frac{1}{\rho^2} \partial_\vartheta \phi \partial_\vartheta \phi + V(\phi) \right]. \quad (12.52)$$

The first and the last term give a finite contribution, since $\partial_\rho f(\rho) \to 0$ and $V(\phi) \to 0$ for $\rho \to \infty$. In contrast, the middle term contributes $\sim \int_0^R d\rho \rho^{-1} f(\rho)$, i.e. a logarithmic diverging term $\ln(\rho/\delta)$ to the linear energy density. The scale $\delta$ has to be determined by the typical extension of the string, and thus the energy density inside the radius $R$ around the string is $E/L \sim \ln[R/(\lambda^{1/2} \eta)]$. If we consider instead of the idealization of an isolated global string the realistic case of a string network, then $R$ should be given by the typical distance of strings.

**Local cosmic strings** We can avoid the (formal) problem of the infinite energy associated with a string if we gauge the model. In this case we obtain the abelian Higgs model,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - V(\phi). \quad (12.53)$$

The kinetic term $D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi$ contains now two contributions which can cancel for $\rho \to \infty$. We require therefore that $A_\mu$ is a pure gauge field for $\rho \to \infty$ with

$$A_\mu \sim -\frac{i e}{\rho} \partial_\mu \ln(\phi/\eta). \quad (12.54)$$

Then $D_\mu \phi$ and $F_{\mu\nu}$ approach zero for $\rho \to \infty$, and the energy density per length of a local string is finite.

A local string carries magnetic flux of the corresponding gauge field. Integrating (12.54) around a circle in a plane with $z =$ const. and large $\rho$ gives

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{A} \cdot dl = -\frac{i}{e} \int_0^{2\pi} d\vartheta \partial_\rho (\ln \rho) = \frac{2\pi n}{e}. \quad (12.55)$$

Thus the magnetic flux carried by local strings is quantised in units of $2\pi/e$. If we go back to cgs units, then the unit of magnetic flux becomes $hc/e$, determined completely by the

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<td>3</td>
<td>0</td>
<td>$\pi_2(\mathcal{M}_0)$</td>
</tr>
<tr>
<td>texture</td>
<td>4</td>
<td>-</td>
<td>$\pi_3(\mathcal{M}_0)$</td>
</tr>
</tbody>
</table>

Table 12.1.: The number $n$ of scalars determines the dimension of the vacuum manifold $\mathcal{M}_0$ ($= S^{n-1}$ for the potential (12.56)) and the dimension $3-n$ of the hypersurface containing the unbroken vacuum.

three fundamental constants of a relativistic quantum theory. A local cosmic string with winding number $n$ carries $n$ flux quanta, analogous to quantized tubes of magnetic flux in a superconductor.

The coupled field equations for $\phi$ and $A_\mu$ can be solved in general only numerically and therefore we only estimate the extension and the energy of a local string. At the core of the string, the vacuum is unbroken, $\langle \phi \rangle = 0$, leading to an potential energy density $V(0) = \lambda \eta^4/4$. The extension of the string is controlled by the Compton wave-lengths

$$\lambda_A = \frac{1}{m_A} = \frac{1}{e v} \quad \text{and} \quad \lambda_\phi = \frac{1}{m_\phi} = \frac{1}{\sqrt{2} \lambda_\phi^2}$$

of the two massive fields $A_\mu$ and $\phi$. Thus the scalar field contributes to the energy of the string

$$E_\phi/L \sim (\lambda \eta^4/4) \times \pi \lambda_\phi^2 \sim \eta^2.$$  

The energy density of the magnetic field contributes $B^2/(8\pi) \times \pi \lambda_A^2$. From (12.55), we find

$$\Phi = \pi \lambda_A^2 B = 2\pi/e \quad \text{for a string with winding number } n = 1 \quad \text{and thus}$$

$$E_A/L \sim (e \lambda_A)^{-2} \sim \eta^2.$$  

As a result, the magnetic and scalar contributions to the energy density are of the same order, $E/L \sim \eta$, given by the vev $\eta$ of the scalar field.

Global monopoles  Let us try to condense our results: If we split a complex scalar fields intro real ones, $\phi = \phi_1 + i\phi_2$, then we considered in $d = 4$ potentials of the type

$$V(\phi) = \frac{\lambda}{4} \left( \sum_{i=1}^n \phi_i^2 - \eta^2 \right)^2.$$  

(12.56)

For $n = 1$, the possible ground-states $\phi^2 = \eta^2$ correspond to the zero-dimensional sphere $S^0$. The single constraint $\phi(x_1, x_2, x_3) = 0$ defines a two-dimensional surface in $\mathbb{R}^3$, which corresponds to a domain wall. The abelian Higgs model with one complex scalar doublet has $n = 2$. Now $\phi_1^2 + \phi_2^2 = \eta^2$ defines an one-dimensional sphere $S^1$ as manifold of the possible ground-states, while a topological defect defined by $\phi_1 = \phi_2 = 0$ is as section of two surfaces a line.

The logical next step is to consider three scalar fields which transform under $SO(3)$. Then the vacuum manifold is the sphere $S^2$ and we expect zero-dimensional topological defects
which are called monopoles. We sketch only the physical picture and discuss first global monopoles. We use spherical coordinates and search for static solutions using as ansatz

$$\phi_i = \eta h(r) \frac{x_i}{r},$$

which satisfies the requirement

$$\phi_i \phi_i = \eta^2$$

for $|x| \to \infty$, if $h(r) \to \pm 1$ for $r \to \infty$. Additionally, the function $h(r)$ should satisfy the boundary condition $h(r) \to 0$ for $r \to 0$ to ensure a non-singular $\phi(0)$. The energy density at large $r$, where $h(r) \approx \eta$ is given by

$$\rho \sim \frac{1}{2} (\partial_i \phi)^2 \sim \frac{3 \eta}{2 r^2}$$

and thus the energy contained inside a sphere of radius $R$ diverges linearly, $E \approx 3\eta R/2$. In contrast to the mild logarithmic divergence in case of global strings, the behavior $E \approx 3\eta R/2$ is disastrous, leading to a fast collapse of the universe.

The natural way-out is to go on to local monopoles and to check if the gauge fields can compensate the scalar gradient energy. Before we do so, we ask however if there is a way to ensure that the total energy of global monopoles is finite. Imagine a monopole ($h(r) = 1$ for $r \to \infty$) and an anti-monopole ($h(r) = -1$ for $r \to \infty$) pair separated by the distance $d$: Their fields $\phi_i(r)$ will cancel for $r \gg d$, leaving over only higher multipole moments $\phi_i \sim r^{-2}$. Thus the energy density of a monopole-antimonopole pair scales as $\rho \sim \eta r^{-4}$ and therefore their total energy $E$ is finite. Separating a monopole-antimonopole pair would cost an infinite amount of energy—instead a new monopole-antimonopole pair will be created as soon as the potential energy between the pair exceeds a certain threshold. We can view this behavior as a simple model for the confinement of colored particles as quarks and gluons in QCD, where the potential energy $V(r)$ for distances $r \gtrsim \Lambda_{\text{QCD}}^{-1}$ scales also linearly.

**t’Hooft-Polyakov monopoles** The simplest model exhibiting local monopoles is the Georgi-Glashow model. It contains a SO(3) gauge field $A^a_{\mu}$ supplemented by a triplet of real Higgs scalars $\phi^a$. Choosing a uniform vev as $\phi^a = (0,0,v)$ leaves a residual U(1) symmetry unbroken, corresponding to rotations around the 3-axis in isospin space. Thus one can view the Georgi-Glashow model as a toy model for the electroweak sector of the SM, where the gauge field $A^3_{\mu}$ plays the role of the photon and the $Z$ boson is missing.

*t’Hooft and Polyakov showed first that this model contains extended classical solutions which have finite energy and correspond to local magnetic monopoles. Using the adjoint representation $(T^a_A)_{bc} = -if^{abc}$ for the scalar triplet of real Higgs scalars $\phi^a$, the covariant derivative becomes

$$D_i \phi^a = \partial_i \phi^a - e \varepsilon^{abc} A^b_i \phi^c$$

or in vector notation $D_i \phi = \partial_i \phi - e A_i \times \phi$.

We fix again the asymptotic behavior of the gauge fields by requiring that the kinetic term $D_i \phi^a$ vanishes for $r \to \infty$. From

$$\partial_i \phi^a \sim \frac{\delta^{ai} - x^i x^a}{r^2},$$


we conclude that $A_i^b$ should be constant, while (12.60) implies that $A_i$ is perpendicular to $\phi$. Evaluating $D_i \phi^a$ using $\varepsilon^{aij} \varepsilon^{akl} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k$ shows that

$$A_i^a = \frac{v}{e} [1 - f(r)] \varepsilon^{aij} \frac{x^j}{r} \tag{12.61}$$

with $f(r) \to 0$ for $r \to \infty$ leads to the desired asymptotic behavior of the covariant derivative. Now the regularity of the solution $A_i^a$ at $r = 0$ requires $f(0) = 1$.

Inserting the two Ansätze (??) and (12.61) into the energy functional of the Georgii-Glashow model and minimizing the energy results in two coupled differential equations for the functions $H(r)$ and $K(r)$. In general, they have to be solved numerically. Therefore we consider only the limit $\lambda/g^2 = m^2_h / (2m_W^2) \to 0$. Then the potential energy $V(\phi)$ is negligible,

$$V(\phi) = \frac{\lambda}{4} \left( \sum_{a=1}^{3} \phi_i^2 - \eta^2 \right)^2 \to 0 \quad \text{for} \quad \lambda \to 0, \tag{12.62}$$

if the fields satisfy the constraint (12.58). Thus the boundary conditions (12.58) remain valid in the limit considered. Neglecting $V(\phi)$, we can use Bogomolnyi’s trick to derive an exact solution. We express the static energy as

$$E = \frac{1}{2} \int d^3x \left[ (B_i^a)^2 + (D_\mu \phi^a)^2 \right] = \frac{1}{2} \int d^3x \left[ (B_i^a \mp D_i \phi^a)^2 \pm 2B_i^a D_i \phi^a \right] \geq 0, \tag{12.63}$$

where we assumed for simplicity that the monopole carries no electric charge, $A_i^0 = E_i^a = 0$. Then we obtain the bound

$$E \geq \int d^3x B_i^a D_i \phi^a \tag{12.64}$$

which is attained, if the fields satisfy $D_i \phi^a = \pm B_i^a$. Now we have to solve only the simpler first-order equation $D_i \phi^a = \pm B_i^a$. Its solution is called a Bogomolnyi-Parad-Sommerfield (BPS) monopole and, generalising, all solitons which minimise the classical action are denoted as BPS states or BPS solitons.

Finally, we want to determine the charge of a BPS monopole. While the massive fields fall off exponentially for $r \to \infty$, the component of $F_{ij}^a$ connected to the massless photon decrease as a power-law. Moreover, the fields (??) and (12.61) are practically uniform at large distances $r$ from the center of the monopole. Thus in a laboratory at $x$, the field $x F_{ij}$ corresponds to the massless photon, while the two states orthogonal to $F_{ij}$ are the massive weak gauge bosons. This implies that the part of $F_{ij}^a$ which corresponds to electromagnetism is parallel to $\phi^a$,

$$F_{ij} = F_{ij}^a \phi^a \tag{12.65}$$

The magnetic part of the SU(2) field-strength tensor is

$$B_i = B_i^a \frac{\phi^a}{v} = \frac{1}{2} \varepsilon_{ijk} F_{jk}^a = \varepsilon_{ijk} \partial_j A_i^a - \frac{1}{2} \varepsilon_{ijk} \varepsilon^{abc} A_j^b A_k^c \tag{12.66}$$

where we inserted the definition of $F_{jk}^a$. If we are far away from the center of the soliton, we can use the asymptotic expression for the fields, setting $f(r) = 0$ and $h(r) = 1$. Performing the differentiations (problem 12.8) we arrive at

$$B_i^a = \frac{1}{g} \frac{m_i n_a}{r^2}. \tag{12.67}$$
Thus the $U(1)$ magnetic field is given by

$$B_i = \frac{1}{g} n_i r^2.$$  \hspace{1cm} (12.68)

Comparing this to the expression $qn_i/(4\pi r^2)$ for the field of a monopole, we conclude that the magnetic charge of the BPS monopole is $q_m = 4\pi/g$.

**Textures** Finally we should comment on the case $n = 4$ which corresponds to the electroweak model. The four equation $\phi_i(x_1, x_2, x_3) = 0$ have in general no solution in $\mathbb{R}^3$. Thus regions which contain the unbroken vacuum $\phi_i = 0$ will be not formed during the electroweak phase transition. Still, correlations can not exist beyond the horizon scale and thus non-zero gradients $\partial_\mu \phi_i$ can be produced. In case of global textures, static solutions with positive energy density can exist. In case of of a broken gauge symmetry, gauge fields will compensate the $\partial_\mu \phi_i$ term in $D_\mu \phi_i$. Thus local textures as predicted by the SM have no observable consequences.

**Homotopy groups and winding numbers** The topological quantum numbers we meet discussing extended classical solutions can be associated to winding numbers of maps between the ground states of a field theory and its configuration space. Let us define the ground-state or vacuum manifold $M_0$ of a theory as the set of all global minima $V(\phi) = 0$ of its potential,

$$M_0 = \{ \phi : V(\phi) = 0 \}.$$  \hspace{1cm} (12.69)

The condition that the potential energy is finite requires that

$$\lim_{|x| \to \infty} \phi(x) = \phi \in M_0.$$  \hspace{1cm} (12.70)

We can compactify $\mathbb{R}^n$ (where $n$ is the number of spatial dimensions) to the sphere $S^n$ using e.g. a stereographic projection as shown in Fig. 12.4. Then we can view the condition that the potential energy is finite as a mapping $S^n \to M_0$. We ask now the question when two such mappings are topologically distinguished?

We consider only the mappings $S^1 \to S^1 \approx \mathbb{R}^2 \{0\}$ which are easiest to visualise. Two closed loops with base point $x_0$, i.e. curves $x(t)$ with $t \in [0 : 1]$ and $x(0) = x(1) = x_0$, are shown in Fig. 12.5. While the black loop is contractable, the red one is wrapped once around $\{0\}$ and therefore not contractable to its base-point $x_0$ by continuous transformations. We define the maps

$$U^{(\nu)} = e^{i \nu \theta} = \left[U^{(1)}\right]^\nu,$$  \hspace{1cm} (12.71)

which count the number of times a loop is wrapped around the origin. The integer $\nu$ is called the winding number of the map and can be rewritten as an integral,

$$\nu = \frac{i}{2\pi} \int_0^{2\pi} d\theta U \partial_\theta U^\dagger.$$  \hspace{1cm} (12.72)

Clearly, this formula reproduces the correct values for the mappings defined above. Moreover, it is useful for the proof that $\nu$ is invariant under continuous transformations. It is sufficient to investigate an infinitesimal change $\delta U$. Unitarity $UU^\dagger = 1$ implies that $\delta U^\dagger U + \delta U^\dagger U = 0$.
12. Phase transitions and topological defects

Figure 12.4.: A stereographic projection maps points $P \in \mathbb{R}^n$ onto points $P' \in S^n$. The north pole $N$ of the sphere $S^n$ corresponds to the sphere $S_{R}^{n-1}$ at spatial infinity, $x_1^2 + \cdots + x_n^2 = R^2 \to \infty$, of $\mathbb{R}^n$.

Figure 12.5.: Two loops in $\mathbb{R}^2 \setminus \{0\}$ with common base point $x_0$: the black one is contractable and has winding number $\nu = 0$, the red loop is wrapped once around $\{0\}$ and has winding number $\nu = -1$.

and $\delta U^\dagger = -U^\dagger \delta U U^\dagger$. Since we will interested later in the non-abelian version of the winding number, we will not use that the $U$ are commuting complex numbers.

We calculate the variation of the integrand in the integral formula (12.72) for $\nu$,  
\[
\delta(U \partial_\vartheta U^\dagger) = \delta U \partial_\vartheta U^\dagger + U \partial_\vartheta \delta U^\dagger = \delta U \partial_\vartheta U^\dagger - U \partial_\vartheta U^\dagger \delta U U^\dagger - \delta U \partial_\vartheta U \delta U^\dagger - U U^\dagger \partial_\vartheta \delta U U^\dagger - U U^\dagger \delta U \partial_\vartheta U^\dagger \\
= -U \left[ \partial_\vartheta U^\dagger \delta U + U^\dagger \partial_\vartheta \delta U \right] U^\dagger = -U \partial_\vartheta \left[ U^\dagger \delta U \right] U^\dagger.
\]  
(12.75)

Here, we inserted $\delta U^\dagger = -U^\dagger \delta U U^\dagger$ and performed the differentiations. Then the first and fourth terms cancel, and finally we combined the remaining two terms using the product rule. In case of the abelian winding number $\nu$ of (12.72),

\[
\delta \nu = \frac{i}{2\pi} \int_0^{2\pi} d\vartheta \, \delta(U \partial_\vartheta U^\dagger) = -\frac{i}{2\pi} \int_0^{2\pi} d\vartheta \, \partial_\vartheta \left[ U^\dagger \delta U \right] = 0. 
\]  
(12.76)
Thus the winding number $\nu$ is an integer which is invariant under infinitesimal deformations of the loop. Since any continuous transformation can be built up out of infinitesimal ones, the maps $S^1 \to S^1$ can be divided into different equivalence classes, each characterised by one value of $\nu \in \mathbb{Z}$. Only maps within each class can be continuously transformed into each other. Mathematicians say that such maps are homotopic. The theory of homotopy groups addresses the question into how many homotopy classes $\pi_n(X)$ the set of maps $S^n \to X$ can be divided.

Using two results from the theory of homotopy groups one can show that a grand unified theory implies the existence of magnetic monopoles. First, the second homotopy group $\pi_2(G/H)$ of the quotient group $G/H$ equals the first homotopy group $\pi_1(H)$. In our case, $H = SU(3) \otimes U(1)$ and $\pi_1(U(1)) = \mathbb{Z}$ is non-trivial. Thus the sequence

$$\pi_2(G/H) = \pi_1(H) = \pi_1(SU(3) \otimes U(1)) = \mathbb{Z}$$

(12.77)

shows that, if $U_{em}(1)$ is unified at higher scales within a larger semi-simple group, then magnetic monopoles should be produced at the GUT phase transition.

**Summary of chapter**

Classical static solutions of theories with SSB can fall into different equivalence classes which are separated by an infinite potential energy barrier. Depending on the number of Higgs fields, this leads to two-, one or zero-dimensional solitons containing the unbroken vacuum. Instantons or bounces are solutions of the classical field equation in Euclidean space which evolve in Euclidean time between two different vacua. If their action is finite, they describe in Minkowski space instantaneous tunneling from the false to the true vacuum.

### Further reading/sources

Topologically non-trivial solutions of the classical field equations are discussed in more detail in [Rub02]. For a calculation of the effective potential of the SM and the resulting tunneling probability see [Esp13] and the references therein.

### Problems

**12.1 Order of the phase transition.**

Determine the critical temperature $T_c$ of the $\lambda \phi^4$ theory from (12.6). Calculate the pressure and the heat capacity above and below $T_c$ and confirm thereby that the phase transition is of second order.

**12.2 Field equation for the bounce solution.**

Show that (12.13) agrees with the Klein-Gordon equation $\Delta \phi = V'(\phi)$ for a radial symmetric $\phi$.

(Use (1.129) to evaluate $\Delta \phi = \nabla^a \nabla_a \phi$.)

**12.3 Scalar instantons.**

Show that the ansatz $\phi(r) = A r/(r^2 + \rho^2)$ solves (12.13). Determine $A$ and the action $S$.

**12.4 Soliton solution of the Sine-Gordon equation.**

Show that (12.35) solves the Sine-Gordon equation and discuss the behavior under Lorentz trans-
formations.

12.5 Domain wall
Estimate the extension of domain wall from dimensional analysis.

12.6 Derrick’s theorem.
Consider the behavior of the energy functional $E[\phi] = T[\phi] + V[\phi]$ of scalar field $\phi$ under scale transformation $x \rightarrow \lambda x$ in $D$ space dimensions. Show that for $D \geq 2$ no stable solutions with finite energy exist.

12.7 Duality of Maxwell
Show that the source-free Maxwell equations are invariant under the duality transformation

\begin{align*}
E' &= E \cos \alpha + B \sin \alpha \\
B' &= -E \sin \alpha + B \cos \alpha.
\end{align*}

Show that the invariance of the Maxwell equations with sources requires the existence of magnetic monopoles.

12.8 Magnetic field of a monopole
Derive Eq. (12.67).
13. Anomalies, instantons and axions

We have seen that any global continuous symmetry of a system described by a Lagrangian $L$ leads to a locally conserved current on the classical level. In order to decide if these classical symmetries do survive quantisation, we have to study if the generating functional $Z[J]$ retains the symmetries of the classical Lagrangian $L$.

Various examples where a quantised system does not share the symmetries of its classical counter-part have been found. As this behaviour came as a surprise, it was called “anomalous” and the non-zero terms violating on the quantum level the classical conservation laws were called “anomalies”. A case we encountered already is the breaking of scale invariance (or conformal symmetry) in the process of renormalising massless QED and QCD. The only other example of anomalous theories in $d = 4$ space-time dimensions are models containing chiral fermions, i.e. theories where left and right-handed fermions interact differently. This anomaly is called axial or chiral anomaly and shows up in loop graphs containing fermions coupled to gauge fields or gravitons.

The fact that a classical symmetries is broken by quantum effects is often described as “the anomaly breaks the classical symmetry”. One should keep in mind that on the quantum level there is no symmetry to start with. Thus there is no Goldstone boson associated to a symmetry broken by an anomaly.

13.1. Axial anomalies

Anomaly from non-invariance of the path integral measure  Anomalies were first discovered performing diagrammatic calculations using canonical quantisation. At a first glance, it looks like the path integral formalism predicts that all classical symmetries of the Lagrangian also hold at the quantum level—and the “failure” to reproduce the anomalies found using canonical quantisation deemed the path integral approach untrustworthy. It was only realised by Fujikawa in 1979 that the integration measure in the path integral may transform non-trivially under a symmetry transformation, leading to an anomaly.

The simplest model exhibiting an axial anomaly is axial electrodynamics, i.e. a fermion coupled via its vector and axial current to two different gauge fields. The Lagrangian for this system reads

$$\mathcal{L} = \bar{\psi}i\gamma^\mu (\partial_\mu + iqV_\mu + igA_\mu \gamma^5)\psi - \frac{1}{4}F^2 - \frac{1}{4}G^2$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad G_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu.$$ 

Performing a combined $U_V(1) \otimes U_A(1)$ gauge transformation induces the following change of
13. Anomalies, instantons and axions

the fields,

$$\psi(x) \to \psi'(x) = \exp\{iq\Lambda(x) + ig\alpha(x)\gamma^5\} \psi(x),$$  \hspace{1cm} (13.3a)

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x) \exp\{-iq\Lambda(x) + ig\alpha(x)\gamma^5\},$$  \hspace{1cm} (13.3b)

$$V_\mu \to V'_\mu = V_\mu - \partial_\mu \Lambda(x),$$  \hspace{1cm} (13.3c)

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \alpha(x).$$  \hspace{1cm} (13.3d)

Since the fermionic field $\psi$ is described by a Grassmann variable, the integration measure $D\psi$ transforms like

$$D\psi \to \left[\det \exp\{iq\Lambda(x) + ig\alpha(x)\gamma^5\}\right]^{-1} D\psi.$$  \hspace{1cm} (13.4)

Thus the product of the fermionic measure changes as

$$D\psi D\bar{\psi} \to D\psi D\bar{\psi} \left[\det \exp\{2iq\alpha(x)\gamma^5\}\right].$$  \hspace{1cm} (13.5)

As result, the variation introduced by a vector $U_V(1)$ gauge transformation cancels in the fermionic measure $D\psi D\bar{\psi}$, while the corresponding change under the axial $U_A(1)$ gauge transformation adds up. We can rewrite this non-zero variation of the integration measure as an effective shift $\delta L$ in the Lagrangian using the identity $\det \exp B = \exp \text{Tr}(B),$

$$\delta L = 2\alpha(x)g \sum_n \phi^*_n(x)\gamma^5 \phi_n(x) = 2\alpha(x)g \sum_n A(x),$$  \hspace{1cm} (13.6)

where $\text{Tr}(B) = \int d^4x \text{tr}(i\delta \mathcal{L})$. For the wave-functions $\phi_n(x)$, we can use any complete set of solutions of the Dirac equation,

$$i\gamma_\mu \phi_n = \lambda_n \phi_n \quad \text{and} \quad \sum_n \phi_n(x)\phi^*_n(y) = \delta(x - y),$$  \hspace{1cm} (13.7)

where $D$ is the gauge-invariant derivative with respect to the two gauge fields. The sum over the eigenfunctions will be UV divergent, and we have therefore to regularise it. A convenient regulator to add is the function

$$f = \exp\left\{\lambda^2_n/M^2\right\} = \exp\left\{-\mathcal{D}^2/M^2\right\},$$  \hspace{1cm} (13.8)

which approaches fast zero for $\lambda^2_n \to -\infty$. The limit $f \to 1$ of the regulator corresponds to $M \to \infty$, which will allow us later an expansion of our result for

$$A(x) = \sum_n \phi^*_n(x)\gamma^5 \exp\left\{-\mathcal{D}^2/M^2\right\} \phi_n(x)$$  \hspace{1cm} (13.9)

in powers of $1/M^2$.

As the vector $U_V(1)$ gauge transformation keeps the integration measure invariant, we ignore it in the following calculations. Thus we need to calculate $\mathcal{D}^2$ including only the axial gauge field $A$,

$$\mathcal{D} \mathcal{D} = \gamma_\mu \gamma_\nu D^\mu D^\nu = D^2 + \frac{g}{2} \epsilon_{\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (13.10)

with

$$D^2 = (\partial_\mu + igA_\mu)(\partial^\mu + igA^\mu) = \partial^2 + 2igA_\mu \partial_\mu + ig(\partial_\mu A^\mu) - g^2 A_\mu A^\mu.$$  \hspace{1cm} (13.11)
Since the trace is invariant to the choice of our wave functions we choose plane-waves $\phi_n = e^{-ikx}$. Then we can replace the differentiations by momenta,
\[
\exp\left(-\frac{D^2}{M^2}\right)\phi_n(x) = \exp\left(-\frac{(k_\mu + gA_\mu)^2}{M^2} + \frac{ig(\partial_\mu A^\mu)}{M^2}\right)\phi_n(x) \quad (13.12)
\]
Taking the continuum limit of the sum and writing out the full regulator, we obtain
\[
A(x) = \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ e^{ikx} \gamma^5 \exp\left\{-\frac{D^2}{M^2}\right\} e^{-ikx} \right] = \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \gamma^5 \exp\left\{-\frac{(k_\mu + gA_\mu)^2}{M^2} + \frac{ig\partial_\mu A^\mu}{M^2} + \frac{g}{2} \frac{\sigma_{\mu\nu} F^{\mu\nu}}{M^2}\right\} \right] \quad (13.13)
\]
The second term in \(\cdots\) is zero because it does not depend on \(k^\mu\) and \(\text{tr}[\gamma^5] = 0\). Shifting variables to the dimensionless \(MK_\mu = k_\mu + gA_\mu\) we obtain
\[
A(x) = M^4 \int \frac{d^4K}{(2\pi)^4} e^{-K^2} \text{tr} \left[ \gamma^5 \exp\left\{\frac{g}{2} \frac{\sigma_{\mu\nu} F^{\mu\nu}}{M^2}\right\} \right] \quad (13.14)
\]
If we expand the exponential in powers of \(1/M^2\), only the terms up to \(\mathcal{O}(M^{-4})\) will survive in the limit \(M \to \infty\). Using the antisymmetry of \(F^{\mu\nu}\), we can also replace \(\sigma_{\mu\nu}\) by \(i\gamma_\mu\gamma_\nu\) and find then
\[
\exp\{\cdots\} = 1 + \frac{ig}{2} \gamma_\mu \gamma_\nu F^{\mu\nu} \frac{1}{M^2} + \frac{1}{2} \left(\frac{ig}{2}\right)^2 \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta F^{\mu\nu} F^{\alpha\beta} \frac{1}{M^4} + \mathcal{O}\left(\frac{1}{M^6}\right) \quad (13.15)
\]
The trace properties of the gamma matrices inform us that the first two terms vanish, while the third results in a term proportional to the totally antisymmetric tensor \(\epsilon_{\mu\nu\alpha\beta}\). Introducing the dual tensor \(\tilde{F}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} F^{\mu\nu}\) and then performing the Gaussian integral over \(K\) we are left with
\[
\text{Tr}(B) = 2i \frac{g^2}{16\pi^2} \int d^4x \alpha(x) \tilde{F}_{\mu\nu} F^{\mu\nu} \quad (13.16)
\]
In a classical theory without gauge fields, a local axial gauge transformation leads to the change
\[
\mathcal{L} \to \mathcal{L} + g(\partial_\mu \alpha) \bar{\psi} \gamma^\mu \gamma^5 \psi \quad (13.17)
\]
Thus the non-invariance of the measure in the path integral leads to violation of the classically conserved axial current,
\[
\delta A_\mu = \partial_\mu (\bar{\psi} \gamma^\mu \gamma^5 \psi) = \partial_\mu (\bar{\psi}_R A_\mu^R - \bar{\psi}_L A_\mu^L) = \frac{g^2}{8\pi} \tilde{F}_{\mu\nu} F^{\mu\nu} \quad (13.18)
\]
This equation is also known as the Adler-Bell-Jackiw anomaly equation.

The extra term introduced in the Lagrangian by an axial gauge transformation is proportional to \(\tilde{F}_{\mu\nu} F^{\mu\nu}\) and thus odd under CP. This interaction term is gauge invariant, has dimension four and corresponds therefore to a renormalisable interaction. The reason why we have not considered it earlier is that it is a total derivative,
\[
F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) = 2\epsilon_{\mu\nu\rho\sigma} \partial^\mu A^\nu \partial^\rho A^\sigma = \partial^\mu (2\epsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma) = \partial^\mu K_\mu \quad (13.19)
\]
Since the four-divergence \(K_\mu\) is not gauge invariant, it cannot be an observable. Therefore it is not excluded that \(K_\mu\) is singular, leading after integration to non-zero effects in the action. Before we discuss in more detail when it is justified to neglect a total derivative term, we will rederive the anomaly using a diagrammatic approach.
Perturbative appearance of anomalies  Historically, the chiral anomaly was first encountered calculating the process $\pi^0 \rightarrow \gamma\gamma$ using perturbation theory. Describing the neutral pion within a non-relativistic quark picture as a $\pi^0 = (uu + dd)/\sqrt{2}$ state, we can view the process as

$$\gamma^\mu \gamma^5 \rightarrow p_1 \quad \gamma^\lambda \rightarrow p_2 \quad \gamma^\mu \gamma^5 \rightarrow p_2 \quad \gamma^\lambda \rightarrow p_1$$

Here, the $\gamma^5$ matrix accounts for the fact that the pion is a pseudoscalar particle.

The two diagrams are connected by the crossing symmetry $\kappa \leftrightarrow \lambda$, $p_1 \leftrightarrow p_2$. The total matrix element of this process is thus given by the sum

$$A_{\kappa\lambda\mu}(p_1, p_2) = S_{\kappa\lambda\mu}(p_1, p_2) + S_{\lambda\kappa\mu}(p_2, p_1),$$

where the matrix element $S_{\kappa\lambda\mu}$ describing the first diagram (neglecting coupling constants) is given by

$$S^{\kappa\lambda\mu} = -(-i)^3 \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \gamma^\kappa \frac{i}{k - \not{p}_1} \gamma^\mu \gamma^5 \frac{i}{k + \not{p}_2} \gamma^\lambda \frac{1}{k} \right].$$

It is sufficient to consider only massless fermions: The anomaly is connected to the UV divergences of these diagrams and in this limit masses play no role.

We check now, if the classical conservation law for the vector and the axial current hold also at the one-loop level. Current conservation implies the following three relations in momentum space,

$$p_1^\kappa A_{\kappa\lambda\mu} = p_2^\lambda A_{\lambda\kappa\mu} = 0$$

$$(p_1 + p_2)^\mu A_{\mu\kappa\lambda} = 0,$$

where (13.22a) are the equations for the conservation of the vector current, and Eq. (13.22b) for the conservation of the axial current. Crossing symmetry implies that these relations must also hold for the two individual amplitudes $S$.

We check first, if the axial current is conserved. In evaluating

$$I_{\kappa\lambda} = (p_1 + p_2)^\mu S_{\kappa\lambda\mu} = -\int \frac{d^4k}{(2\pi)^4} \text{tr} \frac{[\gamma^\kappa (k - \not{p}_1)(\not{p}_1 + \not{p}_2)\gamma^5 (k + \not{p}_2)\gamma^\lambda \not{k}]}{(k - p_1)(k + p_2)^2 k^2},$$

we use

$$(\not{p}_1 + \not{p}_2)\gamma^5 = \not{p}_1\gamma^5 - \gamma^5 \not{p}_2 = -(k - \not{p}_1)\gamma^5 - \gamma^5 (k + \not{p}_2)$$

and $(k \pm \not{p}_1)^2 = (k \pm p_1)^2$, so that we can divide the integral into two parts,

$$I_{\kappa\lambda} = \int \frac{d^4k}{(2\pi)^4} \text{tr} \frac{[\gamma^\kappa \gamma^5 (k + \not{p}_2)\gamma^\lambda \not{k}]}{(k + p_2)^2 k^2} + \int \frac{d^4k}{(2\pi)^4} \text{tr} \frac{[\gamma^\kappa (k - \not{p}_1)\gamma^5 \gamma^\lambda \not{k}]}{(k - p_1)^2 k^2}.$$
13.1. Axial anomalies

create a pseudo-tensor of rank 2 from that, the RHS must be zero. We have thus shown that the axial current is conserved.

Next we verify the conservation of the vector current, following the same line of argument as in the case of the axial current. Starting from

\[ J_{\lambda\mu} = p_1^\nu S_{\kappa\lambda\mu} = -\int \frac{d^4 k}{(2\pi)^4} \frac{\text{tr}[\gamma_\mu (k - p_1) \gamma_5 (k - p_2) \gamma_\lambda p]}{(k - p_1)(k + p_2)^2 k^2}, \] (13.25)

we shift the integration variable \(k' = k + p_2\) and reorder the terms in the trace, obtaining

\[ J_{\lambda\mu} = -\int \frac{d^4 k'}{(2\pi)^4} \frac{\text{tr}[k' - p_1 - p_2 \gamma_\mu \gamma_5 k']}{k'^2 (k' - p_2)^2 k'^2} + \int \frac{d^4 k'}{(2\pi)^4} \frac{\text{tr}[(k' - p_2) (k' - p_2) \gamma_\mu \gamma_5 k']}{k'^2 (k' - p_2)^2}. \] (13.26)

Thus we conclude that \(J_{\lambda\mu}\) also vanishes. For the \(p_2^\lambda S_{\kappa\lambda\mu}\) part, we follow the same argument, but we shift the integration variable by \(k' = k - p_1\) instead.

We have now shown that both the axial current and the vector current are conserved. Using these results in the calculation for the decay width of a \(\pi^0\) leads however to a width which vanishes in the limit of a massless pion. Including the small pion mass leads still to life-time much longer than observed.

Looking back at our calculation, we should check therefore if all our manipulations were legitimate, although we have not regularised the divergent loop integrals. In particular, the superficial divergence of the diagrams \(S_{\kappa\lambda\mu}\) is worse than the logarithmic divergences we have become accustomed to,

\[ S_{\kappa\lambda\mu} = -\int \frac{d^4 k}{(2\pi)^4} \frac{\text{tr}[\gamma_\kappa (k - p_1) \gamma_5 (k - p_2) \gamma_\lambda p]}{k^6} + \text{subleading terms} \] (13.27)

producing a linear divergent term.

An essential ingredient for the derivation of the current conservation was the shift of our integration variable. Such a shift can only be done if the integral is properly convergent (after regularisation, if required) or only logarithmic divergent. By contrast, in the case of a linear divergent integral, the shift \(k' = k - a\)

\[ \int d^4 k' f(k') = \int d^4 k f(k - a) = \int d^4 k \left\{ f(k) - a_\mu \frac{\partial f}{\partial k_\mu} + \ldots \right\} \] (13.28)

changes the value of the integral: Using Gauss’ theorem, we can convert the gradient term into a surface integral which will lead to a finite change of the integral because of \(\Omega \propto k^3\) and \(f' \propto k^{-3}\). You should show in problem [13.1] that the shift \(k' \rightarrow a\mu\) in (13.27) changes \(S_{\kappa\lambda\mu}\) as

\[ S_{\kappa\lambda\mu} \rightarrow S'_{\kappa\lambda\mu} = S_{\kappa\lambda\mu} + \frac{1}{8\pi^2} \epsilon_{\kappa\lambda\mu\nu} a^\nu \] (13.29)

where \(\epsilon_{\kappa\lambda\mu\nu}\) is again the totally anti-symmetric tensor.

We can look back at the corresponding proof of gauge invariance for the vacuum polarisation, Eqs. (8.92–8.95). There we used dimensional regularisation which respects gauge symmetry. Since \(\gamma_5\) is not well-defined for \(d = 4 - \varepsilon\), we cannot apply this regularisation method here. Using Pauli-Villar regularisation as an alternative would break axial symmetry, since it consists of adding massive particles. Thus, from a technical point of view, anomalies
arise, if no regularisation and renormalisation procedures exists which respects the classical symmetry.

Since the shift of the integration variables required to obtain current conservation differ for the vector and axial current, we can absorb only one of the resulting boundary terms by a suitable chosen counter-term. Clearly, we will choose the vector current to be conserved: Otherwise electric charge conservation would be violated, while the axial current is anyway broken by mass effects. So if we add to $T$ a counter term,

$$A_{\kappa\lambda\mu} = S_{\kappa\lambda\mu}(p_1, p_2) + S_{\lambda\kappa\mu}(p_2, p_1) + \frac{1}{4\pi^2}\epsilon_{\kappa\lambda\mu\nu}(p_1^\nu + p_2^\nu), \quad (13.30)$$

our vector current will still be conserved, but the axial current will not

$$(p_1 + p_2)^\mu A_{\kappa\lambda\mu} = \frac{1}{2\pi^2}\epsilon_{\kappa\lambda\mu\nu}p_2^\nu p_1^\mu. \quad (13.31)$$

The added contribution gives a non-zero contribution to the divergence of the axial current which is identical to our previous non-perturbative result,

$$\partial_\mu j_5^\mu = \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}. \quad (13.32)$$

This implies that the perturbative one-loop result for the anomaly is not changed by higher-order corrections. In particular, processes which are dominated by the anomaly like $\pi^0 \rightarrow 2\gamma$ can be calculated reliably in lowest-order perturbation theory, although for $Q^2 = m_\pi^2 \sim \Lambda_{QCD}^2$ the strong coupling constant $\alpha_s(Q^2)$ is certainly not small and corrections of the type shown in are naively expected to be large.

The corresponding result for a non-abelian theory like QCD is

$$\partial_\mu j_5^\mu = \frac{n_f g^2}{16\pi^2} \text{tr} \left( F_{\mu\nu} \tilde{F}^{\mu\nu} \right), \quad (13.33)$$

where $n_f$ denoted the number of quarks flavors.

**Cancellations of anomalies** Anomalies may be useful when they can break ”accidental” global symmetries like baryon and lepton number. As we will see later, this opens the possibility to explain why the universe consists mainly of matter. Similarly, the explicit breaking of global symmetries in the quark Lagrangian can explain why in certain cases no Goldstone bosons are observed.

In contrast, symmetries and the resulting Ward identities between Green functions are crucial for the renormalisability of gauge theories. If these identities are not satisfied, physical observables in renormalisable gauges depend on the gauge-fixing parameter $\xi$, while Lorentz invariance is violated in unitary gauges. The excellent agreement of electroweak precision data with experiment is a strong argument that the underlying theory is renormalisable. As the V-A structure of the electroweak interactions is however similar to our toy model of axial electrodynamics, we can only expect that the anomalies of individual loop diagrams cancel after summing over all contributions.

As an example we analyse an one-loop VVA diagram, i.e. a triangle diagram with two vector and one axial vertices. The anomaly written out for the left-handed fermions is

$$A^{abc} = \left[ \text{tr}(M_a^L M_b^L M_c^L) + \text{tr}(M_a^L M_c^L M_b^L) \right] = \text{tr}(M_a^L \{M_b^L, M_c^L\}) \quad (13.34)$$
where the trace is over SU(2) doublets. The $M$’s are the coupling matrices for all of the three interactions in the loop. If we identify the interactions at the vertex $a$ with $Z$, $b$ with $W^−$ and $c$ with $W^+$ and recall the coupling matrices, then

$$M^L_a = \frac{g}{\cos \vartheta_W} \left( \frac{\tau_3}{2} + \sin^2 \vartheta_W Q \right)$$  \hspace{1cm} (13.35a) \\
$$M^L_b = gT_+ = \frac{g}{2} \tau_+$$  \hspace{1cm} (13.35b) \\
$$M^L_c = gT_- = \frac{g}{2} \tau_-$$  \hspace{1cm} (13.35c)

Requiring that the trace vanishes gives us the following constraint,

$$0 \equiv \text{tr} \left( \left( \frac{\tau_3}{2} + \sin^2 \vartheta_W Q \right) \{ \tau_+ , \tau_- \} \right) = \frac{1}{4} \text{tr} \left( \frac{\tau_3}{2} + \sin^2 \vartheta_W Q \right) = \sum_i Q^L_i ,$$  \hspace{1cm} (13.36)

where we used $\{ \tau_+ , \tau_- \} = 1/4$ and $\text{tr}(\tau_3) = 0$. Thus the anomaly vanishes if the electric charges of all left-handed fermions sum to zero. If we now only look at the leptons and quarks separately, then

$$Q_e + Q_\nu = -1 \neq 0 \quad \text{and} \quad Q_u + Q_d = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \neq 0 .$$

Including both leptons and quarks where we account by the factor three for their colour quantum number we find

$$Q_e + Q_\nu + 3(Q_u + Q_d) = -1 + 3 \frac{1}{3} = 0 .$$

Therefore, chiral anomalies are canceled in the SM within each full fermion generation. In other words, if a single member of a hypothetical fourth generation of fermions would be found, anomaly cancellation would require the existence of a complete set of quarks and leptons. Within the SM, there is no explanation for the miraculous cancellation between the quark and lepton sector, and this has been one of the major motivations to consider GUTs.

The AAA anomaly is automatically zero, since

$$A^{abc} \propto \text{tr} \left[ \tau^a \{ \tau^b , \tau^c \} \right] = 2\delta^{bc} \text{tr} \tau^a = 0 .$$  \hspace{1cm} (13.37)

We have restricted our analysis of the anomaly to an abelian model. In the non-abelian case, the field-strength contains terms linear and quadratic in the gauge fields. As a result, additional anomalies in square and pentagon diagrams appear. However, the absence of anomalies in the triangle diagrams guarantees also their absence in square and pentagon diagrams.

### 13.2. Instantons and the strong CP problem

The effective Lagrangian induced by the chiral anomaly is a surface term. This suggests that a term of the same structure can be obtained as the topological charge of a Yang-Mills theory, following the argument of Bogomol’nyi in Eqs. (12.4.3). If such a topological charge exist, then the unbroken vacuum of a Yang-Mills theory has a non-trivial vacuum structure. Such a non-trivial vacuum structure has however only physical consequences if the corresponding classical tunneling solutions have a finite action. We should therefore search for classical solutions of a pure, Euclidean Yang-Mills theory with $S_{YM} < \infty$. These solutions are the non-abelian analogue of the bounce solution we have considered earlier.
13. Anomalies, instantons and axions

**Instantons** We define an Euclidean Yang-Mills theory by the action

\[ S = \frac{1}{2} \int d^4 x \text{ tr} \{ F_{\mu\nu} F_{\mu\nu} \}, \]  

(13.38)

where \( F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \) \( \epsilon_{\mu\nu\rho\sigma} = \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} \) and derivatives and integrations are with respect to Euclidean coordinates \((x_1, x_2, x_3, x_4 = i x_0)\) with \( x_i \in \mathbb{R} \). Since the metric tensor is \( \eta_{\mu\nu} = \delta_{\mu\nu} \), we do not need to distinguish between lower and upper indices. Therefore, in Euclidean space the dual of the dual field strength tensor is again the field strength tensor, \( \tilde{F}_{\mu\nu} = F_{\mu\nu} \), while in Minkowski space it is the negative of the field strength tensor, \( \tilde{F}_{\mu\nu} = -F_{\mu\nu} \).

We first examine QED, i.e. the case of an abelian YM theory. A finite action, \( \int_{\tau} d^4 x \tilde{F} F < \infty \) (13.39) requires that \( F \) decreases faster than \( \tau^{-2} \) in the limit \( \tau^2 = x_1^2 + x_2^2 + i x_0^2 \to \infty \). But as a classical YM theory contains no scale, \( \tilde{F} F \) must be a polynomial in \( \tau \): Thus \( F \sim O(\tau^{-3}) \) and \( A \sim O(\tau^{-2}) \), and then the total derivative (13.19) behaves as

\[ K^\mu = 2 \epsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma \sim O(\tau^{-5}). \]  

(13.40)

As a result, the surface term in

\[ \int_{\partial \Omega} d^4 \tilde{F} F = \int_{\partial \Omega \mu} K_{\mu} \to 0 \]  

(13.41)

vanishes and we see that in an abelian theory \( \tilde{F} F \) does not influence physical quantities. This argument justifies our usual practise to neglect surface terms in QED.

We now turn to the non-abelian case. Then we can express \( \text{tr} \{ \tilde{F}_{\mu\nu} F_{\mu\nu} \} \) again as a four-divergence,

\[ \partial^\mu K_\mu = 2 \text{ tr} \tilde{F}_{\mu\nu} F_{\mu\nu} \]  

(13.42)

where now

\[ K_\mu = 2 \epsilon_{\mu\nu\rho\sigma} \text{tr} \left( A_\nu F_{\rho\sigma} + \frac{2}{3} i g A_\rho A_\sigma \right). \]  

(13.43)

Choosing a pure gauge field,

\[ A_\mu = \frac{i}{g} (\partial_\mu U) U^{-1}, \]  

(13.44)

at spatial infinity results in \( F_{\mu\nu} = 0 \) for \( \tau \to \infty \) and ensures that the action is finite. On the other hand, a gauge transformation \( U \) which becomes constant for \( \tau \to \infty \), i.e. depends in this limit only on the angles, gives \( A \sim O(\tau^{-1}) \) and thus \( K \sim O(\tau^{-3}) \). As a result, the surface integral may become non-zero in the limit \( \tau \to \infty \). One may wonder if we can gauge away \( A_\mu \propto \partial_\mu U^{-1} \) on \( \tau = \infty \) by performing a suitable gauge transformation \( \tilde{U} \). Since \( \tilde{U} \) has to be regular in all \( \mathbb{R}^4 \), it must be constant a \( \tau = 0 \) and independent of the angles. Thus \( \tilde{U} \) is continuously connected to the identity and can be used only to gauge away \( A_\mu \) in the same homotopy class. Thus the surface term \( \tilde{F} F \) has physical significance, if the gauge fields at \( \tau \to \infty \) are split into non-trivial topological classes.

We now turn to the specific case of SU(2). Any SU(2) matrix can be written as \( U = a + i b \cdot \sigma \) with \( a^2 + |b|^2 = 1 \). Thus SU(2) is isomorphic to \( S^3 \). Compactifying \( \mathbb{R}^4 \) to \( S^3 \), (13.44) defines
the map $S^3 \rightarrow S^3$. The question if non-trivial instanton solutions exist is thus equivalent to the existence of topologically non-trivial mappings $S^3 \rightarrow S^3$, that is to say if $\pi_3(S^3)$ is not the identity.

The generalisation of the winding number (12.72) from $S^1 \rightarrow S^1$ to the case $S^3 \rightarrow S^3$ is

$$\nu = -\frac{1}{24\pi^2} \int dx_1 dx_2 dx_3 \varepsilon_{ijk} \text{tr} \left[ (U \partial_i U^\dagger) (U \partial_j U^\dagger) (U \partial_k U^\dagger) \right],$$  \hspace{1cm} (13.45)

where we used the Cartesian coordinates of $\mathbb{R}^3$. Since $dx_i \partial_i = d\theta_i$, the expression is equally valid using the three angles specifying a point on $S^3$. In order to show that the winding number is invariant under continuous transformations, $\delta \nu = 0$, it is sufficient to consider the variation of a single factor in the trace. Now we can profit from (12.72) where we derived already the variation of this factor in the non-abelian case as

$$\delta (U \partial_i U^\dagger) = -U \partial_i (U^\dagger \delta U) U^\dagger.$$  \hspace{1cm} (13.46)

Inserting this relation into the integrand results in

$$E \equiv \varepsilon_{ijk} \text{tr} \left[ (U \partial_i U^\dagger) (U \partial_j U^\dagger) \delta (U \partial_k U^\dagger) \right] = -\varepsilon_{ijk} \text{tr} \left[ \partial_i U^\dagger U \partial_j U^\dagger U \partial_k (U^\dagger \delta U) \right].$$  \hspace{1cm} (13.47)

Then we perform a partial integration and use that terms in $\partial_k \partial_i U^\dagger$ vanish contracted with $\varepsilon_{ijk}$,

$$E = \varepsilon_{ijk} \text{tr} \left[ \partial_i U^\dagger \partial_k U \partial_j U^\dagger \delta U + \partial_i U^\dagger U \partial_j U^\dagger \partial_k U^\dagger \delta U \right],$$  \hspace{1cm} (13.48)

Thus the winding number $\nu$ is invariant under infinitesimal and thus under continuous transformations.

Next we try to express the winding number as a volume integral over $\text{tr} \{ \tilde{F}_{\mu \nu} F_{\mu \nu} \}$. We write (13.45) as a surface integral

$$\nu = \frac{1}{24\pi^2} \int d\Omega_\mu \varepsilon_{\mu \nu \sigma \tau} \text{tr} \left[ (U \partial_\nu U^\dagger) (U \partial_\sigma U^\dagger) (U \partial_\tau U^\dagger) \right] = -\frac{ig^3}{24\pi^2} \int d\Omega_\mu \varepsilon_{\mu \nu \sigma \tau} \text{tr} \left[ A_\nu A_\sigma A_\tau \right].$$  \hspace{1cm} (13.49)

where we used that $A_\mu$ is a pure gauge field for $\tau \rightarrow \infty$, (13.44). Since then also $F^a_{\sigma \mu} = 0$, we can use (13.43)

$$\nu = \frac{g^2}{32\pi^2} \int d\Omega_\mu K^\mu = \frac{g^2}{16\pi^2} \int d^4x \text{ tr} \{ \tilde{F}_{\mu \nu} F_{\mu \nu} \}. $$  \hspace{1cm} (13.50)

Mathematicians call the winding number of the mapping $S^3 \rightarrow S^3$ the Pontryagin index, while mathematical physicists use often the term Chern-Simon number.

We define instantons as self-dual and anti-self-dual solutions,

$$\tilde{F}_{\mu \nu} = F_{\mu \nu} \quad \text{and} \quad \tilde{F}_{\mu \nu} = -F_{\mu \nu}.$$  \hspace{1cm} (13.51)
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of the classical Yang-Mills equations. Using Bogomol’nyi’s trick, we can show that they correspond to the topologically non-trivial solutions with the lowest energy. We write first

$$\text{tr}\{ (F_{\mu\nu} - \tilde{F}_{\mu\nu})^2 \} = \text{tr}\{ F_{\mu\nu}^2 + \tilde{F}_{\mu\nu}^2 - 2F_{\mu\nu}\tilde{F}_{\mu\nu} \} \geq 0.$$  \hspace{1cm} (13.52)

For the calculation of $\tilde{F}^2$ we use $\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu\kappa\lambda} = 2(\delta_{\rho\kappa}\delta_{\sigma\lambda} - \delta_{\rho\lambda}\delta_{\sigma\kappa})$ and end up with $\tilde{F}^2 = F^2$. Thus

$$\text{tr}\{ F_{\mu\nu}F_{\mu\nu} \} \geq \text{tr}\{ F_{\mu\nu}\tilde{F}_{\mu\nu} \} 0,$$  \hspace{1cm} (13.53)

which is minimized if $F$ is self-dual. We obtain the same bound for an antiself-dual solution, if we choose a plus sign in Eq. (13.52). Integrating over space, we find on the LHS twice the Euclidean action, while the RHS equals $16\pi^2\nu/g^2$. Thus the Euclidean action is bounded by

$$S \geq \frac{8\pi^2\nu}{g^2},$$  \hspace{1cm} (13.54)

and instantons as the non-trivial solutions with the lowest energy have the action $S = 8\pi^2/g^2$.

Our remaining task is to write down an explicit form of the mappings $S^3 \to S^3$ and to show that the definition (13.45) results in an integer winding number. We choose to write $U(x)$ as

$$U^{(0)}(x) = 1$$  \hspace{1cm} (13.55)

$$U^{(1)}(x) = \frac{x_4 + ix \cdot \sigma}{\tau}$$  \hspace{1cm} (13.56)

$$U^{(n)}(x) = \left( \frac{x_4 + ix \cdot \sigma}{\tau} \right)^n$$  \hspace{1cm} (13.57)

where $x_\mu$ is the unit vector $x_\mu = (\sin \chi e, \cos \chi)$. Evaluation of (13.45) for $U^{(1)}(x)$ confirms then $\nu = 1$.

We used for our discussion of instantons in non-abelian Yang-Mills theories the specific example of SU(2). A theorem of Brott states that for any simple Lie group $G$ containing SU(2) the maps $S^3 \to G$ can be deformed continuously to the ones of $S^3 \to SU(2)$. Thus all our results apply identically to the case of strong interactions, SU(3). In the remaining part of this chapter, we will discuss the impact of instantons on the QCD vacuum, while we postpone the electroweak case to chapter 18.

**Tunneling interpretation** We consider the four-dimensional cylinder defined by $x_4 \in [-T : +T]$ and $|x| \leq R$ and sketched in Fig. 13.1 in the limit $T, R \to \infty$. At $x_4 = -T$, we choose $A_\mu$ as a pure gauge field with winding number $\nu_1$, and at $x_4 = T$ with winding number $\nu_2$. On the boundary $|x| = R$, we choose $U$ constant and thus $A_\mu$ is zero. Calculating the total winding number, we find $\nu = \nu_1 - \nu_2$.

$$\frac{g^2}{16\pi^2} \int_\Omega d^4x \text{tr}(F\tilde{F}) = \frac{g^2}{32\pi^2} \int_{\partial\Omega} d^3\sigma \mu K^\mu = \frac{g^2}{32\pi^2} \int d^3x \left[ K^0(t = -\infty) - K^0(t = +\infty) \right]$$  \hspace{1cm} (13.58)

The minus sign appears, because of the opposite orientations of the two caps. Thus the classical solutions we have determined interpolate between a vacua with winding number $\nu_1$ at time $t = -\infty$ and a vacua with winding number $\nu_2$ at time $t = +\infty$. The two vacua are separated by a finite energy barrier, and thus the solutions describe the quantum tunneling between different vacua.
The $\vartheta$ vacuum The tunneling interpretation indicates that the true vacuum of a pure Yang-Mills theory is the superposition of all vacua with fixed winding number $\nu$. Let us call these vacua $|\nu\rangle$ and the true one $|\vartheta\rangle$. Applying a gauge transformation $U \equiv U^{(1)}$ with unit winding number results in

$$U |\nu\rangle = |\nu + 1\rangle.$$  \hfill (13.59)

On the other hand, the Yang-Mills Hamiltonian is invariant under gauge transformations,

$$UHU^\dagger = H,$$  \hfill (13.60)

or $[H, U] = 0$. Thus the true vacuum $|\vartheta\rangle$ is a common eigenstate of $H$ and $U$. Since the vacuum is normalised, the eigenvalue of $U$ has to be a phase,

$$U |\vartheta\rangle = e^{i\vartheta} |\vartheta\rangle.$$  \hfill (13.61)

Its arguments $\vartheta$ is a conserved quantum number. Thus a pure Yang-Mills theory is characterised by two numbers, the coupling $g$ and the new parameter $\vartheta$.

The true vacuum $|\vartheta\rangle$ is a linear superposition of the vacua with fixed winding number given by

$$|\vartheta\rangle = \sum_{n=-\infty}^{\infty} e^{-i\vartheta n} |n\rangle,$$  \hfill (13.62)

since

$$U |\vartheta\rangle = \sum_{n=-\infty}^{\infty} e^{-i\vartheta (n + 1)} = e^{i\vartheta} \sum_{n=-\infty}^{\infty} e^{-i\vartheta n} |n\rangle.$$  \hfill (13.63)

The matrix elements $\langle n' | H | n \rangle$ depend only on the difference $\nu = n' - n$, since (13.60) gives

$$\langle n' + 1 | H | n + 1 \rangle = \langle n' | H | n \rangle.$$  \hfill (13.64)

Using also the parity properties, $P |n\rangle = - |n\rangle$ and $PHP^{-1} = H$, we obtain

$$\langle n' | H | n \rangle = \langle -n' | H | -n \rangle.$$  \hfill (13.65)
Hence the matrix elements $\langle n' | H | n \rangle$ depend only on the absolute value $|\nu| = |n' - n|$. We conclude that instantons lead to an effective potential $V_{\text{eff}}(\vartheta)$ which is periodic and even in $\vartheta$

$$H |\vartheta\rangle = L^3 V_{\text{eff}}(\vartheta) |\vartheta\rangle ,$$  \hspace{1cm} (13.66)

with $V_{\text{eff}}(\vartheta) = V_{\text{eff}}(\vartheta + 2\pi)$ and $V_{\text{eff}}(\vartheta) = V_{\text{eff}}(-\vartheta)$, where $L^3$ is the considered volume. A general argument due to Weinberg shows that points of enhanced symmetry are stationary points of the action. Thus we expect the minimum of $V_{\text{eff}}(\vartheta)$ to coincide with the CP conserving point $\vartheta = 0$.

We included into our definition of the path integral for gauge theories with the help of the Fadeev-Popov trick only gauge fields which are continuously connected with the identity. Thus our next task is to add the effect of the $\vartheta$ vacuum to the path integral. The path integral in the presence of external sources is identical to the vacuum persistence amplitude,

$$\langle \vartheta' | \vartheta \rangle_J = \sum_{n,n'} e^{i(n'\vartheta' - n\vartheta)} \langle n' | n \rangle_J .$$  \hspace{1cm} (13.67)

Now we introduce the difference $\nu = n' - n$ so that we can rewrite the phase as $n'\vartheta' - n\vartheta = n(\vartheta' - \vartheta) + \nu\vartheta'$. But $\langle n' | n \rangle_J$ depends only on $\nu$ and thus we can perform the sum over $n$. This leads to a factor $\delta(\vartheta' - \vartheta)$, which expresses the fact that $\vartheta$ is conserved. Thus

$$\langle \vartheta' | \vartheta \rangle_J = \sum_{\nu} e^{i\nu\vartheta} \int D\Lambda^{(\nu)} e^{-S + JA}$$  \hspace{1cm} (13.68)

where $D\Lambda^{(\nu)}$ denotes the integration over all gauge field configurations with the winding number $\nu$. Replacing $\nu$ with the help of Eq. (13.50) and introducing $D\Lambda = \sum_{\nu} D\Lambda^{(\nu)}$ gives

$$\langle \vartheta' | \vartheta \rangle_J = \int D\Lambda e^{-S + JA} \exp \left\{ \frac{g^2}{16\pi^2} \vartheta \text{tr}(F\tilde{F}) \right\} .$$  \hspace{1cm} (13.69)

Thus instantons induce an additional term $\mathcal{L}_\vartheta$ to the classical Yang-Mills Lagrangian,

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \vartheta \frac{g^2}{16\pi^2} \text{tr}(F\tilde{F}) ,$$  \hspace{1cm} (13.70)

which depends on the arbitrary parameter $\vartheta \in [0, 2\pi[$. In order to discuss observable effects of this additional term, we have to add fermions to the pure Yang-Mills theory we discussed up to now.

**Fermionic contribution to $\vartheta$** The axial anomaly leads to the additional term in the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \frac{\alpha n_f g^2}{16\pi^2} \text{tr}(F\tilde{F}) ,$$  \hspace{1cm} (13.71)

if we perform a chiral $U_A(1)$ transformation $q_{L,R} \rightarrow e^{i\alpha \gamma^5} q_{L,R}$ on the $n_f$ quark fields. This term has the same structure as the instanton contribution, and if we choose

$$\alpha = -\frac{\vartheta}{2n_f}$$  \hspace{1cm} (13.72)

the two terms cancel. Thus for massless quarks the $\vartheta$ parameter is unphysical and we can choose the “simple” $\vartheta = 0$ vacuum.
13.2. Instantons and the strong CP problem

We can understand this if we consider the effect of an instanton transition. For zero masses, we know that
\[ \partial_\mu j_\mu = 16\pi^2 g^2 \text{tr}(F_{\mu\nu} \tilde{F}^{\mu\nu}). \tag{13.73} \]
Integrating gives as change of the axial charge
\[ \Delta Q_5 = Q_5(t = -\infty) - Q_5(t = +\infty) = 2\nu. \tag{13.74} \]
Thus an instanton process \( \nu = \pm 1 \) changes the axial quark number by two units, creating a left-chiral and destroying a right-chiral quark and vice versa. As chirality is a conserved quantum number for massless particles, at least one of the two states connected by the instanton process can therefore not correspond to the vacuum. By contrast, for \( m > 0 \) the mass term mixes left- and right-chiral fields: the quark-antiquark pair can annihilate via the mass term and the states can be identified with the vacuum.

We now consider massive quarks. Then the axial current is additionally to the axial anomaly explicitly broken by the quark masses, since a Dirac mass term transform as
\[ m\bar{q}q \rightarrow me^{2i\alpha \gamma^5} \bar{q}q = \cos(2\alpha)m\bar{q}q + i\sin(2\alpha)m\bar{q}\gamma^5 q \tag{13.75} \]
under a chiral transformation \( q_{L,R} \rightarrow e^{i\alpha \gamma^5} q_{L,R} \). The second term violates \( T \) (CP) invariance, and as a result the cancellation (13.72) will fail. If we consider \( n_f \) flavor of quarks, then the mass matrix \( M_{ij} \) will change as \( M_{ij} \rightarrow e^{2i\alpha}M_{ij} \) and thus
\[ \arg \det M \rightarrow \arg \det M + 2n_f \alpha. \tag{13.76} \]
Therefore only the combination
\[ \bar{\vartheta} \equiv \vartheta + \arg \det M \tag{13.77} \]
is an observable quantity: It is a question of convenience, if we choose real mass matrices and rotate all CP violation via the chiral anomaly into the \( \vartheta \) term. Or if we eliminate \( \mathcal{L}_\vartheta \) and transfer its effect into CP violating complex mass matrices.

We look now for observable consequences of the \( \vartheta \) vacuum in the low-energy interactions of hadrons. Since the \( \vartheta \) term (and the axial anomaly) is a flavor singlet, the change \( c \) in the quark mass,
\[ c \equiv \sin(2\alpha_q)m_q, \tag{13.78} \]
is the same for all quark flavours. In order to shift the effects of the \( \vartheta \) term completely into the mass matrix of the quarks, we need also
\[ \sum_{q=1}^{n_f} 2\alpha_q = -\vartheta. \tag{13.79} \]
Solving for small \( \vartheta \) gives
\[ c = -\frac{\vartheta}{\sum_q m_q^{-1}}, \tag{13.80} \]
and thus the \( \vartheta \) dependent, CP violating part of the mass term becomes
\[ \mathcal{L}^{(\vartheta)}_m = -i\vartheta \left( \sum_q m_q^{-1} \right)^{-1} \sum_q \bar{q}\gamma^5 q. \tag{13.81} \]
For light nucleon and mesons which consist only of $u$ and $d$ quarks this simplifies to
\[ \mathcal{L}_m^{(\vartheta)} = -i\vartheta \frac{m_u m_d}{m_u + m_d} (\bar{u}\gamma^5 u + \bar{d}\gamma^5 d) . \] (13.82)

This mass term generates an CP violating effective pion-nucleon interaction which in turn leads to an electric dipole moment $d_n$ of the neutron. The corresponding limit on $d_n$ bounds the value of $\vartheta$ as $|\vartheta| \leq 2 \times 10^{-10}$. Although the value of $\vartheta$ is a free parameter within the SM, it seems natural to ask for an explanation why $\vartheta$ is so small.

The most straightforward explanation would be that one current quark mass is zero, i.e. that one quark has no Yukawa coupling to the SM Higgs. Then (13.82) shows for $n_f = 2$ clearly that the CP violating effect disappear. This holds also for $n_f > 2$, because $\text{arg} \det M = 0$ if one mass eigenvalue is zero. While it has been debatable if the $m_u \sim 5 \text{MeV}$ deduced from chiral perturbation theory for the current $u$ quark mass might lowered down to $m_u = 0$, this possibility was closed by lattice data around 2005. The question why $\vartheta$ is so small is called the strong CP problem of the SM.

### 13.3. Axions

Peceei and Quinn proposed to promote the parameter $\vartheta$ to a dynamical variable which settles automatically at its minimum zero. The basic ingredient of this proposal is a new massless pseudo-scalar field $a$ called axion which couples to gluons with an interaction of the same structure as the $\vartheta$ term,
\[ \mathcal{L}_a = \frac{1}{2} (\partial_\mu a)^2 - \frac{g^2}{16\pi^2} a F \tilde{F} . \] (13.83)

Since $a$ is a pseudo-scalar, the interaction term $a F \tilde{F}$ conserves CP. Adding $\mathcal{L}_a$ to the effective Lagrangian (13.70) of QCD means that observables depend only on the combination $\vartheta - a/f_a$. If $\mathcal{L}_a$ is invariant under the shift $a \rightarrow a + \text{const.}$, then we can use this arbitrariness to absorb the $\vartheta$ parameter into a redefinition of the field $a$. Such a shift symmetry is typical for a Goldstone boson which has only derivative couplings. Thus the pseudo-scalar particle $a$ should be the Goldstone boson of a spontaneously broken global symmetry, and the $a F \tilde{F}$ interaction suggests to choose this symmetry as a chiral U(1) symmetry. This symmetry is called Peceeci-Quinn symmetry $U_{PQ}(1)$, its Goldstone boson $a$ axion.

Weinberg and Wilczek realised that there is an additional twist in this “simple” proposal: The effective potential (13.66) generated via the $F \tilde{F}$ term has two effects: First, the instanton effects break the shift symmetry, generating a mass term $m_a^2 = \partial^2 V_{\text{eff}}/\partial a^2$ for the axion. Second, the vev of the axion field will relax to the minimum of the potential, $\partial V_{\text{eff}}/\partial a = 0$. But we have argued above that the minimum of $V_{\text{eff}}(\vartheta)$ is situated at $\vartheta = 0$. Thus also in the case of a massive axion the strong CP problem is solved.

Let us illustrate the key points of this idea with a concrete model. We add a complex scalar field $\phi$ and a set of heavy fermions $\psi$ to the SM,
\[ \mathcal{L} = i \tilde{\psi} \gamma^\mu \partial_\mu \psi + \frac{1}{2} \partial_\mu \phi^\dagger \partial^\mu \phi - y_i (\bar{\psi}_L \psi_R \phi + \text{h.c.}) - V(\phi) . \] (13.84)

While $\phi$ is a SM singlet, the fermions are charged under SU(3) and U(1). The Lagrangian is invariant under the global chiral $U_{PQ}(1)$ gauge transformation
\[ \phi \rightarrow e^{i\alpha} \phi \quad \text{and} \quad \psi_{L/R} \rightarrow e^{\pm i\alpha/2} \psi_{L/R} . \] (13.85)
We now break spontaneously the Peccei-Quinn $U_{PQ}(1)$ symmetry, choosing as usually for $V(\phi)$ a Mexican hat potential. Splitting $\phi$ into its vacuum expectation value and the fluctuating fields,

$$\phi = \frac{f_a + \rho e^{ia/f_a}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we generate a mass term for $\rho$ while $a$ remains massless. We assume that $f_a$ is much larger than the energy scale $E$ we are interested in, and thus we can neglect the field $\rho$,

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} a)^2 - m_i \bar{\psi} \gamma^a / f_a \psi - V(\phi) + \mathcal{O}(E^2 / f_a^2).$$

The combined expression is clearly invariant under $U_{PQ}(1)$ transformations $a \to a + \alpha f_a$. Expanding the exponential, we generate mass terms $m_i = y_i f_a / \sqrt{2}$ for the heavy fermions plus fermion-axion interactions,

$$\mathcal{L}_{\text{int}} = -m_i \bar{\psi} \gamma^5 \psi - \frac{m_i}{f_a} a \bar{\psi} \gamma^5 \psi - \ldots$$

The latter lead to a AVV triangle graph for the process $a \to 2g$ which in turn induces via the chiral anomaly the desired $F \tilde{F}$ term. In the same way, an effective coupling $a \to 2\gamma$ coupling is generated. Thus the characteristic feature of an axion is its two-gluon and two-photon coupling.

Finally, we mention that the parameters of the pion and axion sector are connected by

$$m_a f_a \approx m_\pi f_\pi,$$

since the mass of both Goldstone bosons is generated by instantons and they mix $a \leftrightarrow 2g \leftrightarrow \pi^0$.

**Summary of chapter**

The CP-odd term $\tilde{F}_{\mu \nu} F^{\mu \nu}$ is a gauge invariant renormalisable interaction. Terms of this type are produced by instanton transitions between Yang-Mills vacua with different winding numbers and by the chiral anomaly. While we can rotate classically all CP violating phases contained in the quark mass matrices into the single CP violating phase of the CKM matrix, the chiral anomaly leads additionally to the change $\vartheta \to \bar{\vartheta} = \vartheta + \arg \det M$ in the coefficient of the $\tilde{F}_{\mu \nu} F^{\mu \nu}$ term. Since the physics origin of both contributions seem to be disconnected, it is puzzling that they sum up to $|\bar{\vartheta}| < 10^{-10}$. A possible solution is the Peccei-Quinn symmetry which promotes the parameter $\vartheta$ to the field $a = f_a \vartheta$ which settles automatically at the minimum at $\bar{\vartheta} = 0$ of the instanton potential $V_{\text{eff}}$.

**Further reading**

Instantons are discussed in more detail in [Rub02].

Wess-Zumino consistency condition, gravitational anomalies.

**Problems**
13. Anomalies, instantons and axions

13.1 Surface term.

Derive (13.20): Use Gauss’ theorem to convert the gradient term into a surface integral. Derive the trace formula for 6 gamma’s and one $\gamma^5$ and evaluate the trace. Transform the result into Euclidean space and evaluate it.

13.2 Pion decay width.

Calculate the decay width of $\pi^0 \rightarrow 2\gamma$ taking into account the chiral anomaly. Compare the value to the measured one.

13.3 Arg det M in the SM.

Determine arg(det(M)) in the SM and show that it is independent of a possible phase of the vev of the Higgs.

13.4 Euclidean YM.

Derive the connection between $A_{\mu}$ in Minkowski and Euclidean space. Show that this transformation law also leads to a real action for the gauged scalar field theory.
15. Gravity as a gauge theory

We introduce the action and the field equations of gravity, proceeding in a way which stresses the similarity of gravity as a gauge theory with the group GL(4) to Yang-Mills theories. As a bonus, this way allows us to describe also the gravitational interactions of fermions as well as to understand how gravity selects among the many mathematically possible connections on a Riemannian manifold. The linearised Einstein equations can be used to describe the emission and propagation of weak gravitational waves.

15.1. Vielbein formalism and the spin connection

The equivalence principle postulates that in a small enough region around the center of a freely falling coordinate system all physics is described by the laws of special relativity. At larger distances, gravity manifests itself as curvature of space-time. Thus physical laws involving only quantities transforming as tensors on Minkowski space are valid on a curved space-time performing the replacement

\[ \{ \partial_\mu, \eta_{\mu\nu}, d^4x \} \to \{ \nabla_\mu, g_{\mu\nu}, d^4x \sqrt{|g|} \}. \]  

(15.1)

Here, the covariant derivative $\nabla_\mu$ was defined using as connection the Christoffel symbols (or Levi-Civita connection) from Eq. (1.109). We recall that the two requirements $\nabla_\rho g_{\mu\nu} = 0$ (“metric connection”) and $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ (“torsionless connection”) select uniquely this connection. In the following, we want to understand if these conditions are a consequence of Einstein gravity or necessary additional constraints.

A useful framework to address these questions is the Vielbein formalism which is also necessary to include spinors into the framework of general relativity. Since the Levi-Civita connection can be applied only to objects with tensorial indices, the substitution rule $\partial_\mu \to \nabla_\mu$ cannot be applied to the case of spinor representations of the Lorentz group.

**Vielbein formalism** We apply the equivalence principle as physical guideline to obtain the physical laws including gravity: More precisely, we use that we can find at any point $P$ a local inertial frame for which the physical laws become those known from Minkowski space. We demonstrate this first for the case of a scalar field $\phi$. The usual Lagrange density without gravity,

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \]  

(15.2)

is still valid on a general manifold $\mathcal{M}(\{ x^\mu \})$, if we use in each point $P$ locally free-falling coordinates, $\xi^a(P)$. In order to distinguish these two sets of coordinates, we label inertial coordinates by latin letters $(a,b,...)$ while we keep greek indices $\mu, \nu,...$ for arbitrary coordinates. We choose the locally free-falling coordinates $\xi^a$ to be orthonormal. Thus in these coordinates the metric is given by $ds^2 = \eta_{ab}d\xi^a d\xi^b$ with $\eta = \text{diag}(1,-1,-1,-1)$. Then the...
15. Gravity as a gauge theory

action of a scalar field including gravity is

$$S[\phi] = \int d^4 \xi \left[ \frac{1}{2} \eta^{ab} \partial_a \phi \partial_b \phi - V(\phi) \right],$$  \hspace{1cm} (15.3)

where $\partial_a = \partial / \partial \xi_a$. This action looks formally exactly as the one without gravity – however we have to integrate over the manifold $M(\{x^\mu\})$ and all effects of gravity are hidden in the dependence $\xi^a(x^\mu)$.

We introduce now the vierbein (or tetrad) fields $e^a_m$ by

$$d\xi^m = \partial \xi^m / \partial x^\mu \equiv e^m_\mu(x) \, dx^\mu \,.$$ \hspace{1cm} (15.4)

Thus we can view the vierbein $e^m_\mu(x)$ both as the transformation matrix between arbitrary coordinates $x$ and inertial coordinates $\xi$ or as a set of four vectors in $T^*_x M$. In the absence of gravity, we can find in the whole manifold coordinates such that $e^m_\mu(x) = \delta^m_\mu$.

We define analogously the inverse vierbein $e^m_n$ by

$$dx^\mu = \partial x^\mu / \partial \xi^m \equiv e^m_\mu(x) \, d\xi^m.$$ \hspace{1cm} (15.5)

The name is justified by

$$d\xi^m = e^m_\mu \, dx^\mu = e^m_\mu \, e^n_\nu \, d\xi^n,$$ \hspace{1cm} (15.6)

i.e. $e^m_\mu e^n_\nu = \delta^m_n$. Moreover, we can view the vierbein as a kind of square-root of the metric tensor, since

$$ds^2 = \eta_{mn} d\xi^m d\xi^n = \eta_{mn} e^m_\mu e^n_\nu dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu,$$ \hspace{1cm} (15.7)

and thus

$$g_{\mu\nu} = \eta_{mn} e^m_\mu e^n_\nu.$$ \hspace{1cm} (15.8)

Taking the determinant of this equation, we see that the volume element is

$$d^4 \xi = \sqrt{|g|} d^4 x = \det(e^m_\mu) d^4 x \equiv E d^4 x.$$ \hspace{1cm} (15.9)

For later use, we note that latin indices are raised and lowered by the flat metric. It is also possible to construct mixed tensors, having both latin and greek indices. For instance, we can rewrite the energy-momentum tensor as $T_{\mu\nu} = e^m_\mu T_{mn} = e^m_\mu e^n_\nu T_{mn}$.

We have now all the ingredients needed to express (15.3) in arbitrary coordinates $x^\mu$ of the manifold $M$. We first change the derivatives,

$$L = \frac{1}{2} \eta_{mn} e^m_\mu e^n_\nu \partial^\mu \phi \partial^\nu \phi - V(\phi) = \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - V(\phi)$$ \hspace{1cm} (15.10)

and then the volume element in the action,

$$S[\phi] = \int d^4 x \sqrt{|g|} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right].$$ \hspace{1cm} (15.11)

As it should, we reproduced the usual action of a scalar field including gravity. Note that the sole effect of gravitational interactions is contained in the metric tensor and its determinant, while the connection plays no role since $\nabla_\mu \phi = \partial_\mu \phi$. Similarly, the connection drops out of the Lagrangian of a Yang-Mills field, since the field-strength tensor is antisymmetric.
15.1. Vielbein formalism and the spin connection

**Fermions and the spin connection**  We now proceed to the spin-1/2 case. Without gravity, we have
\[ L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \]  (15.12)
with \( \{\gamma^\mu, \gamma^\nu\} = 2\eta^\mu\nu \). Performing a Lorentz transformation, \( \tilde{x}^a = \Lambda^b_a x^b \), the Dirac spinor \( \psi \) transforms as
\[ \tilde{\psi}(\tilde{x}) = S(\Lambda)\psi(x) = \exp \left( -\frac{i}{4} \omega^{\mu\nu} \sigma_{\mu\nu} \right) \psi(x) \]  (15.13)
with \( \omega^{\mu\nu} = -\omega^{\nu\mu} \) as the six parameters and \( \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \) as the six infinitesimal generators of these transformations.

Switching on gravity, we replace \( x^\mu \rightarrow \xi^m \) and \( \gamma^\mu \rightarrow \gamma^m \). General covariance reduces now to the requirement that we have to allow in an inertial system arbitrary Lorentz transformations. The condition that the Dirac equation is invariant under local Lorentz transformations \( \Lambda(x) \) allows us to derive the correct covariant derivative, in a manner completely analogous to the Yang-Mills case: We have to compensate the term introduced by the space-time dependence of \( S(\xi) \) in
\[ \partial_a \psi \rightarrow \tilde{\partial}_a \tilde{\psi}(\tilde{x}) = \Lambda^b_a \partial_b [S(\xi)\psi(x)] \]  (15.14)
by introducing a “latin” covariant derivative
\[ \nabla_a = e^a_\alpha (\partial_\alpha + i\omega_\alpha) \]  (15.15)
and requiring an inhomogeneous transformation law
\[ \omega_\alpha \rightarrow \tilde{\omega}_\alpha = S\omega_\alpha S^{-1} - iS\partial_\alpha S^\dagger \]  (15.16)
for \( \omega \). As a result,
\[ \nabla_a \rightarrow \tilde{\nabla}_a = \Lambda^b_a S\nabla_b S^\dagger \]  (15.17)
and the Dirac Lagrangian is invariant under local Lorentz transformations.

The connection \( \omega_\alpha \) is a matrix in spinor space. Expanding it in the basis \( \sigma^{\mu\nu} \), we find as a more explicit expression for the covariant derivative
\[ \nabla_a = \partial_a + \frac{i}{2} \omega_a^{\mu\nu} \sigma_{\mu\nu} = e^a_\alpha \left( \partial_\alpha + i\frac{1}{2} \omega_\alpha^{\mu\nu} \sigma_{\mu\nu} \right) = e^a_\alpha \left( \partial_\alpha + \frac{1}{2} \omega_\alpha^{\mu\nu} X_{\mu\nu} \right) \]  (15.18)
In the second step we replaced the infinitesimal generators \( \sigma^{\mu\nu} \) specific for the spinor representation by the general generators \( X^{\mu\nu} \) of Lorentz transformations chosen appropriate for the representation that the \( \nabla_a \) act on. The Lie algebra of the Lorentz group implies that the connection \( \omega_a^{\mu\nu} \) is antisymmetric in the indices \( \mu\nu \), if they are both up or down, \( \omega_a^{\mu\nu} = -\omega_a^{\nu\mu} \).

The transformation law (15.16) of the spin connection \( \omega \) under Lorentz transformations \( S \) is completely analogous to the transformation properties (7.14) of a Yang-Mills field \( A^\mu \) under gauge transformations \( U \). One should keep in mind however two important differences: First, a vector lives in a tangent space which is naturally associated to a manifold: In particular, we can associate a vector in \( TP\mathcal{M} \) with a trajectory \( x(\sigma) \) through \( P \). Therefore we have the natural coordinate basis \( \partial_\sigma \) in \( TP\mathcal{M} \) and can introduce vielbein fields. In contrast, matter fields \( \psi(x) \) lives in their group manifold which is attached arbitrarily at each point of the manifold, and the gauge fields act as a connection telling us how we should transport \( \psi(x) \) to
Thus, the vielbein field is zero.

Next we compare this expression to the one using a mixed basis, where the indices are contracted with the usual connection $\Gamma^{\lambda}_{\mu\nu}$ as follows:

\[ \partial_{\mu} A^{\nu} = \partial_{\mu}(e_{\nu}^{a} A^{a}) = e_{\nu}^{a}(\partial_{\mu} A^{a}) + (\partial_{\mu} e_{\nu}^{a}) A^{a}. \tag{15.21} \]

Using this, we can eliminate the second term in (15.20), obtaining

\[ \nabla_{\mu} A^{a} = (\nabla_{\mu} e_{\nu}^{a}) A^{\nu} + \partial_{\mu} A^{a} - (\partial_{\mu} e_{\nu}^{a}) A^{\nu} + e_{\nu}^{a} \Gamma^{\nu}_{\mu\lambda} A^{\lambda}. \tag{15.22} \]

Comparing to (15.19), we can read off how the covariant derivative acts on an object with mixed indices,

\[ \nabla_{\mu} e_{\nu}^{a} = \partial_{\mu} e_{\nu}^{a} - \Gamma^{\lambda}_{\nu\mu} e_{\lambda}^{a} + \omega_{\mu b}^{a} e_{\nu}^{b}. \tag{15.23} \]

More generally, covariant indices are contracted with the usual connection $\Gamma^{\lambda}_{\mu\nu}$ while vierbein indices are contracted with $\omega_{\mu b}^{a}$. In a moment, we will show that the covariant derivative of the vielbein field is zero, $\nabla_{\mu} e_{\nu}^{a} = 0$. Sometimes this property is called tetrade “postulate,” but in fact it follows naturally from the definition of the vierbein field.

In order to derive an explicit formula for the spin connection $\omega_{\mu b}^{a}$, we compare now the covariant derivative of a vector in the two formalisms. First, we write in a coordinate basis

\[ \nabla A = (\nabla_{\mu} A^{\nu}) dx^{\mu} \otimes \partial_{\nu} = (\partial_{\mu} A^{\nu} + \Gamma^{\nu}_{\mu\lambda} A^{\lambda}) dx^{\mu} \otimes \partial_{\nu}. \tag{15.24} \]

Next we compare this expression to the one using a mixed basis,

\[ \nabla A = (\nabla_{\mu} e_{\nu}^{a}) dx^{\mu} \otimes e_{m} = (\partial_{\mu} A^{m} + \omega_{\mu n}^{m} A^{n}) dx^{\mu} \otimes e_{m}. \tag{15.25} \]

Moving $e_{m}^{\sigma}$ to the left and using the Leibniz rule as well as $e_{m}^{\sigma} e_{m}^{m} = \delta_{\nu}^{\sigma}$, it follows

\[ \nabla A = e_{m}^{\sigma}[e_{m}^{m} \partial_{\mu} A^{\nu} + A^{\nu} \partial_{\mu} e_{m}^{m} + \omega_{\mu l}^{m} e_{l}^{\nu} A^{\lambda}] dx^{\mu} \otimes \partial_{\sigma} \tag{15.27} \]

\[ = [\partial_{\mu} e_{m}^{\sigma} + e_{m}^{\sigma} (\partial_{\mu} e_{m}^{m}) A^{\nu} + e_{m}^{\sigma} \omega_{\mu n}^{m} e_{n}^{\nu} A^{\lambda}] dx^{\mu} \otimes \partial_{\sigma}. \tag{15.28} \]

Thus

\[ \Gamma^{\nu}_{\mu\lambda} = e_{m}^{\sigma} \partial_{\mu} e_{m}^{m} + e_{m}^{\sigma} \omega_{\mu n}^{m} e_{n}^{m} = e_{m}^{m} \nabla e_{m}^{\nu}. \tag{15.29} \]
15.2. Action of gravity

or multiplying with two vierbein fields,

\[ \omega^m_{\mu} = e^\sigma_{m} e^n \Gamma^{\nu}_{\mu \lambda} - e^\sigma_{m} \partial_{\mu} e^n_{\nu} \]  
(15.30)

Clearly, the connection \( \Gamma^{\nu}_{\mu \lambda} \) is in general not symmetric, \( \Gamma^{\nu}_{\mu \lambda} \neq \Gamma^{\nu}_{\lambda \mu} \).

Next we show that the covariant derivative of the vierbein vanishes. We start from the covariant derivative of a general mixed two-tensor,

\[ \nabla_\rho X_{\nu \rho} = \partial_\rho X_{\nu \rho} + \Gamma^\lambda_{\nu \mu} X_{\nu \rho} + \omega^a_{\mu \rho} X_{\nu \rho} \]  
(15.31)

Setting now \( X_{\nu \rho} = \eta_{pq} e^q_{\nu} \) and expressing the connection via Eq. (15.29), it follows

\[ \nabla_\rho e_{\nu \rho} = \partial_\rho e_{\nu \rho} + \omega^a_{\rho \nu} + e^\sigma_{\rho \nu} \nabla_\rho e^a_{\sigma} = \partial_\rho e_{\nu \rho} - \partial_\rho e_{\nu \rho} + \omega^a_{\rho \nu} + \omega_{\rho \nu \rho} \]  
(15.32)

because of the antisymmetry of the spin connection. Similarly, one shows that \( \nabla_\rho e_{\nu \rho} = 0 \).

15.2. Action of gravity

**Einstein-Hilbert action** The analogy to the Yang-Mills case suggests as Lagrange density for the gravitational field

\[ \mathcal{L} = \sqrt{|g|} R_{\mu \nu \sigma \tau} R^{\mu \nu \sigma \tau} \]  
(15.34)

This Lagrange density has mass dimension 4 and would thus lead to a dimensionless gravitational coupling constant and a renormalisable theory of gravity. However, such a theory would be in contradiction to Newton’s law. Hilbert chose instead the curvature scalar which has the required mass dimension \( d = 6 \),

\[ \mathcal{L}_{EH} = \sqrt{|g|} R \]  
(15.35)

As we know, we can always add a constant term to the Lagrangian, \( R \rightarrow R - 2\Lambda \), which would act as cosmological constant. The Lagrangian is a function of the metric, its first and second derivatives. \( \mathcal{L}_{EH}(g_{\mu \nu}, \partial_\rho g_{\mu \nu}, \partial_\rho \partial_\sigma g_{\mu \nu}) \). The resulting action

\[ \delta S_{EH}[g_{\mu \nu}] = \int_{\Omega} d^4 x \sqrt{|g|} \left\{ R - 2\Lambda \right\} \]  
(15.36)

is a functional of the metric tensor \( g_{\mu \nu} \), and a variation of the action with respect to the metric gives the field equations for the gravitational field. If we consider gravity coupled to fermions, we have to use the spin connection \( \omega_{\mu} \) in the matter Lagrangian as well as expressing \( \sqrt{|g|} \) and \( R \) through latin quantities, \( \sqrt{|g|} \rightarrow E \) and \( R = R_{\mu \nu} g^{\mu \nu} \rightarrow R_{\mu \nu} \eta^{\mu \nu} \).

We derive the resulting field equations for the metric tensor \( g_{ab} \) directly from the action principle

\[ \delta S_{EH} = \delta \int_{\Omega} d^4 x \sqrt{|g|}(R - 2\Lambda) = \delta \int_{\Omega} d^4 x \sqrt{|g|} (g^{\mu \nu} R_{\mu \nu} - 2\Lambda) = 0 \]  
(15.37)

\(^1\)Recall that the Lagrange equations are modified in the case of higher derivatives which is one reason why we directly vary the action in order to obtain the field equations.
15. Gravity as a gauge theory

We allow for variations of the metric \( g_{\mu\nu} \) restricted by the condition that the variation of \( g_{\mu\nu} \) and its first derivatives vanish on the boundary \( \partial\Omega \).

\[
\delta S_{\text{EH}} = \int_{\Omega} d^4x \left\{ \sqrt{|g|} \, g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{|g|} R_{\mu\nu} \delta g^{\mu\nu} + (R - 2\Lambda) \, \delta \sqrt{|g|} \right\} .
\] (15.38)

Our task is to rewrite the first and third term as variations of \( \delta g^{\mu\nu} \) or to show that they are equivalent to boundary terms.

Let us start with the first term. Choosing normal coordinates, the Ricci tensor at the considered point \( P \) becomes

\[
R_{\mu\nu} = \partial_{\rho} \Gamma^\rho_{\mu\nu} - \partial_{\nu} \Gamma^\rho_{\mu\rho} .
\] (15.39)

Hence

\[
g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \left( \partial_{\rho} \delta \Gamma^\rho_{\mu\nu} - \partial_{\nu} \delta \Gamma^\rho_{\mu\rho} \right) = g^{\mu\nu} \partial_{\rho} \delta \Gamma^\rho_{\mu\nu} - g^{\mu\rho} \partial_{\rho} \delta \Gamma^\nu_{\mu\nu} ,
\] (15.40)

where we exchanged the indices \( \nu \) and \( \rho \) in the last term. Since \( \partial_{\rho} g_{\mu\nu} = 0 \) at \( P \), we can rewrite the expression as

\[
g^{\mu\nu} \delta R_{\mu\nu} = \partial_{\rho} \left( g^{\mu\nu} \delta \Gamma^\rho_{\mu\nu} - g^{\mu\rho} \delta \Gamma^\nu_{\mu\nu} \right) = \partial_{\rho} X^\rho .
\] (15.41)

The quantity \( X^\rho \) is a vector, since the difference of two connection coefficients transforms as a tensor. Replacing in Eq. (15.41) the partial derivative by a covariant one promotes it therefore in a valid tensor equation,

\[
g^{\mu\nu} \delta R_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_{\mu} \left( \sqrt{|g|} X^\mu \right) .
\] (15.42)

Thus this term corresponds to a surface term that vanishes.

Next we rewrite the third term using

\[
\delta \sqrt{|g|} = \frac{1}{2\sqrt{|g|}} \delta |g| = \frac{1}{2\sqrt{|g|}} g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2} \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu}
\] (15.43)

and obtain

\[
\delta S_{\text{EH}} = \int_{\Omega} d^4x \sqrt{|g|} \left\{ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right\} \delta g^{\mu\nu} = 0 .
\] (15.44)

Hence the metric tensor fulfills in vacuum the equation

\[
\frac{1}{\sqrt{|g|}} \frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 ,
\] (15.45)

where we introduced the Einstein tensor \( G_{\mu\nu} \). The constant \( \Lambda \) is called the cosmological constant.

We consider now the combined action of gravity and matter, as the sum of the Einstein-Hilbert Lagrange density \( L_{\text{EH}}/2\kappa \) and the Lagrange density \( L_m \) including all relevant matter fields,

\[
\mathcal{L} = \frac{1}{2\kappa} L_{\text{EH}} + L_m = \frac{1}{2\kappa} \sqrt{|g|} (R - 2\Lambda) + L_m .
\] (15.46)

We will determine the value of the coupling constant \( \kappa \) in the next section, demanding that we reproduce Newtonian dynamics in the weak-field limit. We have already argued that
the source of the gravitational field is the energy-momentum stress tensor which lead to the definition
\[ \frac{2}{\sqrt{|g|}} \delta S_m = T_{\mu \nu} \, . \] (15.47)

Einstein’s field equation follows then as
\[ G_{\mu \nu} + \Lambda g_{\mu \nu} = -\kappa T_{\mu \nu} \, . \] (15.48)

**Palatini action** We start from the Einstein-Hilbert Lagrangian \( \mathcal{L}_{EH}(g_{\mu \nu}, \Gamma^\rho_{\mu \nu}, \partial_\tau \Gamma^\rho_{\mu \nu}) \), while we allow an independent variation of the metric tensor and the connection in the action. We obtain the desired dependence of the Lagrangian expressing the Ricci tensor through the connection and its derivatives,
\[ L_{EH} = \sqrt{|g|} g^{\mu \nu} R_{\mu \nu} = \sqrt{|g|} g^{\mu \nu} \left( \partial_\nu \Gamma^\rho_{\mu \sigma} - \partial_\sigma \Gamma^\rho_{\mu \nu} + \Gamma^\tau_{\rho \sigma} \Gamma^\rho_{\mu \tau} - \Gamma^\tau_{\rho \mu} \Gamma^\rho_{\sigma \tau} \right) \, . \] (15.49)

The variation with respect to the metric,
\[ \delta g S_{EH} = \int_\Omega d^4 x \, \delta \left( \sqrt{|g|} g^{\mu \nu} \right) R_{\mu \nu} = 0 \] (15.50)
gives \( R_{\mu \nu} = 0 \), i.e. the usual Einstein equations in vacuum.

For the variation with respect to the connection we use first the Palatini equation,
\[ \delta \Gamma S_{EH} = \int_\Omega d^4 x \sqrt{|g|} g^{\mu \nu} \delta R_{\mu \nu} = \int_\Omega d^4 x \sqrt{|g|} g^{\mu \nu} \left[ \nabla_\nu (\delta \Gamma^\rho_{\mu \sigma}) - \nabla_\sigma (\delta \Gamma^\rho_{\mu \nu}) \right] \, . \] (15.51)

Applying then the Leibniz rule and relabelling some indices, we find
\[ \delta \Gamma S_{EH} = \int_\Omega d^4 x \sqrt{|g|} \left[ g^{\mu \nu} \delta \Gamma^\rho_{\mu \sigma} - g^{\rho \sigma} \delta \Gamma^\rho_{\mu \nu} \right] \]
\[ + \int_\Omega d^4 x \sqrt{|g|} \left[ (\nabla_\nu g^{\rho \sigma}) \delta \Gamma^\rho_{\mu \sigma} - (\nabla_\sigma g^{\rho \nu}) \delta \Gamma^\rho_{\mu \nu} \right] \, . \] (15.52)

We kept the second line, because we consider an arbitrary connection. Thus we do not know yet if the covariant derivative of the metric vanishes.

Next we perform a partial integration of the first two terms, converting it into a surface term which we can drop. In the remaining part we relabel indices so that we can factor out the variation of the connection,
\[ \delta \Gamma S_{EH} = - \int_\Omega d^4 x \sqrt{|g|} \left[ \delta^\rho_\mu \nabla_\sigma g^{\mu \nu} - \nabla_\rho g^{\mu \nu} \right] \delta \Gamma^\rho_{\mu \nu} \, . \] (15.53)

We use now that the connection is symmetric in the absence of fermion. Then also the variation \( \delta \Gamma^\rho_{\mu \nu} \) is symmetric and the antisymmetric part in the square bracket drops out. Asking that \( \delta \Gamma S_{EH} = 0 \) gives therefore
\[ \frac{1}{2} \delta^\rho_\mu \nabla_\sigma g^{\mu \nu} + \frac{1}{2} \delta^\rho_\mu \nabla_\sigma g^{\mu \nu} - \nabla_\rho g^{\mu \nu} = 0 \] (15.54)
or \( \nabla_\sigma g^{\mu \nu} = \nabla_\sigma g_{\mu \nu} = 0 \). Thus the Einstein-Hilbert action implies the metric compatibility of the connection.
Performing the same exercise with the Einstein-Hilbert plus the matter action considered as functional of the vierbein $e^\mu_m$ and the connection $\omega_\mu$, one finds the following: From the variation $\delta S_{\text{EH}}$ one obtains automatically a metric connection which is however in general not symmetric. The torsion is sourced by the spin-density of fermions. The variation $\delta e S_{\text{EH}}$ gives the usual Einstein equation.

### 15.3. Linearized gravity

We are looking for small perturbations $h_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \ll 1.$$  \hspace{1cm} (15.55)

These perturbations may be caused either by the propagation of gravitational waves or by the gravitational potential of a star. In the first case, current experiments show that we should not hope for $h$ larger than $O(h) \sim 10^{-22}$. Keeping only terms linear in $h$ is therefore an excellent approximation. Choosing in the second case as application the final phase of the spiral-in of a neutron star binary system, deviations from Newtonian limit can become large. Hence one needs a systematic “post-Newtonian” expansion or even a numerical analysis to describe properly such cases.

**Linearized Einstein equations in vacuum** From $\partial_\mu \eta_{\nu\rho} = 0$ and the definition

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} (\partial_\nu g_{\kappa\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\kappa g_{\nu\lambda}).$$  \hspace{1cm} (15.56)

we find for the change of the connection linear in $h$

$$\delta \Gamma^\mu_{\nu\lambda} = \frac{1}{2} \eta^{\mu\kappa} (\partial_\nu h_{\kappa\lambda} + \partial_\lambda h_{\nu\kappa} - \partial_\kappa h_{\nu\lambda}) = \frac{1}{2} (\partial_\nu h^\mu_\lambda + \partial_\lambda h^\mu_\nu - \partial^\mu h_{\nu\lambda}).$$  \hspace{1cm} (15.57)

Here we used $\eta$ to raise indices which is allowed in linear approximation. Remembering the definition of the Riemann tensor,

$$R^\mu_{\nu\lambda\kappa} = \partial_\lambda \Gamma^\mu_{\nu\kappa} - \partial_\kappa \Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\epsilon\lambda} \Gamma^\epsilon_{\nu\kappa} - \Gamma^\mu_{\epsilon\kappa} \Gamma^\epsilon_{\nu\lambda},$$  \hspace{1cm} (15.58)

we see that we can neglect the terms quadratic in the connection terms. Thus we find for the change

$$\delta R^\mu_{\nu\lambda\kappa} = \partial_\kappa \delta \Gamma^\mu_{\nu\lambda} - \partial_\lambda \delta \Gamma^\mu_{\nu\kappa} = \frac{1}{2} \left\{ \partial_\lambda \partial_\nu h^\mu_\kappa + \partial_\lambda \partial_\kappa h^\mu_\nu - (\partial_\nu \partial_\kappa h^\mu_\lambda + \partial_\lambda \partial_\mu h^\mu_\nu - \partial_\kappa \partial^\mu h_{\nu\lambda}) \right\}$$  \hspace{1cm} (15.59)

The change in the Ricci tensor follows by contracting $a$ and $c$,

$$- \delta R_{\nu\kappa} = \delta R^\mu_{\nu\lambda\kappa} = \frac{1}{2} \left\{ \partial_\lambda \partial_\nu h^\mu_\kappa + \partial_\kappa \partial^\lambda h_{\nu\lambda} - \partial_\lambda \partial^\lambda h_{\nu\kappa} - \partial_\kappa \partial_\nu h^\mu_\lambda \right\}.$$  \hspace{1cm} (15.60)

Next we introduce $h \equiv h^\mu_\mu$, $\Box = \partial_\mu \partial^\mu$, and relabel the indices,

$$- \delta R_{\mu\nu} = \frac{1}{2} \left\{ \partial_\mu \partial_\nu h^\rho_\rho + \partial_\nu \partial_\mu h^\rho_\rho - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h \right\}.$$  \hspace{1cm} (15.61)
We now rewrite all terms apart from $\Box h_{\mu\nu}$ as derivatives of the vector

$$\xi_\mu = \partial_\nu h_\mu^\nu - \frac{1}{2} \partial_\mu h,$$

(15.62)
obtaining

$$- \delta R_{\mu\nu} = \frac{1}{2} \{- \Box h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \}.$$  (15.63)

Looking back at the properties of $h_{\mu\nu}$ under gauge transformations, Eq. (4.44), we see that we can gauge away the second and third term. Thus the linearized Einstein equation in vacuum becomes simply

$$\Box h_{\mu\nu} = 0$$

(15.64)
if the harmonic gauge

$$\xi_\mu = \partial_\nu h_\mu^\nu - \frac{1}{2} \partial_\mu h = 0,$$

(15.65)
is chosen. Thus the familiar wave equation holds for all ten independent components of $h_{\mu\nu}$, and the perturbations propagate with the speed of light $c$. Inserting plane waves $h_{\mu\nu} = \varepsilon_{\mu\nu} \exp(ikx)$ into the wave equation, one finds immediately that $k$ is a null vector.

**TT gauge** We can recover our old result for the polarization tensor describing the physical states contained in a gravitational perturbation. We consider a plane wave $h_{\mu\nu} = \varepsilon_{\mu\nu} \exp(ikx)$. After fixing the harmonic gauge $\partial^\mu h_{\mu\nu} = 0$, we can still add four function $\xi_\mu$ with $\Box \xi_\mu = 0$. We can choose them such that four components of $h_{\mu\nu}$ vanish. In the TT gauge, we set $(i = 1, 2, 3)$

$$h_{0i} = 0, \quad h = 0.$$  (15.66)

The harmonic gauge condition becomes $\xi_\alpha = \partial_\beta h_\alpha^\beta$ or

$$\xi_0 = \partial_\beta h_0^\beta = \partial_0 h_0^0 = -i\omega \varepsilon_{00} e^{ikx} = 0,$$

(15.67)
$$\xi_a = \partial_\beta h_a^\beta = \partial_0 h_a^0 = -ik_0 \varepsilon_{0a} e^{ikx} = 0.$$  (15.68)

Thus $\varepsilon_{00} = 0$ and the polarization tensor is transverse, $k^b \varepsilon_{ab} = 0$. If we choose the plane wave propagating in $z$ direction, $k = ke_z$, the $z$ raw and column of the polarization tensor vanishes too. Accounting for $h = 0$ and $\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}$, only two independent elements are left, and we recover our old result,

$$\varepsilon = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  (15.69)

In general, one can construct the polarization tensor in TT gauge by setting first the non-transverse part to zero and then subtracting the trace. The resulting two independent elements are (again for $k = ke_z$) then $\varepsilon_{11} = 1/2(\varepsilon_{xx} - \varepsilon_{yy})$ and $\varepsilon_{12}$.
15. Gravity as a gauge theory

**Linearized Einstein equations with sources** We rewrite first the Einstein equation in an alternative form where the only geometrical term on the LHS is the Ricci tensor. Because of

\[ R_{\mu}^{\mu} - \frac{1}{2} g_{\mu}^{\mu} (R - 2\Lambda) = R - 2(R - 2\Lambda) = -R + 4\Lambda = -\kappa T_{\mu}^{\mu} \]  
(15.70)

we can perform with \( T \equiv T_{\mu}^{\mu} \) the replacement \( R = 4\Lambda - \kappa T \) in the Einstein equation and obtain

\[ R_{\mu\nu} = -\kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + g_{\mu\nu} \Lambda. \]  
(15.71)

This form of the Einstein equations is often useful, when it is easier to calculate \( R \) that \( T \).

Now we move on the determination of the linearized Einstein equations with sources. We found \( \delta R_{\mu\nu} = \Box h_{\mu\nu} \). By contraction follows \( \delta R = \Box h \). Combining both terms gives

\[ \Box \left( h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \right) = 2(\delta R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \delta R) = -2\kappa \delta T_{\mu\nu}. \]  
(15.72)

Since we assumed an empty universe in zeroth order, \( \delta T_{\mu\nu} \) is the complete contribution to the energy-momentum tensor. We omit therefore in the following the \( \delta \) in \( \delta T_{\mu\nu} \).

We introduce as useful short-hand notation the “trace-reversed” amplitude as

\[ \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h. \]  
(15.73)

The harmonic gauge condition becomes then

\[ \partial_\mu \bar{h}_{\mu\nu} = 0 \]  
(15.74)

and the linearized Einstein equation in the harmonic gauge follow as

\[ \Box \bar{h}_{\mu\nu} = -2\kappa \bar{T}_{\mu\nu}. \]  
(15.75)

Because of \( \bar{h}_{\mu\nu} = h_{\mu\nu} \) and Eq. (15.71), we can rewrite this wave equation also as

\[ \Box h_{\mu\nu} = -2\kappa \bar{T}_{\mu\nu}, \]  
(15.76)

with the trace-reversed energy-momentum tensor \( \bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \).

**Newtonian limit** The Newtonian limit corresponds to \( v/c \to 0 \) and thus the only non-zero element of the energy-momentum tensor becomes \( T^{tt} = \rho \). We compare the metric

\[ ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi) \left( dx^2 + dy^2 + dz^2 \right) \]  
(15.77)

to Eq. (15.55) and find as metric perturbations

\[ h_{tt} = 2\Phi \quad h_{ij} = 2\delta_{ij} \Phi \quad h_{ti} = 0. \]  
(15.78)

In the static limit \( \Box \to -\Delta \) and \( v = 0 \), and thus

\[ -\Delta \left( h_{00} - \frac{1}{2} \eta_{00} h \right) = -4\Delta \Phi = -2\kappa \rho. \]  
(15.79)

Hence the linearized Einstein equation has the same form as the Newtonian Poisson equation, and the constant \( \kappa \) equals \( \kappa = 8\pi G \).
15.3. Linearized gravity

**Detection principle of gravitational waves** Consider the effect of a gravitational wave on a free test particle that is initially at rest, \( u^a = (1, 0, 0, 0) \). As long as the particle is at rest, the geodesic equation simplifies to \( \dot{u}^a = \Gamma^a_{\phantom{a}00} \). The four relevant Christoffel symbols are in linearized approximation, cf. Eq. (15.57),

\[
\Gamma^a_{\phantom{a}00} = \frac{1}{2} \left( \partial_0 h^a_0 + \partial_0 h^a_0 - \partial^a h_{00} \right).
\]  

We are free to choose the TT gauge in which all component of \( h_{ab} \) appearing on the RHS are zero. Hence the acceleration of the test particle is zero and its coordinate position is unaffected by the gravitational wave: The TT gauge defines a “comoving” coordinate system.

The physical distance \( l \) of e.g. two test-particles is given by integrating

\[
dl^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta = (h_{\alpha\beta} - \delta_{\alpha\beta}) d\xi^\alpha d\xi^\beta,
\]

where \( g_{\alpha\beta} \) is the spatial part of the metric and \( d\xi \) the spatial coordinate distance between infinitesimal separated test particles. Hence the passage of a periodic gravitational wave, \( h_{ab} \propto \cos(\omega t) \), results in a periodic change of the separation of freely moving test particles. The relative size of this change, \( \Delta L/L \), is given by the amplitude \( h \) of the gravitational wave.

**Linearized action of gravity** We have derived the linearized wave equation describing gravitational waves directly from the Einstein equation. Now we want to obtain the linearized action of gravity. A straightforward but lengthy approach would be to expand the Einstein-Hilbert action to \( O(h^3) \). Instead, we profit from our knowledge of the graviton propagator, which we derived in chapter 4 in Eq. (4.49),

\[
D^\mu
_{\nu;\rho;\sigma}(k) = \frac{1}{k^2 + i\varepsilon} \left( -\eta^\mu\nu \eta^\rho\sigma + \eta^\mu\rho \eta^\nu\sigma + \eta^\mu\sigma \eta^\rho\nu \right) = \frac{P^\mu\nu;\rho\sigma(k)}{k^2 + i\varepsilon}.
\]  

The corresponding Lagrangian is as usually quadratic in the fields, \( \frac{1}{2} h^\mu\nu P_{\nu;\rho;\sigma} \Box h^{\rho\sigma} \). Performing partial integrations and the contractions with the metric tensors gives us as the corresponding action

\[
S = \frac{1}{32\pi G} \int d^4x \left[ \frac{1}{2} \left( \partial_\alpha h_{\beta\gamma} \right)^2 - \frac{1}{4} \left( \partial_\alpha h \right)^2 \right].
\]  

We know that the propagator of a massless particles with helicity \( h > 0 \) can be inverted only, if either the gauge freedom is completely fixed or a gauge-fixing term is added. In contrast to the TT gauge, the harmonic gauge contains still unphysical degrees of freedom. Thus the expression obtained corresponds to the quadratic Einstein-Hilbert Lagrangian in the harmonic gauge plus a gauge-fixing term,

\[
\mathcal{L}^{(2)}_{EH} + \mathcal{L}_{gf} = \frac{1}{32\pi G} \left[ \frac{1}{2} \left( \partial_\alpha h_{\beta\gamma} \right)^2 - \frac{1}{4} \left( \partial_\alpha h \right)^2 \right] = \mathcal{L}^{(2)}_{EH} + \left( \partial^\rho h_{\rho\nu} - \frac{1}{2} \partial_\nu h \right)^2.
\]  

Specializing (15.83) to the TT gauge, we obtain

\[
S_{EH} = \frac{1}{32\pi G} \int d^4x \frac{1}{2} \left( \partial_\mu h_{\mu\nu} \right)^2.
\]  

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We can express an arbitrary polarisation state as the sum over the polarisation tensors for circular polarised waves,

$$ h_{\mu\nu} = \sum_{a=+,\,-} h^{(a)}(a) \varepsilon^{(a)}_{\mu\nu} \quad (15.86) $$

Inserting this decomposition into (15.87) and using $\varepsilon^{(a)}_{\mu\nu} \varepsilon^{(b)}_{\mu\nu} = \delta^{ab}$, the action becomes

$$ S_{TT}^{EH} = \frac{1}{32\pi G} \sum_a \int d^4x \frac{1}{2} \left( \partial_\mu h^{(a)} \right)^2 \quad (15.87) $$

Thus the gravitational action in the TT gauge consists of two scalar degree of freedom, $h^+$ and $h^-$, which determine the contribution of left- and right-circular polarised waves. Apart from the prefactor, the action is the same as the one of two scalar fields. This means that we can short-cut many calculations involving gravitational waves by using simply the corresponding results for scalar fields.

We can understand this equivalence by recalling that the part of the action action quadratic in the fields just enforces the relativistic energy-momentum relation via a Klein-Gordon equation for each field component. The remaining content of (15.83) is just the rule how the unphysical components in $h^{\mu\nu}$ have to be eliminated. In the TT gauge, we have applied already this information, and thus the two scalar wave equations for $h^{(\pm)}$ summarize the Einstein equations at $O(h^2)$.

15.4. Wess-Zumino model and supersymmetry

We have noted already that a symmetry relating bosons and fermions could lead to a cancellation of zero-point energies and radiative corrections. However, the symmetry transformations we considered up to now cannot achieve this since they do not change the spin of a particle. The way-out is to consider symmetries generated by Grassman variables.

We will consider the simplest case of single massless fermion and some scalar fields. An on-shell massless fermion has two degrees of freedom, and we will therefore start to add two scalar fields to match the on-shell fermionic degrees of freedom. Our initial Lagrangian is

$$ \mathcal{L}_0 = \frac{1}{2}(\partial_\mu S)^2 + \frac{1}{2}(\partial_\mu P)^2 + \frac{1}{4} \bar{\chi} \gamma^\mu \partial_\mu \chi $$

where we use Majorana spinors, i.e. the $\chi$ are Grassmann variables. For the following, it is useful to recall the flip properties of Majorana spinors derived in problem 5.9. They imply in particular that the vector current $\bar{\chi} \gamma^\mu \chi$ vanishes. Thus the model has a chiral symmetry of $\chi$ and the standard phase symmetry in $S$ and $P$:

$$ \chi \rightarrow e^{i\beta} \chi \quad , \quad S + iP \rightarrow e^{i\beta}(S + iP) \quad (15.89) $$

A transformation that interchanges $\chi$ with $S$ and $P$, $\chi \rightarrow S, P$, must be parametrized by a Majorana spinor parameter $\alpha$. Moreover, the transformation has to account for the different number of derivatives of bosonic and fermionic fields. In the ansatz

$$ \delta(S, P) = \bar{\alpha} \Gamma \chi \quad , \quad (15.90) $$

the $4 \times 4$-matrix $\Gamma$ in spinor space can be either 1 or $\gamma^5$. In order to match the two real scalar fields to the complex spinor, we made the ansatz $S + iP$ in (15.89). This suggests that $S$ is a
A. Conventions and useful formula

A.4. Gamma function

Definition The Gamma function \( \Gamma(z) \) is defined as the generalization of the factorial \( n! \) with \( \Gamma(n) = (n-1)! \). Thus \( \Gamma(1) = 1 \) and \( z \Gamma(z) = \Gamma(z+1) \). As first step of this generalization, we show that

\[
\Gamma(z) = \int_0^\infty dt \ e^{-t} t^{z-1}
\]  

is an integral representation of the Gamma function valid in the positive half-plane \( \Re z > 0 \):

First, the RHS equals \( \Gamma(1) = 1 \). Second, we can rewrite this integral representation as

\[
\Gamma(z+1) = - \int_0^\infty dt \ \left( \frac{d}{dt} e^{-t} \right) t^z = \int_0^\infty dt \ z e^{-t} t^{z-1} + e^{-t} \bigg|_0^\infty.
\]  

The boundary term vanishes and thus the integral representation also satisfies the recurrence relation \( \Gamma(z+1) = z \Gamma(z) \).

Next we extend \( \Gamma(z) \) into the half-plane \( \Re z > -1 \) by setting \( \Gamma_1(z) = \Gamma(z+1) / z \). The function \( \Gamma_1(z) \) has a simple pole at \( z = 0 \). In the second step, we set \( \Gamma_2(z) = \Gamma(z+2) / [z(z-1)] \), defining thereby the function \( \Gamma_2(z) \) valid in the half-plane \( \Re z > -2 \) with simple poles at \( z = 0 \) and -1. By induction, we can extend thus the Gamma function as an analytic function in the whole complex plane except for simple poles at \( z = 0, -1, -2, \ldots \).

Euler's beta function is defined by

\[
B(a,b) \equiv \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dt \ t^{a-1}(1-t)^{b-1} = \int_0^\infty dt \ \frac{t^{a-1}}{(1+t)^{a+b}}.
\]  

The integral representations are valid for \( \Re(a) > 0, \Re(b) > 0 \). We can use the first one to derive the values of the Gamma function for half-integer arguments: Substituting \( t = \sin^2 \vartheta \) gives

\[
\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = 2 \int_0^{\pi/2} \sin^2 \vartheta \ (\sin \vartheta)^{2a-1}(\cos \vartheta)^{2b-1},
\]  

what results in \( \Gamma(1/2) = \sqrt{\pi} \) for \( a = b = 1/2 \). Using then the recurrence relation, we can generate from \( \Gamma(1) = 1 \) and \( \Gamma(1/2) = \sqrt{\pi} \) the values for all (positive) integer and half-integer arguments: \( \Gamma(n) = (n-1)! \), \( \Gamma(3/2) = \sqrt{\pi}/2 \), \( \Gamma(5/2) = 3\sqrt{\pi}/4 \), etc.

Logarithmic derivative The (first) logarithmic derivative of the Gamma function is called

\[
\psi_1(z) = \frac{d \ln \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}.
\]  

We define moreover the special value

\[
\psi_1(1) = \frac{\Gamma'(1)}{\Gamma(1)} = -\gamma
\]  

as the Euler-Mascheroni constant \( \gamma \). Differentiating \( z \Gamma(z) = \Gamma(z+1) \) we obtain \( \Gamma(z + z \Gamma'(z) = \Gamma'(z+1) \) or

\[
1 + \frac{z \Gamma'(z)}{\Gamma(z)} = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{z \Gamma'(z+1)}{\Gamma(z+1)}.
\]  

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This gives us a recurrence relation for the psi function,

$$\psi_1(z + 1) = \frac{1}{z} + \psi_1(z), \quad (A.36)$$

which becomes for the special case of integers $n = 0, 1, 2, \ldots$

$$\psi_1(n + 1) = \frac{1}{n} + \psi_1(n) = \frac{1}{n} + \frac{1}{n-1} + \ldots + \psi_1(1). \quad (A.37)$$

Thus we can express the psi function for integer values through the Euler-Mascheroni constant and a finite harmonic serie,

$$\psi_1(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}. \quad (A.38)$$

What is left to determine is the value of $\gamma$. Consider the asymptotic behavior of $\psi_1(x)$ for $x \to \infty$ using Stirling’s formula,

$$\ln \Gamma(x + 1) = \left(x + \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln 2\pi + O(x^{-1}). \quad (A.39)$$

Differentiating this expression results in

$$\psi_1(x + 1) = \ln x + \frac{1}{2x} + O(x^{-2}). \quad (A.40)$$

In the limit $x \to \infty$, we find thus $\psi_1(x + 1) = \ln x$. Inserting this equality into $\psi_1(n+1)$, we obtain an expression for the Euler-Mascheroni constant,

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) = 0.5772 \ldots \quad (A.41)$$

(Suprisingly it is not known, if this number is rational or not.)

**Expansion near a pole** We are now in the position to derive formula like (2.118) for the expansion of the Gamma function around its poles. We start expanding $\Gamma(1 + \varepsilon)$ around one,

$$\Gamma(1 + \varepsilon) = \Gamma(1) + \varepsilon \Gamma'(1) + O(\varepsilon^2) = 1 + \varepsilon \Gamma(1) \psi_1(1) + O(\varepsilon^2) = 1 - \varepsilon \gamma + O(\varepsilon^2). \quad (A.42)$$

From $z \Gamma(z) = \Gamma(z + 1)$ it follows

$$\Gamma(\varepsilon) = \frac{\Gamma(1 + \varepsilon)}{\varepsilon} = \frac{1}{\varepsilon} - \gamma + O(\varepsilon). \quad (A.43)$$

Applying $z \Gamma(z) = \Gamma(z + 1)$ again, we obtain

$$\Gamma(-1 + \varepsilon) = -\frac{\Gamma(\varepsilon)}{1 - \varepsilon} = -(1 + \varepsilon - \varepsilon^2 + \ldots) \left[\frac{1}{\varepsilon} - \gamma + O(\varepsilon)\right] = -\frac{1}{\varepsilon} - 1 + \gamma + O(\varepsilon). \quad (A.44)$$

For general $n$, the formula

$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\varepsilon} + \psi_1(n + 1) + O(\varepsilon)\right], \quad (A.45)$$

holds for the expansion of the Gamma function near a pole.
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