NTNU

Cancellation of infrared divergences in QCD

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Abstract

The scattering process $e^+e^- \rightarrow$ hadrons is considered at next-to-leading order, $O(\alpha_s)$. It is demonstrated that the cross sections for both the real gluon emission and virtual gluon exchange have infrared singularities. Using dimensional regularization it is shown that the divergent terms cancel exactly when adding the contributions, such that the total cross section for $e^+e^- \rightarrow$ hadrons is finite at this order. This result, and the general implications of infrared divergences and their cancellation, is discussed in light of the Kinoshita-Lee-Nauenberg theorem and the concept of infrared safety.

Contents

List of figures

List of tables

List of symbols and notation

g_s, α_s The coupling constant of quantum chromodynamics; $\alpha_s \equiv g_s^2/4\pi$.

e, α_{em} The *positive* elementary electric charge and coupling constant of quantum electrodynamics; $\alpha_{\text{em}} \equiv e^2/4\pi$. The electron charge is $-e$.

e_q The electric quantum number of a quark q ; the quark charge is ee_q .

z^* The complex conjugate of the number $z = x + iy$: $z^* = x - iy$. Applies elementwise to vectors, matrices etc.

$\Re\{z\}$ The real part of the complex number z : $\Re\{z\} = (z + z^*)/2$.

$\Im\{z\}$ The imaginary part of the complex number z : $\Im\{z\} = (z - z^*)/2i$.

$\boldsymbol{I}$ The identity matrix.

A^T The transpose of the matrix A : $(A^T)_{ij} = (A)_{ji}$.

$A^\dagger$ The Hermitian adjoint of the matrix A : $A^\dagger = (A^*)^T$.

$\eta^{\mu\nu}$ The metric tensor in Minkowski space, with signature $(+, -, -, -)$.

$\boldsymbol{a}$ The spatial part of a 4-vector $a = (a^0, \boldsymbol{a}) = (a^0, a^1, a^2, a^3)$: $\boldsymbol{a} = (a^1, a^2, a^3)$.

$\boldsymbol{a} \cdot \boldsymbol{b}$ The scalar product of two spatial vectors: $\boldsymbol{a} \cdot \boldsymbol{b} = a^i b^i$.

$\boldsymbol{a}^2$ The scalar product of a spatial vector with itself: $\boldsymbol{a}^2 = \boldsymbol{a} \cdot \boldsymbol{a}$.

$\tilde{a}$ The length of a spatial vector: $\tilde{a} = \sqrt{\boldsymbol{a}^2}$. Also used for Euclidean lengths of vectors in arbitrary dimensions.

$a \cdot b$ The inner product of two 4-vectors: $a \cdot b = a^\mu b_\mu = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}$.

a^2 The inner product of a 4-vector with itself: $a^2 = a \cdot a = a^\mu a_\mu = (a^0)^2 - \boldsymbol{a}^2$.

x_i The energy fraction of particle i in the COM frame: $x_i = 2p_i \cdot Q/Q^2$ with $Q = \sum_i p_i$.

γ^μ The Dirac gamma matrices.

$\overline{\psi}$ The Dirac adjoint $\overline{\psi} = \psi^\dagger \gamma^0$ of a Dirac spinor ψ .

$\not{a}$ The Feynman slash, denoting the contraction of a 4-vector and the gamma matrices: $\not{a} = a_\mu \gamma^\mu$.

$u^s(p)$ Dirac spinor for a fermion with spin state s and momentum p .

$v^s(p)$ Dirac spinor for an antifermion with spin state s and momentum p .

a^μ A component of a 4-vector $a = (a^0, a^1, a^2, a^3)$, specified by a greek index $\mu \in \{0, 1, 2, 3\}$, known as a Lorentz index. The index may be lowered by contraction with the metric tensor: $a_\mu = \eta_{\mu\nu} a^\nu$.

a^i A spatial component of a 4-vector $a = (a^0, a^1, a^2, a^3)$, specified by a latin index $i \in \{1, 2, 3\}$.

N_c The number of color degrees of freedom for quarks. For real-world quarks, $N_c = 3$.

λ^a The Gell-Mann matrices, listed in equation (A.15).

T^a The generators of the fundamental representation of $\mathrm{SU}(N_c)$. For $N_c = 3$ defined as $T^a = \lambda^a/2$

f^{abc} The structure constants of the generator algebra: $[T^a, T^b] = i f^{abc} T^c$.

T_F The Dynkin index of the generator algebra: $\mathrm{Tr}\left[T^a T^{b\dagger}\right] = T_F \delta^{ab}$. For $N_c = 3$ and $T^a = \lambda^a/2$, $T_F = 1/2$.

C_F The color factor of the generator algebra: $\left(T^a T^{a\dagger}\right)_{ik} = C_F \delta_{ik}$. For $N_c = 3$ and $T^a = \lambda^a/2$, $C_F = 4/3$.

$D_F^{\mu\nu}(p)$ The Feynman propagator for a gauge boson with momentum p .

$S_F(p)$ The Feynman propagator for a fermion with momentum p .

$\mathcal{M}$ A general scattering amplitude.

$\overline{|\mathcal{M}|^2}$ The average over initial spin and color states and sum over final spin and color states of $|\mathcal{M}|^2$.

The Einstein summation convention applies to all kinds of indices unless otherwise stated, e.g. $a_i b_i = \sum_i a_i b_i$. For Lorentz indices, there must always be one upper and one lower summation index: $a^\mu a_\mu = \sum_\mu a^\mu a_\mu = \sum_{\mu\nu} \eta_{\mu\nu} a^\mu a^\nu$. Units are chosen such that $\hbar = c = 1$.

Preface

This report is the outcome of my work in the physics specialization project, TFY4510, carried out during the fall semester of 2014 under supervision of Prof. Michael Kachelrieß at the Department of Physics at the Norwegian University of Science and Technology (NTNU). The objective of the project has been to learn about the phenomenon of infrared divergences and how they cancel when defining observables satisfying particular requirements, mainly in the context of quantum chromodynamics and with electron-positron annihilation to quarks as the process of study.

Infrared divergences have not been treated in detail during the course of my studies, and the topic was therefore largely unknown to me before embarking on this project. The main motivation for studying this is to prepare for a possible Master's thesis studying annihilations in a wino-like Dark Matter candidate model.

As the project work has consisted solely of gaining acquaintance with the subject matter, there are no new results to present. The report therefore amounts to a thorough presentation of established theoretical results, reflecting my level of knowledge as of finishing this project. The aim has been to give a comprehensive and largely self-contained presentation of the main ideas and derivations, including a significant amount of background material that I had to learn or review in order to get through the calculations. Much of this has been relegated to appendices in an attempt to obtain a readable and reasonably well-structured text, but I still consider it an integral part of my project work.

1 Introduction

Quantum chromodynamics (QCD) is the theory describing the strong interactions between the *quarks* and *gluons*, the fundamental particles which make up baryons such as the proton and neutron, and mesons such as the pion. The quarks and gluons carry a quantum number which is specific to QCD, known as *color charge*. QCD is a quantum field theory described by the Lagrangian [1, pp. 21-22]

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}_{q,i}\left(i\not{\partial} - m_q\right)\delta_{ij}\psi_{q,j} - g_s\overline{\psi}_{q,i}T^a_{ij}\not{A}^a\psi_{q,j} - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\,, \tag{1.1}$$

Here, ψ_q is the Dirac spinor field for a quark flavor labeled q , with mass m_q , while $F^a_{\mu\nu}$ is the field strength tensor for the gauge field A^a_μ describing the gluons. The indices $i, j \in \{1, 2, 3\}$ label the color charge basis states of the quarks, while the index $a \in \{1, \dots, 8\}$ labels color charge basis states for the gluons. The terms are further discussed in appendix A.

Quarks also carry electric charge and are subject to interactions in quantum electrodynamics. These interactions are not present in equation (1.1) but are discussed in appendix A.2. Finally, quarks interact weakly, but weak interactions will not be discussed in this text.

1.1 Perturbative quantum chromodynamics

When applying the usual tools of perturbative quantum field theory to the QCD Lagrangian, the Feynman rules in appendix A.3 appear. Compared to for example quantum electrodynamics (QED), there is however a significant complication when predicting experimental outcomes using this formalism: perturbative quantum field theory describes scattering processes where the initial and final states consist of free particles in the fundamental fields in the Lagrangian, but free quarks or gluons are never observed in experiments. On the contrary, color charged particles appear only in bound states such as baryons and mesons, which are color singlets. This property is known as color confinement.

One way to get an understanding of this is to look at the effective coupling constant of QCD, obtained by renormalizing the theory. The procedure of renormalization is not discussed here, but is treated in any text on quantum field theory, and discussed in detail for the case of QCD in [1, pp. 80-105]. One main result is that the parameters appearing in the bare Lagrangian in equation (1.1), such as g_s , do not equal to the values measured in experiments. In fact, the measured parameters will depend on the energy scale at which the process occurs.

For the QCD coupling constant, one usually considers $\alpha_s = g_s^2/4\pi$ rather than g_s itself, following the example of the fine structure constant used in QED, $\alpha_{\mathrm{em}} = e^2/4\pi$. To leading order in perturbation theory, one finds the following expression for the running of the coupling [1, p. 31]:

$$\alpha_s\left(Q^2\right) = \frac{1}{\beta_0 \ln\left(Q^2/\Lambda_{QCD}^2\right)} . \tag{1.2}$$

Here, Q is the momentum transfer in the reaction of interest, $\beta_0 > 0$ is a constant, and $\Lambda_{QCD} \sim \mathcal{O}(200)$ MeV is an experimentally determined parameter called the QCD scale.

Performing a computation in perturbative quantum field theory essentially amounts to making a series expansion in the coupling constant α_s , and only computing the first few terms. This only makes sense if $\alpha_s \ll 1$ (and since equation (1.2) is only a leading order approximation derived from the same perturbation theory, this expression is only valid in the same regime). But from equation (1.2), it is clear that $\alpha_s \ll 1$ requires $Q^2 \gg \Lambda_{QCD}^2$. This shows that perturbative QCD may be applicable to high-energy, short-distance processes, but cannot be used at lower energies and larger distances where the coupling is strong. It is precisely these low-energy processes that account for the binding of quarks and gluons into hadrons which can be detected, and it is therefore not at all obvious how to connect calculations in perturbative QCD to experiment.

There are in general two possible solutions to this problem. One is to factorize a QCD process into a perturbative part which describes the specific short-distance interactions relevant for the process, and a universal non-perturbative part describing the transition from perturbatively free quarks and gluons to hadrons, starting from the perturbative final state. The non-perturbative part cannot be calculated analytically from the Lagrangian, but may be measured experimentally. The justification of this approach relies on the fact that the perturbative and non-perturbative phenomena occur on very different energy, time and distance scales, in many cases isolating them from affecting each other to a significant degree. For a number of quantities there exist formal proofs that the factorization is well-defined [2], but such issues are beyond the scope of this text.

The other approach, which is intimately related to the main topic of this text, is to define the measured observables such that they are insensitive to the effects of long-distance interactions. Such observables are called *infrared safe*. This concept will be defined and discussed in chapter 3.

Table 1.1: Masses of quarks and leptons. The two uncertainties on the top mass give statistical and systematical uncertainties, respectively. The data is retrieved from [4].

Particle	Mass m/MeV
u	$2.3^{+0.7}_{-0.5}$
d	$4.8^{+0.5}_{-0.3}$
s	95 ± 5
c	$(1.275 \pm 0.025) \cdot 10^3$
b	$(4.18 \pm 0.03) \cdot 10^3$
t	$(173.21 \pm 0.51 \pm 0.71) \cdot 10^3$
e	$0.510998928 \pm 0.000000011$
μ	$105.6583715 \pm 0.0000035$
τ	1776.82 ± 0.16

An example of an infrared safe quantity which will be discussed in great detail is the total cross section for the production of hadrons from the annihilation of a positron and an electron. To leading order in electroweak interactions, the electron and positron annihilate to a virtual photon or Z-boson, which then forms a quark-antiquark pair: $e^+e^- \to \gamma^*/Z \to q\bar{q}$. So far this process does not involve QCD, and is described perturbatively in electroweak theory. The $q\bar{q}$-state then undergoes hadronization via non-perturbative processes. To leading order in both electroweak theory and QCD, this is the *only* perturbative e^+e^--scattering process that undergoes hadronization, so the total cross section for $e^+e^- \to \gamma^*/Z \to q\bar{q}$ must be identical to the total cross section for $e^+e^- \to$ hadrons at this order. Although this cross section gives no details about which particles or momenta to expect in the hadronic final state, it is interesting by virtue of being a testable prediction derived using only perturbative techniques.

An unrelated, but important feature of perturbative QCD is seen by considering the masses of fundamental particles, shown in table 1.1. With Λ_{QCD} on the order of 200 MeV , it is clear that $Q^2 \gg \Lambda^2_{QCD}$ also implies that $Q^2 \gg m_u^2, m_d^2, m_s^2, m_e^2, m_\mu^2$. Assuming that no c, b, t quarks and τ leptons are produced (this is guaranteed if $Q^2 < (2m_c)^2$), it is therefore a very good approximation in perturbative QCD to assume that all particles are massless. This greatly simplifies many calculations.

This introduction to perturbative QCD has been mostly based on heuristic reasoning and appeal to intuition. For a more rigorous discussion, see e.g. [3].

1.2 Infrared divergences

One may obtain higher-order approximations of the cross section for $e^+e^- \to$ hadrons by including QCD interactions between the particles in the perturbative final state. The lowest order correction involves a free gluon emitted from one of the quarks, such that the process considered now is $e^+e^- \to q\bar{q}g$. Another possibility is to let the $q\bar{q}$-pair emit and absorb a virtual gluon before leaving the perturbative domain, thus going to higher order in the amplitude for $e^+e^- \to q\bar{q}$. Although these processes have different final states on the perturbative level, they both contribute to the total cross section for $e^+e^- \to$ hadrons.

When computing these amplitudes, however, a serious complication arises: the gluon is massless and can be emitted with arbitrarily small momentum, and as will be shown, the amplitude for $e^+e^- \to q\bar{q}g$ diverges as the gluon energy tends to 0 . The divergence survives the phase space integral, leaving the cross section for the process infinite. This phenomenon is an example of an *soft* divergence, that is, a divergence due to contributions from a low-momentum limit. In the approximation of massless quarks, the amplitude also diverges when the gluon is emitted parallel to the quark; this is called a *collinear* divergence. Together, they are known as *infrared* divergences. These divergences are fundamentally different from the *ultraviolet* divergences often encountered in loop integrals in quantum field theory, which stem from high-energy contributions. The procedure of renormalization used to handle the ultraviolet divergences does *not* eliminate infrared divergences.

However, an infrared divergence also appears in $e^+e^- \to q\bar{q}$ at next-to-leading order, when the energy of the virtual gluon tends to 0 . This divergence has the opposite sign of the divergences in $e^+e^- \to q\bar{q}g$, and when computing the total cross section of $e^+e^- \to$ hadrons, these divergences cancel exactly, leaving cross section finite. The main objective of this project is to show this cancellation by explicitly calculating the necessary amplitudes and cross sections and regularizing the infinities. This calculation is carried out in chapter 2.

The specific process of cancellation discussed here is just an example of a more general phenomenon. Infrared divergences always show up in theories with massless particles, they appear at all orders, and in many cases it is possible to show rigorously that they also cancel at each order when computing quantities involving sums over properly defined sets of initial and final states. From a physical perspective, it is imperative that singularities can be avoided in one way or the other when computing physically measurable quantities, and this is where the concept of infrared safety becomes relevant. The general theorems and implications of infrared divergences and infrared safety, as well as their interpretation in the context of QCD, will be discussed in chapter 3.

2

Cancellation of infrared divergences in $e^+e^- \to$ hadrons

This chapter contains the mathematical derivations demonstrating the cancellation of QCD infrared divergences in e^+e^- annihilation to hadrons. First, the amplitude for the finite lowest order contribution, $e^+e^- \to q\bar{q}$, is computed, and then a formalism is developed for efficient computation of the necessary cross sections. This is used along with dimensional regularization to show how infrared divergences in $e^+e^- \to$ hadrons cancel to first order in α_s . The presentation closely follows [1].

To get a preliminary intuition for the cancellation mechanism, consider first the scattering amplitude for the process $e^+e^- \to q\bar{q}$, which can be expanded in the QCD coupling α_s as

$$\mathcal{M}_{q\bar{q}} = \mathcal{M}_{q\bar{q}}^{(0)} + \alpha_s \mathcal{M}_{q\bar{q}}^{(1)} + \cdots . \tag{2.1}$$

As will be shown, $\mathcal{M}_{q\bar{q}}^{(0)}$ is finite, while $\mathcal{M}_{q\bar{q}}^{(1)}$ contains infrared singularities. Similarly, the process $e^+e^- \to q\bar{q}g$ can be expanded as

$$\mathcal{M}_{q\bar{q}g} = \sqrt{\alpha_s}\mathcal{M}_{q\bar{q}g}^{(0)} + \cdots . \tag{2.2}$$

Compared to equation (2.1), this amplitude contains an extra factor of g_s , or equivalently $\sqrt{\alpha_s}$, due to the free gluon emission. As will be shown, $\mathcal{M}_{q\bar{q}g}^{(0)}$ also contains infrared singularities.

Note in passing that, as a general rule in this text, a subscript like $q\bar{q}$ or $q\bar{q}g$ on a quantity denotes the final state of the process the quantity pertains to. This unambiguously identifies the process in question, since the only initial state considered is e^+e^- . A superscript like (0) or (1) means the that the quantity is computed at a specific order in the QCD perturbation expansion, with leading order starting at (0) , and all powers of α_s factored out to facilitate straightforward comparison of the order of terms from different expansions.

These amplitudes correspond to processes with different final states and cannot be added directly, but both contribute to the total cross section for

$e^+e^- \to$ hadrons. Using equation (A.1), one can write

$$\sigma_{\text{hadrons}} = \sigma_{q\bar{q}} + \sigma_{q\bar{q}g} + \cdots \ , \tag{2.3}$$

$$\sigma_{q\bar{q}} = \frac{1}{4I} \int d\Phi_2 \left|\mathcal{M}_{q\bar{q}}\right|^2 \ , \tag{2.4}$$

$$\sigma_{q\bar{q}g} = \frac{1}{4I} \int d\Phi_3 \left|\mathcal{M}_{q\bar{q}g}\right|^2 \ . \tag{2.5}$$

Expanding the squared amplitudes to first order in α_s , one finds

$$\left|\mathcal{M}_{q\bar{q}}\right|^2 = \left|\mathcal{M}^{(0)}_{q\bar{q}}\right|^2 + 2\alpha_s \Re\left\{\mathcal{M}^{(0)}_{q\bar{q}} \mathcal{M}^{(1)*}_{q\bar{q}}\right\} + \cdots \ , \tag{2.6}$$

$$\left|\mathcal{M}_{q\bar{q}g}\right|^2 = \alpha_s \left|\mathcal{M}^{(0)}_{q\bar{q}g}\right|^2 + \cdots \ . \tag{2.7}$$

Although absolute squares of amplitudes must be positive, individual interference terms such as the one in equation (2.6) may be negative. For the infrared divergences to cancel to order α_s , one must find that

$$\int d\Phi_2 \, 2\Re\left\{\mathcal{M}^{(0)}_{q\bar{q}} \mathcal{M}^{(1)*}_{q\bar{q}}\right\} + \int d\Phi_3 \left|\mathcal{M}^{(0)}_{q\bar{q}g}\right|^2 = \text{finite} \ . \tag{2.8}$$

Of course, this will only show cancellation to first order in α_s . The next term in equation (2.6) is $\alpha_s^2 |\mathcal{M}^{(1)}_{q\bar{q}}|^2$, and it will contain positive infrared singularities, as will the leading order term for $e^+e^- \to q\bar{q}gg$. These may then cancel against interference terms of order α_s^2 in equation (2.7), and so on. Only the cancellation to first order will be rigorously derived here, but the general issue will be discussed in chapter 3.

2.1 Leading order amplitude for $e^+e^- \to q\bar{q}$

The leading order contribution to electron-positron annihilation to hadrons, $e^+e^- \to \gamma^* \to q\bar{q}$, is given by the Feynman diagram in figure 2.1. Strictly speaking, the similar process with the virtual photon γ^* replaced by a Z boson also participates, but this contribution is strongly attenuated except around the pole of the Z boson propagator at momentum transfer $Q^2 = M_Z^2 \approx (91\,\text{GeV})^2$ [4, 5, p. 337]. Here, this contribution will be neglected, thus implicitly assuming a momentum transfer far from this resonance.

The diagram in figure 2.1 consists of QED vertices only, and using the rules in appendix A the amplitude is straightforward to write down:

$$\begin{aligned} i\mathcal{M}^{(0)}_{q\bar{q}} &= \bar{v}^{s'}(\ell^+)\left(ie\gamma_\mu\right)u^s(\ell^-)\, iD_F^{\mu\nu}(Q)\,\bar{u}^r(q)\left(-iee_q\gamma_\nu\right)\delta_{ij}v^{r'}(\bar{q}) \\ &= \left[ie\bar{v}^{s'}(\ell^+)\,\gamma_\mu u^s(\ell^-)\right] iD_F^{\mu\nu}(Q)\left[-i\delta_{ij}ee_q\bar{u}^r(q)\,\gamma_\nu v^{r'}(\bar{q})\right] , \end{aligned} \tag{2.9}$$

where $\ell^-, \ell^+, q, \bar{q}$ are the 4-momenta of the electron, positron, quark and antiquark, respectively, while $Q = \ell^- + \ell^+ = q + \bar{q}$ is the four-momentum transfer,

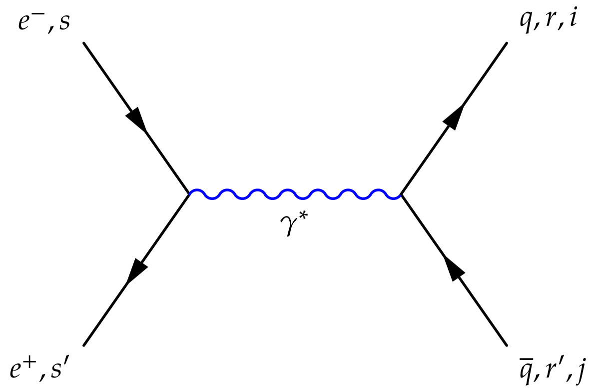


Figure 2.1: Leading order diagram for $e^-e^+ \to q\overline{q}$. The labels s, s', r, r' denote spin polarizations, while i, j are color indices.

and $D_F^{\mu\nu}(Q)$ is the (gauge-dependent) photon propagator from equation (A.7), repeated here for convenience:

$$iD_F^{\mu\nu}(Q) = i\frac{-\eta^{\mu\nu} + (1-\xi)\, Q^\mu Q^\nu / Q^2}{Q^2 + i\varepsilon} \,. \tag{2.10}$$

The $u, v, \overline{u}, \overline{v}$ are Dirac spinors, explained in appendix B.2. The δ_{ij} enforces color conservation on the photon-quark-antiquark vertex.

For later convenience, the lepton and hadron currents J_L and J_X can be defined as the incoming and outgoing currents coupling to the photon propagator, respectively. For this diagram, one finds

$$(J_L)_\mu = ie\overline{v}^{s'}(p^+)\,\gamma_\mu u^s(p^-) \;, \tag{2.11}$$

$$\left(J_{q\overline{q}}^{(0)}\right)_\nu = -i\delta_{ij} e e_q \overline{u}^r(q)\,\gamma_\nu v^{r'}(\overline{q}) \;. \tag{2.12}$$

As several different hadron currents will be considered throughout the report, they are labeled with subscripts and superscripts identifying the process and expansion order they pertain to. The lepton current will be the same in all computations, so it is simply referred to J_L .

Contracting the momentum transfer Q^μ with the lepton and hadron currents gives

$$\begin{aligned} Q^\mu (J_L)_\mu &\propto \overline{v}^{s'}(\ell^+)\left(\not{\ell}^+ + \not{\ell}^-\right) u^s(\ell^-) \\ &= \overline{v}^{s'}(\ell^+)\,(-m_e + m_e)\, u^s(\ell^-) = 0 \,, \end{aligned} \tag{2.13}$$

$$\begin{aligned} Q^\nu \left(J_{q\overline{q}}^{(0)}\right)_\nu &\propto \overline{u}^r(q)\,(\not{q} + \not{\overline{q}})\, v^{r'}(\overline{q}) \\ &= \overline{u}^r(q)\left(m_q - m_q\right) v^{r'}(\overline{q}) = 0 \,. \end{aligned} \tag{2.14}$$

Here the Dirac equations in momentum space,

$$(\not{p} - m)\, u\,(p) = (\not{p} + m)\, v\,(p) = 0\,, \tag{2.15}$$

$$\overline{u}\,(p)\,(\not{p} - m) = \overline{v}\,(p)\,(\not{p} + m) = 0\,, \tag{2.16}$$

were used. They are discussed in detail in appendix B.2 as equations (B.35), (B.36), (B.60) and (B.61).

Thus, the gauge dependent term in the photon propagator will not contribute to the amplitude, as predicted by gauge invariance, and one can therefore elect to work in the Feynman gauge $\xi = 1$, where

$$iD_F^{\mu\nu}\,(Q) = i\frac{-\eta^{\mu\nu}}{Q^2}\,. \tag{2.17}$$

The amplitude may then be written

$$i\mathcal{M}_{q\bar{q}}^{(0)} = -\frac{i}{Q^2}\,(J_L)_\mu \left(J_{q\bar{q}}^{(0)}\right)^\mu\,. \tag{2.18}$$

In most experimental setups, the beam of particles in the initial state is unpolarized and the spin and color states of the final particles are never measured. The squared amplitude describing this process is thus obtained by averaging over initial state spins, and summing over final state spins and colors, the squared amplitudes for the spin- and color-dependent amplitude, as follows:

$$\begin{aligned} \overline{\left|\mathcal{M}_{q\bar{q}}^{(0)}\right|^2} &= \frac{1}{4}\sum_{s,s',r,r',i,j}\left|\mathcal{M}_{q\bar{q}}^{(0)}\right|^2 \\ &= \frac{1}{Q^4}L_{\mu\nu}H_{q\bar{q}}^{(0)\mu\nu}\,, \end{aligned} \tag{2.19}$$

where the overbar denotes matrix elements that are spin- and color-averaged as described. The lepton and hadron tensors $L_{\mu\nu}$ and $H_{q\bar{q}}^{\mu\nu}$ are defined as

$$L_{\mu\nu} = \frac{1}{4}\sum_{s,s'}(J_L)_\mu\,(J_L^*)_\nu\,, \tag{2.20}$$

$$H_{q\bar{q}}^{\mu\nu} = \sum_{r,r',i,j}\left(J_{q\bar{q}}\right)^\mu \left(J_{q\bar{q}}^*\right)^\nu\,. \tag{2.21}$$

To compute these tensors, it is convenient to make use of the completeness relations for Dirac basis spinors, given in equations (B.71) and (B.72) and repeated here for convenience:

$$\sum_s u^s\,(p)\,\overline{u}^s\,(p) = \not{p} + m\,, \tag{2.22}$$

$$\sum_r v^r\,(p)\,\overline{v}^r\,(p) = \not{p} - m\,. \tag{2.23}$$

The lepton tensor evaluates to

$$
\begin{aligned}
4L_{\mu\nu} &= e^2 \sum_{s,s'} \left[\bar{v}^{s'}(\ell^+)\,\gamma_\mu u^s(\ell^-)\right]\left[\bar{v}^{s'}(\ell^+)\,\gamma_\nu u^s(\ell^-)\right]^\dagger \\
&= e^2 \sum_{s,s'} \bar{v}^{s'}(\ell^+)\,\gamma_\mu u^s(\ell^-)\,\bar{u}^s(\ell^-)\,\gamma_\nu v^{s'}(\ell^+) \\
&= e^2 \sum_{s,s'} \mathrm{Tr}\left[\bar{v}^{s'}(\ell^+)\,\gamma_\mu u^s(\ell^-)\,\bar{u}^s(\ell^-)\,\gamma_\nu v^{s'}(\ell^+)\right] \\
&= e^2 \sum_{s,s'} \mathrm{Tr}\left[v^{s'}(\ell^+)\,\bar{v}^{s'}(\ell^+)\,\gamma_\mu u^s(\ell^-)\,\bar{u}^s(\ell^-)\,\gamma_\nu\right] \\
&= e^2 \mathrm{Tr}\left[\left(\not{\ell}^+ - m_e\right)\gamma_\mu\left(\not{\ell}^- + m_e\right)\gamma_\nu\right] .
\end{aligned}
\tag{2.24}
$$

Here, equation (B.59) is used in the first step. The resulting expression consists of two sandwiches of spinors and spinor-space matrices, which both evaluate to spinor-space scalars, and it is therefore trivially equivalent to the spinor-space trace of the same expression, as shown in step two. The cyclicity of the trace (equation (B.7)) is then exploited in the third step to reorganize the expression such that the completeness relations can be used to give the final expression.

Using the same manipulations, the hadron tensor is found to be

$$
H^{(0)\mu\nu}_{q\bar{q}} = N_c (ee_q)^2 \mathrm{Tr}\left[\left(\not{q} + m_q\right)\gamma^\mu\left(\not{\bar{q}} - m_q\right)\gamma^\nu\right] . \tag{2.25}
$$

The extra factor of N_c comes from the sum over color states in the final state. Following appendix A, the number of colors will be kept general throughout the computations, such that

$$
\sum_{i,j} \delta_{ij}\delta_{ij} = N_c . \tag{2.26}
$$

Using results from appendix B.1, the traces may be computed:

$$
\begin{aligned}
L_{\mu\nu} &= \frac{e^2}{4}\ell^{+\rho}\ell^{-\sigma}\mathrm{Tr}\left[\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\right] - e^2 m_e^2 \mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right] \\
&= e^2\left[\ell^{+\rho}\ell^{-\sigma}\left(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\rho\sigma}\eta_{\mu\nu} + \eta_{\rho\nu}\eta_{\mu\sigma}\right) - m_e^2\eta_{\mu\nu}\right] \\
&= e^2\left[\ell^+_\mu\ell^-_\nu + \ell^-_\mu\ell^+_\nu - \left(\ell^+\cdot\ell^- + m_e^2\right)\eta_{\mu\nu}\right] \\
&= e^2\left[\ell^+_\mu\ell^-_\nu + \ell^-_\mu\ell^+_\nu - \left(Q^2/2\right)\eta_{\mu\nu}\right] ,
\end{aligned}
\tag{2.27}
$$

$$
H^{(0)\mu\nu}_{q\bar{q}} = 4N_c\left(ee_q\right)^2\left[q^\mu\bar{q}^\nu + \bar{q}^\mu q^\nu - \left(Q^2/2\right)\eta^{\mu\nu}\right] . \tag{2.28}
$$

Here, the following identites have been used, which follow from the relativistic energy-momentum relation (equation (B.32)):

$$
\begin{aligned}
Q^2 &= \left(\ell^+ + \ell^-\right)^2 \\
&= \left(\ell^+\right)^2 + \left(\ell^-\right)^2 + 2\ell^+\cdot\ell^- \\
&= 2\left(\ell^+\cdot\ell^- + m_e^2\right) ,
\end{aligned}
\tag{2.29}
$$

$$
Q^2 = 2\left(q\cdot\bar{q} + m_q^2\right) . \tag{2.30}
$$

Using the equations (2.29) and (2.30), the final expression for the matrix element then becomes

$$\overline{\left|\mathcal{M}^{(0)}_{q\bar{q}}\right|^2} = \frac{4N_c\left(e^2e_q\right)^2}{Q^4}\Big[2\,(\ell^+\cdot q)\,(\ell^-\cdot\bar{q}) + 2\,(\ell^-\cdot q)\,(\ell^+\cdot\bar{q}) + Q^2\left(m_e^2+m_q^2\right)\Big]\,. \tag{2.31}$$

2.2 Cross sections for $e^+e^- \to$ hadrons

It should be clear from the discussion above that the factorization of the matrix element into lepton and hadron tensors $L_{\mu\nu}$ and $H^{\mu\nu}$ is not limited to $\overline{|\mathcal{M}_{q\bar{q}}|^2}$. On the contrary, such factorizations are possible in any process where the interaction between two fermion currents is mediated by a single photon in all included diagrams. In the case of $e^+e^- \to$ hadrons, it is thus a general feature of the matrix elements to any order in QCD, as long as only the leading order in QED is considered. Moreover, the lepton tensor $L_{\mu\nu}$ will be the same in all these calculations. Given the magnitude of the QED coupling at $\alpha_{\text{em}} \sim 10^{-2}$, this should be a good approximation. Therefore, only leading-order contributions in QED will be considered here.

Thus, the cross section for $e^+e^- \to X$, where X is a state of quarks and gluons, can in general be written

$$\sigma_X = \frac{1}{2Q^2}\cdot\frac{1}{Q^4}L_{\mu\nu}\int d\Phi_X\, H^{\mu\nu}_X\,, \tag{2.32}$$

where only the hadron tensor and phase space measure depends on the final state. The flux factor in equation (A.2) has been used together with the approximation of massless particles, discussed in section 1.1.

From this point, all calculations will assume massless particles, $m_e = m_q = 0$. Moreover, all calculations will take place in $d = 4-2\varepsilon$ spacetime dimensions to facilitate dimensional regularization, using the results in appendix C. Hence, the metric tensor contracts to $\eta^\mu_\mu = d$, and the coupling constants will be accompanied by a mass scale μ^ε in accordance with equation (C.3).

Due to gauge invariance in QED, one must have $Q_\mu H^{\mu\nu} = Q_\nu H^{\mu\nu} = 0$ and $Q_\mu L^{\mu\nu} = Q_\nu L^{\mu\nu} = 0$, such that the $(1-\xi)Q^\mu Q^\nu$-term in the photon propagator drops out. In section 2.1, this was explicitly shown on the level of currents in equations (2.13) and (2.14). When going to higher order, the cancellation is not guaranteed to happen for individual diagrams, but when summing all diagrams of a given order the gauge-dependent term must disappear.[1]

[1] In this particular case the gauge-dependent term would drop out even if it was only canceled by one of the tensors, and it was confirmed in section 2.1 that $L_{\mu\nu}$ does the job. However, gauge invariance must hold for all QED processes, including for example the process where the matrix element factorizes as $\propto H_{\mu\nu}H^{\mu\nu}$, so *any* current coupling to a photon must cancel the gauge-dependent term when summing all contributions at a given order.

In equation (2.32) the momenta of the final state particles are integrated out, so the only free variable is Q^2 , and the only quantities available to carry the Lorentz indices of $H^{\mu\nu}$ are $\eta^{\mu\nu}$ and Q^μ . Hence

$$\int d\Phi\, H^{\mu\nu} = H_1\left(Q^2\right)\eta^{\mu\nu} + H_2\left(Q^2\right)Q^\mu Q^\nu\,. \tag{2.33}$$

Contracting this with Q^μ and imposing gauge invariance shows that $H_1(Q^2) = -Q^2 H_2(Q^2)$. Thus, defining $H(Q^2) = -H_1(Q^2)/(d-1)$, one has

$$\int d\Phi\, H^{\mu\nu} = \frac{1}{d-1}\left(-\eta^{\mu\nu} + \frac{Q^\mu Q^\nu}{Q^2}\right)H\left(Q^2\right)\,. \tag{2.34}$$

This can be solved for $H(Q^2)$ by contracting with $\eta_{\mu\nu}$, since

$$-\,\eta_{\mu\nu}\left(-\eta^{\mu\nu} + \frac{Q^\mu Q^\nu}{Q^2}\right) = d-1\,, \tag{2.35}$$

such that

$$H\left(Q^2\right) = -\eta_{\mu\nu}\int d\Phi\, H^{\mu\nu}\,. \tag{2.36}$$

Substituting equation (2.34) into equation (2.32) and using the gauge invariance condition for $L_{\mu\nu}$, one finds

$$\sigma_X = \frac{1}{2(d-1)Q^6}\left(-\eta^{\mu\nu}L_{\mu\nu}\right)H_X\left(Q^2\right)\,. \tag{2.37}$$

Using equation (2.27) and equation (2.29) gives

$$\eta^{\mu\nu}L_{\mu\nu} = \mu^{2\varepsilon}e^2Q^2\left(1-\varepsilon\right)\,. \tag{2.38}$$

The cross section can thus be expressed

$$\begin{aligned}\sigma_X &= \frac{\mu^{2\varepsilon}e^2}{2Q^4}\frac{1-\varepsilon}{3-2\varepsilon}H_X\left(Q^2\right)\\ &= \frac{2\pi\alpha_{\text{em}}\mu^{2\varepsilon}}{Q^4}\frac{1-\varepsilon}{3-2\varepsilon}H_X\left(Q^2\right)\,.\end{aligned} \tag{2.39}$$

Here, $\alpha_{\text{em}} = e^2/4\pi$. This way, the computation of any cross section contributing to the total cross section for $e^+e^- \to$ hadrons at leading order in QED is reduced to finding the corresponding hadron tensor, contracting it with the metric tensor and integrating over the relevant phase space.

Using this formalism, the cross section for $e^+e^- \to q\bar{q}$ at leading order is straightforward to compute. The first step is to find

$$\eta_{\mu\nu}H^{(0)\mu\nu}_{q\bar{q}} = 4N_c\mu^{2\varepsilon}\left(ee_q\right)^2Q^2\left(1-\varepsilon\right)\,. \tag{2.40}$$

This does not depend explicitly on any of the outgoing momenta, so the phase space integral is trivial by specializing to the center-of-momentum (COM) frame and using equation (C.35):

$$\begin{aligned} H^{(0)}_{q\bar{q}}\left(Q^2\right) &= -\int d\Phi_2^{(d)} \eta_{\mu\nu} H^{(0)\mu\nu}_{q\bar{q}} \\ &= \frac{4N_c \mu^{2\varepsilon}\left(ee_q\right)^2 Q^2}{4\pi} \left(\frac{\pi}{\tilde{q}^2}\right)^{\varepsilon} \frac{\tilde{q}}{\sqrt{Q^2}} \frac{(1-\varepsilon)\,\Gamma\,(1-\varepsilon)}{\Gamma\,(2-2\varepsilon)} \,. \end{aligned} \tag{2.41}$$

Here, $\tilde{q}^2 = \boldsymbol{q}^2 = (q^0)^2 - m_q^2 = (q^0)^2$. In the COM frame, $\tilde{q} = q^0 = \sqrt{Q^2}/2$. Hence,

$$H^{(0)}_{q\bar{q}}\left(Q^2\right) = 2\alpha_{\text{em}} N_c e_q^2 Q^2 \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{(1-\varepsilon)\,\Gamma\,(1-\varepsilon)}{\Gamma\,(2-2\varepsilon)} \,. \tag{2.42}$$

Putting it all together, the cross section for $e^+e^- \to q\bar{q}$ at leading order becomes

$$\sigma^{(0)}_{q\bar{q}} = \frac{4\pi\alpha_{\text{em}}^2 \mu^{2\varepsilon} N_c e_q^2}{Q^2} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{(1-\varepsilon)^2}{3-2\varepsilon} \frac{\Gamma\,(1-\varepsilon)}{\Gamma\,(2-2\varepsilon)} \,. \tag{2.43}$$

As a side remark, note that the cross section has mass dimension $M^{2(\varepsilon-1)}$, or equivalently, $2-2\varepsilon = d-2$ length dimensions, and is thus an area in the usual sense only for $\varepsilon = 0$. This is a consequence of appending the mass scale μ^{ε} to all couplings in the computation, and if d is literally interpreted as the spacetime dimension, it is consistent with the definition of the scattering cross section as the proportionality constant relating the number of scattering events in a differential spacetime region to the number of targets present in this region times the number of scatterers passing through the region per cross sectional area of the incident beam, as explained in e.g. [6, p. 159].[2] On the other hand, in [1] the mass dimension μ is only appended to the couplings in the hadron current and not those in the lepton current, and thus the cross section is always 2-dimensional. This is unproblematic in this particular case, but the principle seems difficult to generalize and this approach will not be followed here. The final results are of course identical in the $\varepsilon \to 0$ limit.

2.3 Real gluon emission: $e^+e^- \to q\bar{q}g$ at leading order

The leading order Feynman diagrams for the process $e^+e^- \to q\bar{q}g$ is shown in figure 2.2. Following the discussion in the last section, only the hadron tensor needs to be computed in order to evaluate the cross section. Applying the Feynman rules from appendix A, the sum of the hadron currents in these two

[2] In $d-1$ spatial dimensions, the cross section of a beam traveling along a single dimension is a $d-2$-dimensional quantity, and so is the scattering cross section.

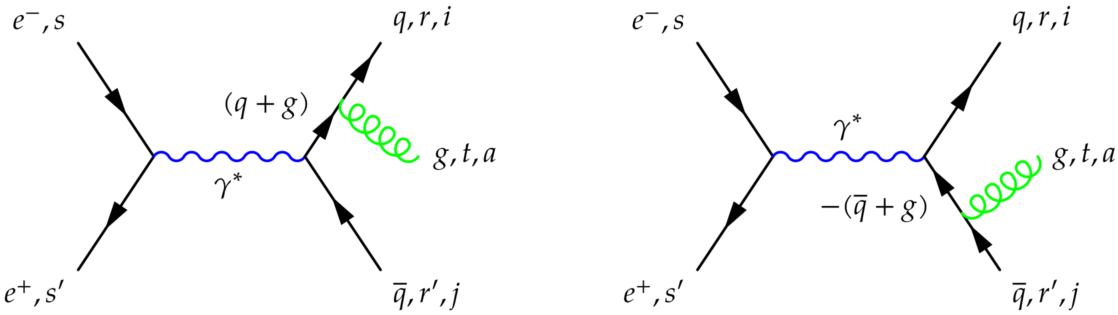


Figure 2.2: Leading order diagrams for $e^-e^+ \to q\bar{q}g$. The labels s, s', r, r', t denote spin polarizations, while i, j, a are color indices. The momentum of the internal quark flows in the same direction as the fermion current, as indicated by the arrowhead.

diagrams is found to be

$$\begin{aligned}\sqrt{\alpha_s}\left(J^{(0)}_{q\bar{q}g}\right)_\mu &= -i\mu^{2\varepsilon} e e_q g_s T^a_{ij} \varepsilon^{*\sigma}_t(g) \\ &\times \bar{u}^r(q)\left[\gamma_\sigma \frac{\not{q}+\not{g}}{(q+g)^2}\gamma_\mu + \gamma_\mu \frac{-(\bar{\not{q}}+\not{g})}{(\bar{q}+g)^2}\gamma_\sigma\right] v^{r'}(\bar{q}) \ .\end{aligned} \tag{2.44}$$

Here, g is the 4-momentum of the gluon. As explained on page 14, all powers of α_s are factored out of quantities defined at a specific order in the perturbation expansion.

This time, the hadron tensor includes a sum over the polarization and color states of the gluon in addition to everything else:

$$\begin{aligned}\alpha_s H^{(0)\mu\nu}_{q\bar{q}g} &= \alpha_s \sum_{r,r',i,j,t,a} \left(J^{(0)}_{q\bar{q}g}\right)^\mu \left(J^{(0)*}_{q\bar{q}g}\right)^\nu \\ &= \alpha_s \sum_t \varepsilon^*_{t,\sigma}(g)\,\varepsilon_{t,\rho}(g) \sum_{r,r',i,j,a} \left(J^{(0)}_{q\bar{q}g}\right)^{\mu\sigma} \left(J^{(0)*}_{q\bar{q}g}\right)^{\nu\rho} \ .\end{aligned} \tag{2.45}$$

Here, the gluon polarization vector was factored out to isolate the polarization sum, which is evaluated in equation (A.14). However, using equations (B.25), (B.26), (B.36) and (B.60) and the vanishing mass of all particles, one finds

$$\begin{aligned}\left(J^{(0)}_{q\bar{q}g}\right)^{\mu\sigma} g_\sigma &\propto \bar{u}^r(q)\left[\not{g}\frac{\not{q}+\not{g}}{(q+g)^2}\gamma^\mu - \gamma^\mu \frac{\bar{\not{q}}+\not{g}}{(\bar{q}+g)^2}\not{g}\right] v^{r'}(\bar{q}) \\ &= \bar{u}^r(q)\left[\frac{\not{g}\not{q}}{2q\cdot g}\gamma^\mu - \gamma^\mu \frac{\bar{\not{q}}\not{g}}{2\bar{q}\cdot g}\right] v^{r'}(\bar{q}) \\ &= \bar{u}^r(q)\left[\left(\boldsymbol{I} - \frac{\not{q}\not{g}}{2q\cdot g}\right)\gamma^\mu - \gamma^\mu\left(\boldsymbol{I} - \frac{\not{g}\bar{\not{q}}}{2\bar{q}\cdot g}\right)\right] v^{r'}(\bar{q}) = 0 \,,\end{aligned} \tag{2.46}$$

so the second term in equation (A.14) falls out, and the polarization sum may be replaced by

$$\sum_t \varepsilon^*_{t,\sigma}(g)\,\varepsilon_{t,\rho}(g) \to -\eta_{\sigma\rho}\,. \tag{2.47}$$

This could be anticipated from the discussion in appendix A.4: the final state contains only a single gluon, so there is no external gluon pair coupling to a triple-gluon vertex in any diagram. The result is that

$$\alpha_s H^{(0)\mu\nu}_{q\bar{q}g} = -\alpha_s \sum_{r,r',i,j,a} \left(J^{(0)}_{q\bar{q}g}\right)^{\mu\sigma} \left(J^{(0)*}_{q\bar{q}g}\right)^{\nu}_{\sigma}\,. \tag{2.48}$$

This expression also includes a sum over products of the T^a_{ij} , which equals the trace from equation (A.20):

$$\sum_{i,j,a} T^a_{ij}\left(T^a_{ij}\right)^* = \mathrm{Tr}\left[T^a T^{a\dagger}\right] = C_F N_c\,. \tag{2.49}$$

The only remaining sum is that over the spinor indices. As in section 2.1, this can be rewritten as a trace over gamma matrices.

$$\begin{aligned} H^{(0)\mu\nu}_{q\bar{q}g} &\propto \sum_{r,r'} \bar{u}^r(q)\left[\gamma^\sigma \frac{\not{q}+\not{g}}{2q\cdot g}\gamma^\mu - \gamma^\mu \frac{\bar{\not{q}}+\not{g}}{2\bar{q}\cdot g}\gamma^\sigma\right] v^{r'}(\bar{q}) \\ &\qquad \times \bar{v}^{r'}(\bar{q})\left[\gamma^\nu \frac{\not{q}+\not{g}}{2q\cdot g}\gamma_\sigma - \gamma_\sigma \frac{\bar{\not{q}}+\not{g}}{2\bar{q}\cdot g}\gamma^\nu\right] u^r(q) \\ &= \mathrm{Tr}\Bigg[\not{q}\left(\gamma^\sigma \frac{\not{q}+\not{g}}{2q\cdot g}\gamma^\mu - \gamma^\mu \frac{\bar{\not{q}}+\not{g}}{2\bar{q}\cdot g}\gamma^\sigma\right) \\ &\qquad \times \bar{\not{q}}\left(\gamma^\nu \frac{\not{q}+\not{g}}{2q\cdot g}\gamma_\sigma - \gamma_\sigma \frac{\bar{\not{q}}+\not{g}}{2\bar{q}\cdot g}\gamma^\nu\right)\Bigg] \end{aligned} \tag{2.50}$$

The hadron tensor will be contracted with $\eta_{\mu\nu}$ when finding the cross section, and doing it right away will simplify the traces. Defining

$$t_{qq} = \mathrm{Tr}\left[\not{q}\gamma^\sigma(\not{q}+\not{g})\gamma^\mu\bar{\not{q}}\gamma_\mu(\not{q}+\not{g})\gamma_\sigma\right]\,, \tag{2.51}$$
$$t_{q\bar{q}} = \mathrm{Tr}\left[\not{q}\gamma^\sigma(\not{q}+\not{g})\gamma^\mu\bar{\not{q}}\gamma_\sigma(\bar{\not{q}}+\not{g})\gamma_\mu\right]\,, \tag{2.52}$$
$$t_{\bar{q}q} = \mathrm{Tr}\left[\not{q}\gamma^\mu(\bar{\not{q}}+\not{g})\gamma^\sigma\bar{\not{q}}\gamma_\mu(\not{q}+\not{g})\gamma_\sigma\right]\,, \tag{2.53}$$
$$t_{\bar{q}\bar{q}} = \mathrm{Tr}\left[\not{q}\gamma^\mu(\bar{\not{q}}+\not{g})\gamma^\sigma\bar{\not{q}}\gamma_\sigma(\bar{\not{q}}+\not{g})\gamma_\mu\right]\,, \tag{2.54}$$

the contracted hadron tensor can be written

$$\begin{aligned} -\alpha_s\eta_{\mu\nu}H^{(0)\mu\nu}_{q\bar{q}g} &= C_F N_c \mu^{4\varepsilon}\left(e e_q g_s\right)^2 \\ &\quad \times \left[\frac{t_{qq}}{(2q\cdot g)^2} + \frac{t_{\bar{q}\bar{q}}}{(2\bar{q}\cdot g)^2} - \frac{t_{q\bar{q}}+t_{\bar{q}q}}{(2q\cdot g)(2\bar{q}\cdot g)}\right]\,, \end{aligned} \tag{2.55}$$

where $Q = q + \bar{q} + g$.

Here, the infrared divergences appear for the first time. The contracted hadron tensor, and hence the scattering cross section $\sigma^{(0)}_{q\bar{q}g}$, diverge as $q \cdot g \to 0$ or $\bar{q} \cdot g \to 0$. By inspecting the calculations performed so far, one may also realize that the denominators in equation (2.55) would consist of these factors even without assuming massless quarks. Writing

$$q \cdot g = E_q E_g \left(1 - \beta_q \cos\theta_{qg}\right) \ , \ \beta_q = \sqrt{1 - \frac{m_q^2}{E_q^2}} \ , \tag{2.56}$$

it is clear that the divergences occur as $E_g \to 0$. Additionally, if $m_q = 0$ such that $E_q = 0$ is allowed and $\beta_q = 1$, the expression will diverge as $\theta_{qg} \to 0$ or $E_q \to 0$. The singularities at zero energy are examples of soft singularities, while the singularity at zero angle is a collinear singularity.

The traces can be simplified and computed using equation (B.22):

$$\begin{aligned} t_{qq} &= 4\,(1-\varepsilon)^2\,\mathrm{Tr}\left[\not{q}\,(\not{q}+\not{g})\,\bar{\not{q}}\,(\not{q}+\not{g})\right] \\ &= 16\,(1-\varepsilon)^2\, q_\mu\,(q+g)_\nu\,\bar{q}_\rho\,(q+g)_\sigma\left[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right] \\ &= 16\,(1-\varepsilon)^2\left[2\,(q\cdot(q+g))\,(\bar{q}\cdot(q+g)) - q\cdot\bar{q}\,(q+g)^2\right] \\ &= 32\,(1-\varepsilon)^2\left[(q\cdot g)\,(\bar{q}\cdot g)\right] \ . \end{aligned} \tag{2.57}$$

The trace $t_{\bar{q}\bar{q}}$ can be found by replacing q with $\bar{q}$ in t_{qq} , so $t_{\bar{q}\bar{q}} = t_{qq}$. The cross terms are more complicated, but using equation (B.20), one finds

$$\begin{aligned} t_{q\bar{q}} + t_{\bar{q}q} &= 16\,(q+g)_\rho\,(\bar{q}+g)_\delta\left(q_\tau\bar{q}_\lambda + \bar{q}_\tau q_\lambda\right) \\ &\quad \times\Big[\,(-2+\varepsilon\,(1+\varepsilon))\,\eta^{\rho\delta}\eta^{\tau\lambda} + \varepsilon\,(1-\varepsilon)\left(\eta^{\rho\tau}\eta^{\delta\lambda} + \eta^{\rho\lambda}\eta^{\delta\tau}\right)\Big] \\ &= 16\Big[\,(-2+\varepsilon\,(1+\varepsilon))\,2\,(\bar{q}\cdot q)\,((\bar{q}+g)\cdot(q+g)) \\ &\qquad + \varepsilon\,(1-\varepsilon)\,2\,(\bar{q}\cdot(\bar{q}+g))\,(q\cdot(q+g)) \\ &\qquad + \varepsilon\,(1-\varepsilon)\,2\,(q\cdot(\bar{q}+g))\,(\bar{q}\cdot(q+g))\,\Big] \\ &= -32\,(1-\varepsilon)\Big[\,(\bar{q}\cdot q)\,Q^2 - 2\varepsilon\,(q\cdot g)\,(\bar{q}\cdot g)\,\Big] \ . \end{aligned} \tag{2.58}$$

Here, the relations $q^2 = \bar{q}^2 = g^2 = 0$ and $Q^2 = 2q \cdot \bar{q} + 2q \cdot g + 2\bar{q} \cdot g$ were used to simplify the expression. Inserting the traces in equation (2.55) gives

$$\begin{aligned} -\alpha_s \eta_{\mu\nu} H^{(0)\mu\nu}_{q\bar{q}g} &= 8 C_F N_c \mu^{4\varepsilon}\left(e e_q g_s\right)^2 (1-\varepsilon) \\ &\quad \times\left[(1-\varepsilon)\left(\frac{\bar{q}\cdot g}{q\cdot g} + \frac{q\cdot g}{\bar{q}\cdot g}\right) + \frac{(q\cdot\bar{q})\,Q^2}{(q\cdot g)\,(\bar{q}\cdot g)} - 2\varepsilon\right] \ . \end{aligned} \tag{2.59}$$

To simplify the phase space integral, the COM frame energy fractions x_i are used. They are defined in equation (C.36), and repeated here for convenience:

$$x_i = \frac{2p_i \cdot Q}{Q^2} \ . \tag{2.60}$$

For the massless quarks and gluons, this implies that

$$1 - x_q = \frac{2\bar{q}\cdot g}{Q^2}\,, \tag{2.61}$$

$$1 - x_{\bar{q}} = \frac{2q\cdot g}{Q^2}\,, \tag{2.62}$$

$$1 - x_g = \frac{2q\cdot \bar{q}}{Q^2}\,, \tag{2.63}$$

$$x_q + x_{\bar{q}} + x_g = 2\,. \tag{2.64}$$

Inserting this and using equation (C.37) to evaluate the phase space integral in equation (2.36) gives

$$\begin{aligned}
\alpha_s H^{(0)}_{q\bar{q}g}\left(Q^2\right) = {} & \frac{\alpha_{\mathrm{em}}\alpha_s C_F N_c e_q^2 Q^2}{\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{2\varepsilon}\frac{(1-\varepsilon)}{\Gamma\left(2-2\varepsilon\right)} \\
& \times \int_0^1 dx_q \int_{1-x_q}^1 dx_{\bar{q}}\, \frac{1}{\left[\left(1-x_q\right)\left(1-x_{\bar{q}}\right)\left(x_q+x_{\bar{q}}-1\right)\right]^{\varepsilon}} \\
& \times \left[(1-\varepsilon)\left(\frac{1-x_{\bar{q}}}{1-x_q}+\frac{1-x_q}{1-x_{\bar{q}}}\right)+\frac{2\left(1-x_g\right)}{\left(1-x_q\right)\left(1-x_{\bar{q}}\right)}-2\varepsilon\right]
\end{aligned} \tag{2.65}$$

The collinear and soft quark divergences now appear in the limit where either $x_q \to 1$ or $x_{\bar{q}} \to 1$, while the soft gluon divergence manifest as the double pole when both limits are taken simultaneously. To calculate the integral, substitute $x_q = x, x_{\bar{q}} = 1 - vx$. The Jacobian is then x , and the integral becomes

$$\begin{aligned}
& \int_0^1 dx \int_0^1 dv \frac{x}{\left[x^2\left(1-x\right)v\left(1-v\right)\right]^{\varepsilon}} \\
& \times \left[(1-\varepsilon)\left(\frac{xv}{1-x}+\frac{1-x}{xv}\right)+\frac{2\left(1-v\right)}{\left(1-x\right)v}-2\varepsilon\right] \\
= {} & 2\left(1-\varepsilon\right)\frac{\Gamma\left(2-\varepsilon\right)\Gamma\left(1-\varepsilon\right)\Gamma\left(-\varepsilon\right)}{\Gamma\left(3-3\varepsilon\right)} \\
& + 2\frac{\Gamma\left(2-\varepsilon\right)\Gamma\left(-\varepsilon\right)^2}{\Gamma\left(2-3\varepsilon\right)} - 2\varepsilon\frac{\Gamma\left(1-\varepsilon\right)^3}{\Gamma\left(3-3\varepsilon\right)} \\
= {} & \frac{\Gamma\left(1-\varepsilon\right)^3}{\Gamma\left(1-3\varepsilon\right)}\frac{2}{1-3\varepsilon}\left(\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}-\frac{1}{2-3\varepsilon}\left(\frac{1}{\varepsilon}-2-2\varepsilon\right)\right) \\
= {} & \frac{\Gamma\left(1-\varepsilon\right)^3}{\Gamma\left(1-3\varepsilon\right)}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}-\frac{107}{4}\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\right)\,.
\end{aligned} \tag{2.66}$$

Each term in the integral factorizes into a product of two Euler β-function integrals, equation (C.16). All of the integrals are convergent for $\varepsilon < 0$, which is typical of infrared divergences: they are regularized when going to higher spacetime dimensions $d > 4$. The expressions have been manipulated further using equation (C.12).

This gives

$$\begin{aligned}
\alpha_s H^{(0)}_{q\bar{q}g}\left(Q^2\right) &= \frac{\alpha_{\mathrm{em}}\alpha_s C_F N_c e_q^2 Q^2}{\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{2\varepsilon}\frac{(1-\varepsilon)\,\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \\
&\quad\times\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-3\varepsilon)}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}-\frac{107}{4}\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\right) \\
&= H^{(0)}_{q\bar{q}}\left(Q^2\right)\frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \\
&\quad\times\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-3\varepsilon)}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}+\mathcal{O}(\varepsilon)\right) .
\end{aligned} \tag{2.67}$$

Thus, the scattering cross section for $e^+e^- \to q\bar{q}g$ at leading order is

$$\begin{aligned}
\alpha_s\sigma^{(0)}_{q\bar{q}g} &= \frac{2\alpha_{\mathrm{em}}\alpha_s\mu^{2\varepsilon}C_F N_c e_q^2}{Q^2}\left(\frac{4\pi\mu^2}{Q^2}\right)^{2\varepsilon}\frac{(1-\varepsilon)^2}{3-2\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \\
&\quad\times\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-3\varepsilon)}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}+\mathcal{O}(\varepsilon)\right) \\
&= \sigma^{(0)}_{q\bar{q}}\frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-3\varepsilon)}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}+\mathcal{O}(\varepsilon)\right)
\end{aligned} \tag{2.68}$$

These expressions show explicitly the how the infrared divergences manifest as poles as $\varepsilon \to 0$. The total $e^+e^- \to$ hadrons cross section can only be finite at $\mathcal{O}(\alpha_s)$ if these poles are canceled exactly by another contribution at the same order.

2.4 Virtual gluon exchange: $e^+e^- \to q\bar{q}$ at next-to-leading order

The next-to-leading order diagrams for $e^+e^- \to q\bar{q}$ are shown in figure 2.3, with only the hadronic part included. As explained by equation (2.6), the $\mathcal{O}(\alpha_s)$-contribution to the cross section is given by the interference term between these diagrams and the leading order diagram calculated in section 2.1.

In the formalism used here, the first step is to write the total hadron current as an expansion in α_s . To order $\mathcal{O}(\alpha_s)$,

$$\left(J_{q\bar{q}}\right)_\mu = \left(J^{(0)}_{q\bar{q}} + \alpha_s J^{(1)}_{q\bar{q}} + \cdots\right)_\mu . \tag{2.69}$$

Here, $J^{(0)}_{q\bar{q}}$ is the leading-order current from equation (2.12), and $\alpha_s J^{(1)}_{q\bar{q}}$ is the sum of the currents from the diagrams in figure 2.3. The function $H(Q^2)$ from

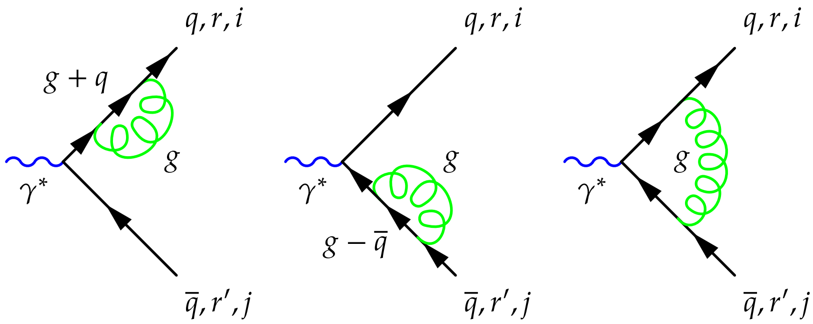


Figure 2.3: Diagrams for the hadron current in $e^-e^+ \to q\bar{q}$ at one loop. The labels r, r' denote spin polarizations, while i, j are color indices. The momentum of the internal quarks flow in the same direction as the fermion current, as indicated by the arrowhead.

equation (2.36) can also be expanded:

$$H_{q\bar{q}}\left(Q^2\right) = H_{q\bar{q}}^{(0)}\left(Q^2\right) + \alpha_s H_{q\bar{q}}^{(1)}\left(Q^2\right) + \cdots , \tag{2.70}$$

where $H_{q\bar{q}}^{(0)}$ is the leading-order term given by equation (2.42). By combining the definition of the hadron tensor $H_{q\bar{q}}^{\mu\nu}$ from equation (2.21), and of $H(Q^2)$ from equation (2.36), one finds that the next term is an interference term between the leading and next-to-leading order currents:

$$\begin{aligned} \alpha_s H_{q\bar{q}}^{(1)}\left(Q^2\right) = -\alpha_s \eta_{\mu\nu} \int d\Phi_2^{(d)} \sum_{r,r',i,j} \bigg[\left(J_{q\bar{q}}^{(0)}\right)^\mu \left(J_{q\bar{q}}^{(1)*}\right)^\nu \\ + \left(J_{q\bar{q}}^{(1)}\right)^\mu \left(J_{q\bar{q}}^{(0)*}\right)^\nu \bigg] \\ = -\alpha_s \int d\Phi_2^{(d)} \sum_{r,r',i,j} 2\Re\left\{ \left(J_{q\bar{q}}^{(0)*}\right)^\mu \left(J_{q\bar{q}}^{(1)}\right)_\mu \right\} . \end{aligned} \tag{2.71}$$

In the first two diagrams in figure 2.3, the virtual gluon loops back onto the quark it was emitted from. The corresponding factor in the amplitude is known as the quark self-energy $-i\Sigma(q)$, and for massless quarks it is given in the Feynman gauge as

$$\begin{aligned} -i\Sigma\left(q\right) &= -\mu^{2\varepsilon} g_s^2 T_{ik}^a T_{kj}^a \int \frac{d^d g}{(2\pi)^d} \frac{\eta^{\mu\nu}}{g^2} \frac{\gamma_\mu \left(\not{g} + \not{q}\right) \gamma_\nu}{\left(g + q\right)^2} \\ &= 2C_F \delta_{ij} \mu^{2\varepsilon} g_s^2 \left(1 - \varepsilon\right) \int \frac{d^d g}{(2\pi)^d} \frac{\not{g} + \not{q}}{g^2 \left(g^2 + 2q \cdot g\right)} , \end{aligned} \tag{2.72}$$

where equations (A.19) and (B.22) and the fact that q is an external, on-shell momentum were used. The integral in equation (2.72) can be rewritten using the results in appendix C:

$$\begin{aligned}
&\int \frac{d^d g}{(2\pi)^d} \frac{\not{g} + \not{q}}{g^2 \left(g^2 + 2q \cdot g\right)} \\
&= \int \frac{d^d g}{(2\pi)^d} \int_0^1 d\alpha \frac{\not{g} + \not{q}}{\left(\alpha \left(g^2 + 2q \cdot g\right) + (1-\alpha)\, g^2\right)^2} \\
&= \int \frac{d^d g}{(2\pi)^d} \int_0^1 d\alpha \frac{\not{g} + \not{q}}{(g + \alpha q)^4} \\
&= \int_0^1 d\alpha\, (1-\alpha)\, \not{q} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} \\
&= \frac{i\not{q}}{2} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^4} \\
&= \frac{i\not{q}}{2} \frac{\Omega_d}{(2\pi)^d} \int_0^\infty d\tilde{k}_E\, \tilde{k}_E^{d-5} \,.
\end{aligned} \tag{2.73}$$

Here, the denominator was first rewritten using the Feynman parameter integral from equation (C.18), and then by completing the square and using $q^2 = 0$. The integration variable was changed to $k = g + \alpha q$ and the term proportional to $\not{k}$ discarded, since it is antisymmetric under reflection through the origin and thus must vanish upon integration. Finally, the k-integral was rotated to Euclidean space as per equation (C.7) and rewritten as a single-variable integral by exploiting the isotropy of the integrand and using equation (C.9).

The final expression in equation (2.73) reveals that this integral is undefined for any value of the dimension d : it has an infrared divergence due to contributions as $\tilde{k}_E \to 0$ for $d \leq 4$, and an ultraviolet divergence due to contributions as $\tilde{k}_E \to \infty$ for $d \geq 4$. This is related to the fact that a massless, on-shell quark does not provide an intrinsic scale for the integral, since $q^2 = 0$. Following the conventions in dimensional regularization, such an integral is *defined* to be zero [7, p. 172]. Somewhat heuristically, this can be justified by splitting the integral at an arbitrary scale Λ and assuming different values of d in the two regimes such that both integrals are convergent:

$$\int_0^\infty d\tilde{k}_E\, \tilde{k}_E^{d-5} = \int_0^\Lambda d\tilde{k}_E\, \tilde{k}_E^{d-5} + \int_\Lambda^\infty d\tilde{k}_E\, \tilde{k}_E^{d'-5} = \frac{\Lambda^{-2\varepsilon}}{-2\varepsilon} - \frac{\Lambda^{-2\varepsilon'}}{-2\varepsilon'} \,. \tag{2.74}$$

Here, $\varepsilon < 0$ to regulate the infrared divergence, while $\varepsilon' > 0$ to regulate the ultraviolet divergence. Of course, the right hand side > would only equal the left hand side if $\varepsilon' = \varepsilon$, and by analytic continuation one might thus say that the integral vanishes.

This means that the first two diagrams in figure 2.3 do not contribute, and $J^{(1)}_{q\bar{q}}$ is given by the third diagram only. This diagram is known as the vertex correction.

This result was derived in the Feynman gauge, but holds in any covariant gauge. The gauge-dependent term in the gluon propagator gives a term proportional to the integral

$$\begin{aligned}
&\int \frac{d^d g}{(2\pi)^d} \frac{\not{g}\,(\not{g}+\not{q})\,\not{g}}{g^4\left(g^2+2q\cdot g\right)} \\
&\quad = \int \frac{d^d g}{(2\pi)^d} \int_0^1 d\alpha \frac{2\,(1-\alpha)\,\not{g}\,(\not{g}+\not{q})\,\not{g}}{(g+\alpha q)^6} \\
&\quad = -\int_0^1 d\alpha\, 2\left(2\alpha\,(1-\alpha)+\frac{1-\varepsilon}{2-\varepsilon}\,(1-\alpha)^2\right)\not{q}\int \frac{d^d k}{(2\pi)^d}\frac{1}{k^4}\,.
\end{aligned} \tag{2.75}$$

Here, the denominator was rewritten using equation (C.19), integration variable changed to $k = g + \alpha q$, and terms proportional to an odd power of k discarded. The gauge-dependent term thus contains the same k-integral as equation (2.73), and is therefore set to zero by the same argument. Strictly speaking, this could have been inferred from the absence of a scale in the denominator of equation (2.75), without further manipulations.

An important result used when simplifying the expression above is the following: consider the integral

$$\int \frac{d^d k}{(2\pi)^d} f\left(k^2\right) k^\mu k^\nu = \eta^{\mu\nu} \int \frac{d^d k}{(2\pi)^d} f\left(k^2\right) g\left(k^2\right)\,. \tag{2.76}$$

where the function f is isotropic in k-space as indicated by the k^2-dependence. The Lorentz indices of the integrated quantity can only be carried by $\eta^{\mu\nu}$, and it must therefore be possible to rewrite the integral as shown on the right, with an unknown function g . Contracting both sides with $\eta_{\mu\nu}$ shows that one can set $g(k^2) = k^2/d$. Hence, in integrals over all of k-space, one may always substitute

$$k^\mu k^\nu \to \frac{1}{d} k^2 \eta^{\mu\nu}\,, \tag{2.77}$$

provided the other factors in the integrand are isotropic. In particular, one may perform the following replacements:

$$\not{k}\not{q}\not{k} \to -\frac{1-\varepsilon}{2-\varepsilon} k^2 \not{q}\,, \tag{2.78}$$

$$(k\cdot q)\,(k\cdot\bar{q}) \to \frac{1}{2\,(2-\varepsilon)} k^2\,(q\cdot\bar{q})\,, \tag{2.79}$$

where the former was used in the last step in equation (2.75), and the latter will become important below.

The hadron current $\alpha_s J^{(1)}_{q\bar{q}}$ of the third diagram in figure 2.3 is given by

$$\alpha_s \left(J^{(1)}_{q\bar{q}}\right)_\mu = -\mu^{3\varepsilon} e e_q g_s^2 T^a_{ik} T^a_{kj} \int \frac{d^d g}{(2\pi)^d} \frac{\bar{u}^r\,(q)\,\Gamma_\mu v^{r'}\,(\bar{q})}{g^2\,(g+q)^2\,(g-\bar{q})^2}\,, \tag{2.80}$$

where[3]

$$
\begin{aligned}
\mathit{\Gamma}_\mu &= \gamma_\rho \left(\slashed{g} + \slashed{q}\right) \gamma_\mu \left(\slashed{g} - \slashed{\bar{q}}\right) \gamma_\sigma \left(\eta^{\rho\sigma} - (1-\xi)\,\frac{g^\rho g^\sigma}{g^2}\right) \\
&= \gamma^\sigma \left(\slashed{g} + \slashed{q}\right) \gamma_\mu \left(\slashed{g} - \slashed{\bar{q}}\right) \gamma_\sigma - \frac{1-\xi}{g^2}\slashed{g}\left(\slashed{g} + \slashed{q}\right) \gamma_\mu \left(\slashed{g} - \slashed{\bar{q}}\right) \slashed{g}\,.
\end{aligned}
\tag{2.81}
$$

Hence, the interference in equation (2.71) can be written

$$
\begin{aligned}
\sum_{r,r',i,j} -2\left(J_{q\bar{q}}^{(0)*}\right)^\mu \left(\alpha_s J_{q\bar{q}}^{(1)}\right)_\mu &= 2iC_F N_c \mu^{4\varepsilon}\left(e e_q g_s\right)^2 \\
&\qquad \times \int \frac{d^d g}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu \slashed{q}\mathit{\Gamma}_\mu \slashed{\bar{q}}\right]}{g^2\,(g+q)^2\,(g-\bar{q})^2}\,.
\end{aligned}
\tag{2.82}
$$

The trace can be split into two terms corresponding to the two terms in $\mathit{\Gamma}_\mu$. The first trace can be evaluated using equation (B.20):

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma^\mu \slashed{q}\gamma^\sigma \left(\slashed{g} + \slashed{q}\right) \gamma_\mu \left(\slashed{g} - \slashed{\bar{q}}\right) \gamma_\sigma \slashed{\bar{q}}\right] \\
&\quad = 16 q_\rho \left(g+q\right)_\tau \left(g-\bar{q}\right)_\delta \bar{q}_\lambda \\
&\qquad \times \Big[\, (-2 + \varepsilon\,(1+\varepsilon))\,\eta^{\rho\delta}\eta^{\tau\lambda} + \varepsilon\,(1-\varepsilon)\left(\eta^{\rho\tau}\eta^{\delta\lambda} + \eta^{\rho\lambda}\eta^{\delta\tau}\right) \Big] \\
&\quad = 16\Big[\, (-2 + \varepsilon\,(1-\varepsilon))\,(q\cdot(g-\bar{q}))\,(\bar{q}\cdot(g+q)) \\
&\qquad\quad + \varepsilon\,(1-\varepsilon)\,(q\cdot(g+q))\,(\bar{q}\cdot(g-\bar{q})) \\
&\qquad\quad + \varepsilon\,(1-\varepsilon)\,(q\cdot\bar{q})\,((g+q)\cdot(g-\bar{q}))\, \Big] \\
&\quad = 8\,(1-\varepsilon)\left[Q^4 - 2g\cdot(q-\bar{q})\,Q^2 - 4\,(g\cdot q)\,(g\cdot\bar{q}) + \varepsilon g^2 Q^2\right]
\end{aligned}
\tag{2.83}
$$

Here, the relations $q^2 = \bar{q}^2 = 0$ and $Q^2 = 2q\cdot\bar{q}$ were used to simplify the expression. The second trace, corresponding to the gauge-dependent term in $\mathit{\Gamma}_\mu$, is found by applying equation (B.24):

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma^\mu \slashed{q}\slashed{g}\left(\slashed{g} + \slashed{q}\right) \gamma_\mu \left(\slashed{g} - \slashed{\bar{q}}\right) \slashed{g}\slashed{\bar{q}}\right] \\
&\quad = -2\mathrm{Tr}\left[\left(\slashed{g} + \slashed{q}\right)\slashed{g}\slashed{q}\left(\slashed{g} - \slashed{q}\right)\slashed{g}\slashed{\bar{q}}\right] \\
&\qquad + 2\varepsilon\mathrm{Tr}\left[\slashed{q}\slashed{g}\left(\slashed{g} + \slashed{q}\right)\left(\slashed{g} - \slashed{q}\right)\slashed{g}\slashed{\bar{q}}\right]
\end{aligned}
\tag{2.84}
$$

To proceed, note that since $q^2 = \bar{q}^2 = 0$,

$$
\left(\slashed{g} + \slashed{q}\right)\slashed{g}\slashed{q} = \left(\slashed{g} + \slashed{q}\right)\left(\slashed{g} + \slashed{q}\right)\slashed{q} = \left(g+q\right)^2\slashed{q}\,.
\tag{2.85}
$$

[3] Note that the gauge parameter ξ here specifies the gauge used for QCD, which is not the same as the gauge used for QED and discussed in sections 2.1 and 2.2. Although the final result should be invariant with respect to the QCD gauge also, the gauge-dependent term is kept here for generality and as an extra check on the computations.

Similarly identities hold for $\not{q}\not{g}(\not{g}+\not{q})$ and $(\not{g}-\not{\overline{q}})\not{g}\not{\overline{q}}$. Hence,

$$\begin{aligned}
&\mathrm{Tr}\left[\gamma^\mu \not{q}\not{g}\,(\not{g}+\not{q})\,\gamma_\mu\,(\not{g}-\not{\overline{q}})\,\not{g}\not{\overline{q}}\right] \\
&\qquad = -2\,(1-\varepsilon)\,(g+q)^2\,(g-\overline{q})^2\,\mathrm{Tr}\left[\not{q}\not{\overline{q}}\right] \\
&\qquad = -4\,(1-\varepsilon)\,(g+q)^2\,(g-\overline{q})^2\,Q^2\,.
\end{aligned} \tag{2.86}$$

The factors $(g+q)^2$ and $(g-\overline{q})^2$ cancel against the same factors in the denominator in the g-integral, so the gauge-dependent term is proportional to

$$\int \frac{d^d g}{(2\pi)^d}\frac{1}{g^4}\,, \tag{2.87}$$

which is set to zero for the same reason as the self-energy integrals. Thus, the gauge-dependency drops out.

The denominator in equation (2.82) can be rewritten using equation (C.19):

$$\begin{aligned}
&\frac{1}{g^2\,(g+q)^2\,(g-\overline{q})^2} \\
&\qquad = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{2}{\left[\alpha\,(g+q)^2+\beta\,(g-\overline{q})^2+(1-\alpha-\beta)\,g^2\right]^3} \\
&\qquad = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{2}{\left[(g+\alpha q-\beta\overline{q})^2+\alpha\beta Q^2\right]^3}\,.
\end{aligned} \tag{2.88}$$

The g-integral is then solved by changing integration variable to $k = g+\alpha q-\beta\overline{q}$. Substituting this variable change into equation (2.83) gives

$$\begin{aligned}
&\mathrm{Tr}\left[\gamma^\mu \not{q}\gamma^\sigma\,(\not{g}+\not{q})\,\gamma_\mu\,(\not{g}-\not{\overline{q}})\,\gamma_\sigma\not{\overline{q}}\right] \\
&\qquad = 8\,(1-\varepsilon)\Big[\,(1-\alpha-\beta+(1-\varepsilon)\,\alpha\beta)\,Q^4 \\
&\qquad\qquad - 2k\cdot\left[q-\overline{q}-(1-\varepsilon)\,(\alpha q-\beta\overline{q})\right]Q^2 \\
&\qquad\qquad - 4\,(k\cdot q)\,(k\cdot\overline{q})+\varepsilon k^2 Q^2\Big] \\
&\qquad \to 8\,(1-\varepsilon)\,Q^2\left[(1-\alpha-\beta+(1-\varepsilon)\,\alpha\beta)\,Q^2-\frac{(1-\varepsilon)^2}{2-\varepsilon}k^2\right]\,,
\end{aligned} \tag{2.89}$$

where the third line was obtained by using the replacement from equation (2.79) and discarding the term proportional to k , both justified by the fact that the expression will be integrated over all of k-space. The relevant k-integrals evaluate to

$$\begin{aligned}
\int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2+\alpha\beta Q^2\right]^3} &= (-i)^{d-1} \frac{\Omega_d}{(2\pi)^d} \int_0^\infty d\tilde{k}_E \frac{\tilde{k}_E^{d-1}}{\left[\tilde{k}_E^2+\alpha\beta Q^2\right]^3} \\
&= \frac{(-i)^{d-1} \left(\alpha\beta Q^2\right)^{(d/2)-3}}{(4\pi)^{d/2}\,\Gamma\left(d/2\right)} \int_0^1 dx\, x^{2-(d/2)} \left(1-x\right)^{(d/2)-1} \\
&= \frac{i}{2\left(4\pi\right)^2} \left(-\frac{4\pi}{Q^2}\right)^{\varepsilon} \Gamma\left(1+\varepsilon\right) \frac{1}{\left(\alpha\beta\right)^{\varepsilon}} \frac{1}{\alpha\beta Q^2}
\end{aligned} \tag{2.90}$$

$$\begin{aligned}
\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{\left[k^2+\alpha\beta Q^2\right]^3} &= (-i)^{d-1} \frac{\Omega_d}{(2\pi)^d} \int_0^\infty d\tilde{k}_E \frac{\tilde{k}_E^{d+1}}{\left[\tilde{k}_E^2+\alpha\beta Q^2\right]^3} \\
&= \frac{(-i)^{d-1} \left(\alpha\beta Q^2\right)^{(d/2)-2}}{(4\pi)^{d/2}\,\Gamma\left(d/2\right)} \int_0^1 dx\, x^{1-(d/2)} \left(1-x\right)^{(d/2)} \\
&= \frac{i}{2\left(4\pi\right)^2} \left(-\frac{4\pi}{Q^2}\right)^{\varepsilon} \Gamma\left(1+\varepsilon\right) \frac{1}{\left(\alpha\beta\right)^{\varepsilon}} \frac{2-\varepsilon}{\varepsilon}
\end{aligned} \tag{2.91}$$

Here, equations (C.8) and (C.9) were used to get single-variable integrals, and the integration variable was changed to $x = (1+\tilde{k}_E^2/\alpha\beta Q^2)^{-1}$ to put them on Euler β-function form, equation (C.16).

Combining the results in equations (2.88) to (2.91), the interference term becomes:

$$\begin{aligned}
&\alpha_s \sum_{r,r',i,j} -2 \left(J_{q\bar{q}}^{(0)*}\right)^{\mu} \left(\alpha_s J_{q\bar{q}}^{(1)}\right)_{\mu} \\
&\quad = -16\alpha_{\mathrm{em}}\alpha_s \mu^{2\varepsilon} C_F N_c e_q^2 Q^2 \left(-\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \left(1-\varepsilon\right)\Gamma\left(1+\varepsilon\right) \\
&\qquad \times \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{\left(\alpha\beta\right)^{\varepsilon}} \left[\frac{1-\alpha-\beta+\left(1-\varepsilon\right)\alpha\beta}{\alpha\beta} - \frac{\left(1-\varepsilon\right)^2}{\varepsilon}\right]
\end{aligned} \tag{2.92}$$

The α- and β-integrals decouple into Euler β-function integrals upon substituting $\beta = (1-\alpha)v$:

$$
\begin{aligned}
&\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha\beta)^\varepsilon} \left[\frac{1-\alpha-\beta+(1-\varepsilon)\,\alpha\beta}{\alpha\beta} - \frac{(1-\varepsilon)^2}{\varepsilon} \right] \\
&= \int_0^1 d\alpha \int_0^1 dv\, \alpha^{-\varepsilon} (1-\alpha)^{1-\varepsilon} v^{-\varepsilon} \left[\frac{1-v}{\alpha v} + (1-\varepsilon)\left(2-\frac{1}{\varepsilon}\right)\right] \\
&= \left(\frac{\Gamma(-\varepsilon)^2}{\Gamma(2-2\varepsilon)} + \frac{\Gamma(1-\varepsilon)^2}{\Gamma(3-2\varepsilon)} (1-\varepsilon)\left(2-\frac{1}{\varepsilon}\right)\right) \\
&= \frac{1}{2}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)} \left(\frac{2}{\varepsilon^2(1-2\varepsilon)} - \frac{1}{\varepsilon}\right) \\
&= \frac{1}{2}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 + 16\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right) .
\end{aligned}
\tag{2.93}
$$

There is no explicit q- or $\overline{q}$-dependency left, so the phase space integral is trivial in the COM frame using equation (C.35), and one finds

$$
\begin{aligned}
\alpha_s H_{q\overline{q}}^{(1)}\left(Q^2\right) = \Re\Bigg\{ &\frac{-\alpha_{\mathrm{em}}\alpha_s C_F N_c e_q^2 Q^2}{\pi} \left(\frac{4\pi\mu^2}{Q^2}\right)^{2\varepsilon} (-1)^\varepsilon \\
&\times \frac{(1-\varepsilon)\,\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \frac{\Gamma(1+\varepsilon)\,\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)} \\
&\times \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 + 16\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right)\Bigg\}
\end{aligned}
\tag{2.94}
$$

$$
\begin{aligned}
= H_{q\overline{q}}^{(0)}\left(Q^2\right) &\frac{-\alpha_s C_F}{\pi} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \Re\left\{(-1)^\varepsilon\right\} \\
&\times \frac{\Gamma(1+\varepsilon)\,\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 + \mathcal{O}(\varepsilon)\right) .
\end{aligned}
$$

The next-to-leading order contribution to the cross section for $e^+e^- \to q\overline{q}$ can thus be written

$$
\begin{aligned}
\alpha_s \sigma_{q\overline{q}}^{(1)} = \sigma_{q\overline{q}}^{(0)} &\frac{-\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \Re\left\{(-1)^\varepsilon\right\} \\
&\times \frac{\Gamma(1+\varepsilon)\,\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 + \mathcal{O}(\varepsilon)\right) .
\end{aligned}
\tag{2.95}
$$

This clearly diverges as $\varepsilon \to 0$. Before proceeding to show that divergences cancel in the total $e^+e^- \to$ hadrons cross section, it is a good idea to pause for a moment and consider what kinds of divergences the poles in this equation (2.95) actually represent. By comparing with equation (C.16), it is clear that equation (2.90) is convergent in an interval including $\varepsilon = 0$. The result then enters the first term in the integrand in equation (2.93), and the integral of this term is convergent only for $\varepsilon < 0$. Thus, the first term on the fourth line in equation (2.93) must represent infrared divergences.

On the other hand, equation (2.91) is obviously logarithmically divergent for $d = 4$, and only converges for $\varepsilon > 0$. The result is used in the second term in the integrand in equation (2.93), and this integral places no further restrictions on ε for convergence. Hence, the second term in the fourth line in equation (2.93) (the $-1/\varepsilon$-term) represents an ultraviolet divergence.

However, the procedure of cancellation about to be demonstrated actually removes all of these divergences, although it was never expected to work for ultraviolet divergences, which are normally removed by renormalization. However, having QCD vertex corrections renormalizing the QED coupling is physically unacceptable, since these corrections would apply to quarks but not leptons, and thus enable processes violating conservation of electric charge. It may seem like the ultraviolet divergence is really an infrared divergence in disguise, and it will certainly be treated as such when showing the cancellation. Section 2.6 contains a short discussion of why this may be considered plausible.

2.5 Cancellation: $e^+e^- \to$ hadrons at next-to-leading order

The total cross section for $e^+e^- \to$ hadrons at $\mathcal{O}(\alpha_s)$ is given by the sum up to $\mathcal{O}(\alpha_s)$ of the cross sections of all processes $e^+e^- \to X$ that can be defined at this order. Here, X is any state of quarks and gluons. There are no other such processes than the two considered so far, and hence the cross section may be written

$$
\begin{aligned}
\sigma_{\text{hadrons}} &= \sigma_{q\bar{q}}^{(0)} + \alpha_s\left(\sigma_{q\bar{q}g}^{(0)} + \sigma_{q\bar{q}}^{(1)}\right) + \mathcal{O}\left(\alpha_s^2\right) \\
&= \sigma_{q\bar{q}}^{(0)}\Bigg\{1 + \frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-3\varepsilon)}\Bigg[\left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} + \mathcal{O}(\varepsilon)\right) \\
&\qquad - \Re\left\{(-1)^{\varepsilon}\right\}\frac{\Gamma(1+\varepsilon)\Gamma(1-3\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 + \mathcal{O}(\varepsilon)\right)\Bigg] \\
&\qquad + \mathcal{O}\left(\alpha_s^2\right)\Bigg\}\,.
\end{aligned}
\tag{2.96}
$$

The factor $(-1)^{\varepsilon}$ can be expanded as follows:

$$
\begin{aligned}
(-1)^{\varepsilon} &= e^{i\pi\varepsilon} \\
&= 1 + i\pi\varepsilon - \frac{\pi^2}{2}\varepsilon^2 - \frac{i\pi^3}{6}\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right)\ ,
\end{aligned}
\tag{2.97}
$$

such that

$$
\Re\left\{(-1)^{\varepsilon}\right\} = 1 - \frac{\pi^2}{2}\varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right)\ . \tag{2.98}
$$

The three Γ-functions in the second term in equation (2.96) may be expanded using equation (C.15). For the denominator, the geometric series expansion $(1-x)^{-1} = 1 + x + x^2 + \cdots$ must also be used:

$$\begin{aligned}
\frac{1}{\Gamma(1-2\varepsilon)} &= \left[1 + 2\gamma_E\varepsilon + \left(\frac{\pi^2}{3} + 2\gamma_E^2\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right)\right]^{-1} \\
&= 1 - \left(2\gamma_E\varepsilon + \left(\frac{\pi^2}{3} + 2\gamma_E^2\right)\varepsilon^2\right) \\
&\quad + \left(2\gamma_E\varepsilon + \left(\frac{\pi^2}{3} + 2\gamma_E^2\right)\varepsilon^2\right)^2 + \mathcal{O}\left(\varepsilon^3\right) \\
&= 1 - 2\gamma_E\varepsilon - \left(\frac{\pi^2}{3} - 2\gamma_E^2\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \ .
\end{aligned} \tag{2.99}$$

Multiplying this with the straightforward expansions of the factors in the numerator gives

$$\frac{\Gamma(1+\varepsilon)\,\Gamma(1-3\varepsilon)}{\Gamma(1-2\varepsilon)} = 1 + \frac{\pi^2}{2}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \ , \tag{2.100}$$

and thus,

$$\Re\left\{(-1)^{\varepsilon}\right\}\frac{\Gamma(1+\varepsilon)\,\Gamma(1-3\varepsilon)}{\Gamma(1-2\varepsilon)} = 1 + \mathcal{O}\left(\varepsilon^3\right) \ . \tag{2.101}$$

This is exactly the result needed for the divergent terms in σ_{hadrons} to vanish, and hence the $\varepsilon \to 0$ limit can be taken, giving the finite result

$$\begin{aligned}
\sigma_{\text{hadrons}} &= \sigma_{q\bar{q}}^{(0)}\left(1 + \frac{3\alpha_s C_F}{4\pi} + \mathcal{O}\left(\alpha_s^2\right)\right) \\
&= \frac{4\pi\alpha_{\text{em}}^2 N_c e_q^2}{3Q^2}\left(1 + \frac{3\alpha_s C_F}{4\pi} + \mathcal{O}\left(\alpha_s^2\right)\right) \ .
\end{aligned} \tag{2.102}$$

Thus, it has been shown that the total cross section for $e^+e^- \to$ hadrons in the massless quark approximation is without soft and collinear divergences at $\mathcal{O}(\alpha_s)$, despite the fact that the individual contributions from real gluon emission and virtual gluon exchange both suffer from this.

2.6 The ultraviolet divergence

As mentioned in section 2.4, the virtual gluon exchange cross section $\alpha_s\sigma_{q\bar{q}}^{(1)}$ contains what appears to be an ultraviolet divergence, which nevertheless cancels by the same mechanism as the infrared divergences. Some insight on this conundrum can be gained by once again considering the self-energy diagrams, postponing for the moment the definition stating that the scaleless integral vanishes. Using equation (2.74), the self-energy may be written in the Feynman gauge as

$$\begin{aligned}
-i\Sigma\left(q\right) &= i\not{q}\frac{\alpha_s C_F \delta_{ij}}{4\pi}\left[\left(\frac{4\pi\mu^2}{\Lambda^2}\right)^{\varepsilon'}\frac{1}{\varepsilon'\Gamma\left(1-\varepsilon'\right)}-\left(\frac{4\pi\mu^2}{\Lambda^2}\right)^{\varepsilon}\frac{1}{\varepsilon\Gamma\left(1-\varepsilon\right)}\right] \\
&= i\not{q}\frac{\alpha_s C_F \delta_{ij}}{4\pi}\left[\frac{1}{\varepsilon'}-\frac{1}{\varepsilon}+\mathcal{O}\left(\varepsilon'-\varepsilon\right)\right] .
\end{aligned} \tag{2.103}$$

By convention, half of the self-energy is canceled by wave function renormalization [1, p. 111, 8, p. 36]. The contributing current from the two first diagrams in figure 2.3 is thus

$$\begin{aligned}
\alpha_s\left(J_\Sigma\right)_\mu &= \mu^\varepsilon e e_q \overline{u}^r\left(q\right)\left[\frac{\left(-i\Sigma\left(q\right)\right)}{2}\frac{\not{q}}{q^2}\gamma_\mu+\gamma_\mu\frac{-\not{\overline{q}}}{\overline{q}^2}\frac{\left(-i\Sigma\left(-\overline{q}\right)\right)}{2}\right]v^{r'}\left(\overline{q}\right) \\
&= \frac{i\alpha_s C_F \delta_{ij}\mu^\varepsilon e e_q}{4\pi}\left[\frac{1}{\varepsilon'}-\frac{1}{\varepsilon}+\mathcal{O}\left(\varepsilon'-\varepsilon\right)\right]\overline{u}^r\left(q\right)\gamma_\mu v^{r'}\left(\overline{q}\right) .
\end{aligned} \tag{2.104}$$

As in equation (2.74), the unprimed ε is used in the term that was regularized with $\varepsilon < 0$ and contains the infrared divergence, while ε' is used in the ultraviolet divergent term. The interference term entering $\alpha_s\sigma^{(1)}_{q\overline{q}}$ is then

$$\begin{aligned}
&\alpha_s\sum_{r,r',i,j}-2\left(J^{(0)*}\right)^\mu\left(J_\Sigma\right)_\mu \\
&\qquad = -8\alpha_{\text{em}}\alpha_s C_F N_c \mu^{2\varepsilon}Q^2\left[\frac{1}{\varepsilon'}-\frac{1}{\varepsilon}+\mathcal{O}\left(\varepsilon'-\varepsilon\right)\right] .
\end{aligned} \tag{2.105}$$

The term in the vertex correction interference term that was regularized in with $\varepsilon > 0$, and hence contains the ultraviolet divergence, is found by including only the second term under the integral in equation (2.92). Expanding such that only the divergence is explicitly written, it reads

$$\left[\alpha_s\sum_{r,r',i,j}-2\left(J^{(0)*}\right)^\mu\left(J^{(1)}_{q\overline{q}}\right)_\mu\right]_{UV} = 8\alpha_{\text{em}}\alpha_s C_F N_c \mu^{2\varepsilon}Q^2\left[\frac{1}{\varepsilon'}+\mathcal{O}\left(1\right)\right] . \tag{2.106}$$

A primed ε' is used since the divergence is ultraviolet.

Summing these two terms leads to a remarkable result: the ultraviolet divergence is canceled exactly, but replaced by an infrared divergence with the exact same expression. It was this infrared divergence, then, that was canceled by adding the real gluon emission and the virtual gluon exchange cross sections together – the ultraviolet divergence had already been canceled by summing the different diagrams for virtual gluon exchange.

The same result is derived in an arbitrary gauge in [8, pp. 39-41], using Pauli-Villars regularization. Dimensional regularization makes the mechanism at work here somewhat opaque in comparison, but an interpretation is that the definition $\int d^d k\ (k^2)^{-n} = 0$ in effect admits analytic continuation of integrals regularized for $\varepsilon > 0$ to $\varepsilon < 0$, such that a pole appearing to be ultraviolet may in fact be infrared. This is also discussed in e.g. [9, p. 498].

3

Discussion

The appearance and cancellation of infrared divergences are ubiquitous phenomena in theories with massless particles. There is a lot to learn about physically measurable quantities, both in quantum field theories in general and in QCD in particular, from understanding the nature of these singularities and under which conditions they cancel. These issues will be discussed in this chapter.

3.1 Infrared divergences in general quantum field theories

A scattering process emitting massless particles in the final state, such as depicted in figure 2.2, will in general have a factors like the following in the denominator of the amplitude:

$$\begin{aligned}(k+p)^2 - m^2 &= 2k \cdot p \\ &= 2E_k E_p (1 - \beta_p \cos\theta_{kp}), \quad \beta_p = \sqrt{1 + \frac{m^2}{E_p^2}}\,.\end{aligned} \tag{3.1}$$

Here, k and p denote the momenta of the massless particle and the particle it was emitted from, respectively, and m is the mass of the latter. Since these are the momenta of particles in the final state, they are on-shell with $k^2 = 0$ and $p^2 = m$, giving the simplified form. The on-shell condition for the massless particle allows $k = 0$, or equivalently, $E_k = 0$, and unless some factors cancel against factors in the numerator, the amplitude will clearly have a singularity here. Moreover, integrating over phase space will not eliminate the divergence when working in $d = 4$ spacetime dimensions, hence the cross section inherits the singularity. When $m \to 0$ such that $\beta_p \to 1$, singularities also appear as $E_p \to 0$ or $\theta_{kp} \to 0$. These are the soft and collinear singularities, investigated in detail for a specific example in the previous chapter. However, this simple argument shows that the phenomenon is universal: in massless theories, scattering cross sections blow up when a massless particle is emitted with low momentum or parallel to the emitter.

This is actually not too surprising. In the $k \to 0$-limit, the virtual particle with momentum $k + p$ that existed before the emission is put on-shell – it becomes real. This may be interpreted as the particle being free to propagate for an unlimited time and distance before the emission actually happens, and this violates the fundamental assumptions used when setting up the perturbative expansion in quantum field theories. Specifically, when deriving the LSZ reduction formula, which relates scattering amplitudes to the Greens functions that can be derived directly from the Hamiltonian, it is assumed that the fields are free of interactions in the far past and future, $t \to \pm\infty$, due to the increasing distance between the particles [6, pp. 111-116]. In the limit of low energy, however, the wavelength of a massless particle becomes infinite, so this freedom can never be obtained. In this sense, final states with a fixed number of massless particles are not well-defined in perturbative quantum field theory.

This can also be seen by considering the similar process where the massless particle is exchanged as a virtual particle rather than emitted, as in figure 2.3. The $k \to 0$-limit puts all three particles in the loop on the mass-shell, and hence the loop interaction may take an arbitrarily long time and cover arbitrarily long distances. Thus, the distinction between exchange and emission of the massless particle vanishes. This indicates that when computing a quantity incorporating the $k = 0$-limit of one of these processes, the other should also be taken into account.

One can also consider this from an experimentalists perspective. A detector has a finite energy resolution, so there is no way to actually detect a particle with arbitrarily low momentum in an experiment. From this point of view, the singularities in the cross section are not a problem at all, the mistake was rather to integrate over the phase space regions where the singularities appear. The contribution from momenta smaller than some cutoff should instead be regarded as a part of the cross section for the similar process *without* emission of the massless particle, since this is what it would look like to the detector. Particles below the detection limit are called *soft* particles.

These arguments are readily extended to cover collinear singularities when both particles are massless. In this case, the distinction between a particle with momentum p and two collinear particles with with momenta λp and $(1 - \lambda)p$ vanishes.

These arguments explain how, for massless theories, the concept of external states with a definite number of particles must be abandoned in order to avoid infrared singularities. Although each amplitude in the perturbation expansion is constructed from such states, a physically measurable quantity can only be computed by including the contribution from a particular external state on equal footing with all states obtained by adding any number of soft and collinear particles to it. The external states of processes without infrared singularities therefore do not consist of isolated particles, but rather of *jets* made up of any number of nearly parallel particles with total momentum equal to the particles they replace, along with a cloud of soft particles that escape detection.

A formalism making the mathematical statement of these ideas straightforward was introduced by [10]. A general measurement from a scattering experiment with massless particles may be expressed as

$$\mathcal{I} = \sum_n \frac{1}{n!} \int d\Phi_n \frac{d\sigma_n}{d\Phi_n} \mathcal{S}_n \left(p_1, \dots, p_n\right) \ . \tag{3.2}$$

Here, n labels the number of particles in a perturbative final state, and $d\sigma_n/d\Phi_n$ is the differential cross section for scattering into this state. The measurement is assumed to not distinguish different kinds of particles, hence there is only a single differential cross section per n, and each term is divided by $n!$ to avoid overcounting. The measured quantity is defined by the weights $\mathcal{S}_n(p_1, \dots, p_n)$. For the total cross section, the weights are simply $\mathcal{S}_n(p_1, \dots, p_n) = 1$. The requirement for *infrared safe* measurements is that the weights are equal in all soft and collinear limits, that is,

$$\mathcal{S}_{n+1} \left(p_1, \dots, (1-\lambda) \, p_n, \lambda p_n\right) = \mathcal{S}_n \left(p_1, \dots, p_n\right) \ , \tag{3.3}$$

and similar for all other pairs of momenta. For $0 < \lambda < 1$, this ensures that a single particle is considered equal to two collinear particles carrying the same total momentum. The case $\lambda = 0$ takes care of the degeneracy due to soft particles.

As an example of an infrared safe observable, consider the Sterman-Weinberg jet [11].[1] Here, the same process is considered as in chapter 2, $e^+e^- \to \text{hadrons}$. An event is registered when all but a fraction $\varepsilon \ll 1$ of the total energy $\sqrt{Q^2}$ is emitted within a pair of back-to-back cones of half-angle $\delta \ll 1$, lying within a larger, fixed pair of back-to-back cones of solid angle $\Omega \ll 1$ (with $\pi\delta^2 \ll \Omega$) oriented at an angle θ to the incoming e^+e^- beams. The observable is the scattering rate for such events. It is infrared safe: a particle splitting into two nearly parallel particles does not affect whether an event will be registered or not as long as the angle between them is sufficiently small, and neither does the emission of a particle with sufficiently low energy.

The Kinoshita-Lee-Nauenberg theorem

The discussion above is presented in a heuristic fashion, but the claims are supported by rigorous proofs. The Kinoshita-Lee-Nauenberg theorem (KLN) [12, 13] states that in any quantum field theory, the total transition rate between states of equal energy is free of infrared divergences in the limit of massless particles. Formally, the theorem can be stated as follows (similar to the presentation in [7, pp. 364-370, 14, pp. 440-447]):

[1] This particular observable is not often used in current experiments, but is used as an example here because it easy to understand.

Let $D(E_0, \varepsilon)$ denote the set of states with energy E in the interval $E_0 - \varepsilon < E < E_0 + \varepsilon$, and S be the S-matrix of the theory. Then, the total transition rate

$$P(E_0, \varepsilon) = \sum_{a,b \in D(E_0,\varepsilon)} |\langle a|S|b\rangle|^2 \tag{3.4}$$

is free of infrared singularities in the limit of massless particles.

Proofs of the theorem may be found in all of the listed references.

The significance of the theorem is that it shows that for observable quantities satisfying some criterion, *all* quantum field theories are free of infrared divergences in the limit of massless particles. Moreover, this holds at each order in perturbation theory. However, in general one is not only required to sum over *final* states, as done in the calculation in chapter 2, but also over *initial* states. This was avoided in the $e^+e^- \to$ hadrons cross section because the initial state did not participate in QCD interactions.

The KLN theorem involves summation over all states with the same total energy, which is a somewhat unfamiliar configuration, and much more inclusive than the observables described by equation (3.3). However, in [15, pp. 548-553], the proof is modified to show that the same result holds for configurations such as Sterman-Weinberg jets and other measurements satisfying equation (3.3) as well. The requirement to sum over initial states then enters via the differential cross sections, which must be defined by averaging over the squared amplitudes and flux factors for all possible initial states related to the nominal initial state by additional soft or collinear particles.

It is worth noting that in QED, if the electron mass is non-zero, infrared safe quantities can be obtained by only summing over final states. This is known as the Bloch-Nordsieck theorem [16]. In QCD, however, this does *not* hold, as shown by e.g. [17], so summation over initial states will in general be required.

3.2 The optical theorem and cutting rules

A convenient way of visualizing and interpreting the relation between infrared divergent quantities whose sum is finite is obtained using the optical theorem. For two-particle scattering, the optical theorem can be stated as follows (adapted from [18, p. 455]):

$$\Im\left\{\mathcal{M}\left(k_1 k_2 \to k_1 k_2\right)\right\} = 2I \sum_X \sigma\left(k_1 k_2 \to X\right) \ , \tag{3.5}$$

where k_1, k_2 are the momenta of the two incoming particles and I is the flux factor from equation (A.2). The theorem states that the imaginary part of the forward scattering amplitude for a two-particle state is proportional to the sum over the total cross section for this initial state. A forward scattering amplitude is a scattering amplitude where the final state is identical to the initial state.

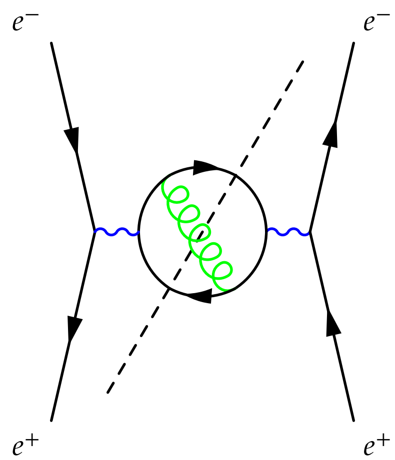


Figure 3.1: Forward scattering $e^+e^- \to e^+e^-$ at $\mathcal{O}(\alpha_{\text{em}}^2\alpha_s)$. The dashed line shows an example cut.

An intuitive understanding of this theorem may be obtained by considering forward scattering Feynman diagrams which include loops where the kinematic conditions allow the particles to be put on-shell. Figure 3.1 shows an example of a diagram for $e^+e^- \to e^+e^-$ at $\mathcal{O}(\alpha_{\text{em}}^2\alpha_s)$. When performing the loop integrals in this amplitude, some of the virtual particles will be put on-shell in certain regions of the space of loop momenta. However, when a propagator goes on-shell, an imaginary delta function appears:

$$\Im\left\{\frac{1}{x+i\varepsilon}\right\} = -\pi\delta\left(x\right) \ . \tag{3.6}$$

The result is that when solving the loop momentum integrals, the regions where particles in the loop go on-shell contributes an imaginary part to the scattering amplitude. Moreover, the delta function makes this contribution integral appear very similar to a phase space integral, and in fact it will be proportional to the cross section of the scattering process $e^+e^- \to X$ at $\mathcal{O}(\alpha_{\text{em}}^2\alpha_s)$, where X is the final state consisting of the on-shell particles in the loop. The state X must contain all particles along some *cut* through the diagram, and it must be possible to put all these particles on-shell simultaneoulsy. An example of a cut is shown by the dashed line in figure 3.1.

Thus, to calculate the imaginary part of the amplitude for a single Feynman diagram such as the one in figure 3.1, one may proceed as follows:

1. Identify all possible cuts through the intermediate states in the diagram such that the particles along the cut may be put on-shell simultaneously.

2. For each cut, calculate the cross section of the interference between the half-diagram to the left of the cut and the mirror image of the half-diagram to the right of the cut.

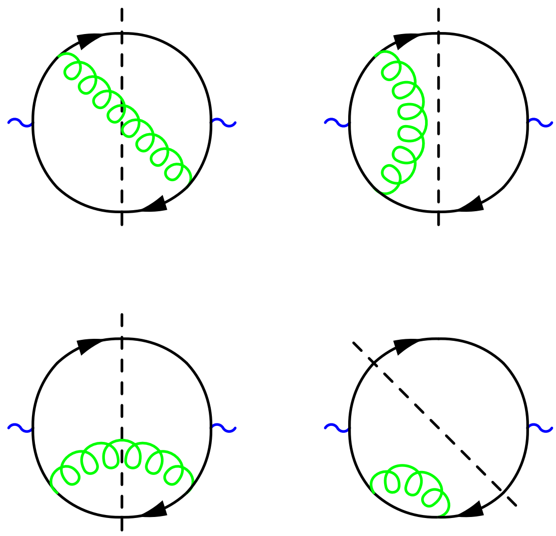


Figure 3.2: Cut diagrams for $\gamma^* \to$ hadrons at $\mathcal{O}(\alpha_s)$. The similar diagrams with the quark and antiquark swapped are not drawn.

3. Add all these cross sections together, and multiply with half of the flux factor for the initial state.

This is known as the cutting rules, and may be thought of as the optical theorem in action, broken down to specific orders and diagrams. The claims will not be proved here, but they are not difficult to demonstrate for simple processes such as the one in figure 3.1, see e.g. [18, pp. 453-466].

By virtue of the optical theorem and cutting rules, the diagram in figure 3.1 is very closely related to the cross sections computed in chapter 2. To minimize clutter, the external states e^+e^- will be amputated in the following, such that cuts of forward scattering diagrams for γ^* at $\mathcal{O}(\alpha_{\text{em}}\alpha_s)$ are considered instead. Figure 3.2 shows the two different ways to cut each of the two possible loop configurations at this order. The top line contains the same loop as figure 3.1 cut in two different ways: the first cut corresponds to the interference terms between the two diagrams in the real gluon emission diagrams in figure figure 2.2, and the second cut corresponds to the interference between the leading order diagram in figure 2.1 and the vertex correction in figure 2.3. The bottom line contains a similar loop where the gluon is emitted and absorbed by the same quark. Here, the first cut corresponds to the squared amplitude of one of the real emission diagrams in figure 2.2, while the second corresponds to the interference between the leading order diagram in figure 2.1 and a self-energy insertion in figure 2.3.

Since QED and QCD are renormalizable theories, the amplitude for a loop diagram such as the one in figure 3.1 is expected to be finite.[2] The cutting rules relate the imaginary part of this amplitude to the sum of cross sections computed in chapter 2, containing infrared divergences, and this provides an alternative argument for why the divergences should cancel when the cross sections are summed. Moreover, it admits an alternative interpretation of the origin infrared divergences: they appear in quantities corresponding to incomplete loop momentum integrals. This picture is of course not more fundamental than other interpretations presented in this chapter, but it provides a neat way to organize the diagrams and identify groups of terms whose sum must be finite, since they are different cuts of the same loop. This picture also strengthens the viewpoint that in the infrared limit, the distinction between real emission and virtual exchange vanishes: these two kinds of cuts correspond to adjacent regions in the space of loop momenta, and the infrared limit corresponds to the boundary between these regions.

Cut diagrams are used in the original proof of the KLN theorem in [12]. The cut diagrams explained here are not sufficient for this purpose, however, since they cannot handle infrared divergences in the initial state.

3.3 Infrared safety in QCD

The discussion in the previous sections applies to quantum field theories in general. When considering quantum chromodynamics in particular, there is an additional concern related to the increasing strength of the QCD coupling at long distances, and in particular confinement. In particular, the arguments relying on the properties of detectors do not translate directly to QCD, since the final states in perturbative QCD are never observed in experiment, as explained in section 1.1.

On the other hand, the concept of jets containing any number of nearly parallel particles with definite total momentum is most welcome: this is description is quite appropriate to the hadronic final states observed in QCD scattering experiments, although the internal dynamics of the QCD jets, leading to hadronization, seems to be more complex than in theories such as QED.

The essential observation described in section 3.1, explaining infrared divergences and leading to the definition of infrared safe observables and jets, is that sensible measurements must be insensitive to interactions over long time and distance scales, since otherwise it is not possible to separate the interacting fields from the free external states. In QCD the idea of free external states made up of fundamental particles is invalid, but if an observable is truly insensitive to long-distance interactions, the composition of the external states should not

[2]There is no need to worry about infrared divergences in figure 3.1 diagram, since the process is not considered at an order where photon emission or exchange in the external states cannot be accomodated.

matter. Thus, infrared safe quantities calculated in perturbative QCD may be compared directly to experiment without considering any non-perturbative dynamics.

This is not to say that infrared safe observables are the only interesting observables in QCD. In particular, any observable distinguishing different kinds of hadrons obviously depends on long-distance interactions, and does not fit into the scheme described by equation (3.3). The solution in this case is to factorize the process into a perturbative and a non-perturbative part as described in section 1.1, where the perturbative part depends on the particular high-energy, short-distance interactions involved, while the non-perturbative part is universal and completely independent of short-distance physics. In this approach the perturbative part must still be infrared safe in order to be calculable in perturbation theory, and all the long-distance sensitivity is placed in the non-perturbative part [3].

4

Summary

In the total cross section for the scattering process $e^+e^- \rightarrow$ hadrons , the contributions at $\mathcal{O}(\alpha_s)$ have been shown to contain infrared singularities, both from the real gluon emission and virtual gluon exchange diagrams. However, by using dimensional regularization to safely manipulate the infinities, it has been demonstrated that the divergent terms cancel, such that the total cross section is finite.

By considering the causes of these singularities, one is led to the conclusion that well-defined measurements computed in perturbative quantum field theory cannot in general have a definite number of massless particles in the initial and final states. In order to obtain finite observables one must instead take care to define them such that they are insensitive to low-energy, long-distance interactions. Observables satisfying this property are called *infrared safe*. This naturally leads to the concept of *jets*, collections of an arbitrary number of nearly parallel particles with definite total momentum, which can be related to experimental measurements. The Kinoshita-Lee-Nauenberg theorem demonstrates that for suitably defined observables, infrared divergences will always cancel, to all orders.

In the particular case of quantum chromodynamics, infrared safety defines the set of observables for which theoretical values may be calculated from perturbation theory alone. However, due to the non-perturbative strength of the QCD coupling at low energies and long distances, and in particular the phenomenon of confinement, many relevant observables will necessarily depend on long-distance interactions. By factorizing the process into a perturbative and a non-perturbative part, however, the short-distance physics may still be described by infrared safe calculations in perturbation theory.

A

Feynman rules for QED and QCD

This appendix lists the Lagrangian and Feynman rules for quantum electrodynamics (QED) and quantum chromodynamics (QCD), mainly following the conventions in [1]. Using the parametrization from [19], the sign conventions are $\eta_s = \eta_e = \eta_G = +1$.

A.1 General remarks

A Feynman diagram is a pictorial representation of a scattering process, which can be translated directly to a term in the power series expansion of the associated scattering amplitude $\mathcal{M}$, where the expansion parameter is the coupling constant of the theory, e.g. $\alpha_{\text{em}} = e^2/4\pi$ for QED and $\alpha_s = g_s^2/4\pi$ for QCD. In the diagram, particles are represented by lines joined together at vertices, and the number of vertices in the diagram is related to the order of the term it represents. The initial and final states in the scattering process are represented by open-ended lines at each end of the diagram, called external lines, while lines terminating at a vertex in both ends are called internal lines and are said to represent *virtual* particles. A typical example, describing a process similar to the one discussed in section 2.1, is shown in figure A.1.

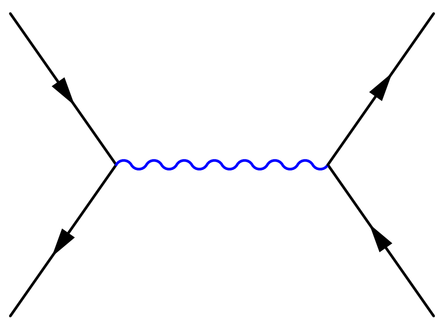
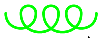


Figure A.1: An example of a Feynman diagram

The convention here is that time runs from left to right, so the initial states come in from the left and the final states exit to the right. Four different kinds of lines are used to represent the different particles in QED and QCD:

A fermion of any kind. The arrowhead shows the direction of the associated fermion current, hence lines with arrows pointing back in time represent the corresponding antifermion. Since the current is conserved, any fermion line pointing towards a vertex must be balanced by a line pointing away from it.

A photon, the gauge boson of QED.



A gluon, the gauge boson of QCD.

A ghost, an unphysical particle appearing in QCD when working in certain gauges. The ghosts only appear as internal lines. Ghosts obey fermionic statistics, and are therefore drawn with an arrowhead.

The translation from a diagram to a term in the scattering amplitude is done according to the following general rules

- Each external line is replaced by a solution to the equation of motion for the corresponding free field, e.g. a Dirac spinor for a fermion line.
- Each internal line is replaced by the propagator for the corresponding free field.
- Each vertex is replaced by a vertex factor derived from the interaction Lagrangian of the theory.
- 4-momentum is conserved at each vertex. Any momenta left undetermined by this are integrated over using the measure $d^4p/(2\pi)^4$, or in d spacetime dimensions, $d^dp/(2\pi)^d$.
- For each closed loop of fermion lines, a factor of -1 is appended.
- If identical fermions exist in the external states, a diagram obtained by crossing two of the external fermion lines while leaving the initial and final states unchanged acquires a factor of -1 compared to the uncrossed diagram.
- For fermions, the order of the terms must be correct in order for the spinors and matrices to contract correctly. For each connected string of fermion lines, the external state spinors, vertex factors and propagators

must be written down in the order they appear when following the string from end to end *against* the direction of the arrowheads. For a closed fermion loop, the starting and ending point is arbitrary, but a trace must be taken over the final product. (Alternatively, all spinor-related quantities may be written with explicit spinor indices that match at each vertex.)

- In some cases, a symmetry factor must be appended if the number of different ways to connect the vertices and obtain an identical diagram is not as large as the prefactor from the Taylor expansion of the action exponential and any numerical coefficients from the interaction term in the Lagrangian. This is a rather technical point that is easiest to understand through practice. See [15, pp. 265-267] for a comprehensive review.

When the scattering amplitude $\mathcal{M}$ has been computed to the desired order, the differential cross section $d\sigma$ for the process is found taking the absolute square of the amplitude, dividing by the flux factor $4I$, and appending the phase space measure $d\Phi_n$ for a final state with n particles:

$$d\sigma = \frac{1}{4I} |\mathcal{M}|^2 \, d\Phi_n \, . \tag{A.1}$$

In this text only two-particle collisions are considered. The flux factor is then [1, p. 427, 6, pp. 158-160]

$$\begin{aligned} 4I &= 4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2} \\ &\approx 2s, \quad \text{when } \; m_1^2, m_2^2 \ll s = (p_1 + p_2)^2 \, . \end{aligned} \tag{A.2}$$

Phase space measures are discussed in appendix C.2.

A.2 QED

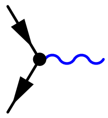
The Lagrangian for the quantum electrodynamics of a fermion with with mass m_f and electric charge ee_f is

$$\begin{aligned} \mathcal{L}_{\text{QED}} &= \overline{\psi}_f \left(i\not{D} - m_f \right) \psi_f - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &= \overline{\psi}_f \left(i\not{\partial} - m_f \right) \psi_f - ee_f \overline{\psi}_f \not{A} \psi_f - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \, , \end{aligned} \tag{A.3}$$

where $\psi_f(x)$ is the spinor field of the fermion, A_μ is the gauge field for the photon, the photon field strength tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, , \tag{A.4}$$

Table A.1: Building blocks of QED Feynman diagrams

Incoming lines $p \longrightarrow$	Outgoing lines $p \longrightarrow$	Internal lines $p \longrightarrow$	Vertex
 $= u\,(p)$	 $= \overline{u}\,(p)$	 $= iS_F\,(p)$	 μ $= -iee_f\gamma_\mu$
 $= \overline{v}\,(p)$	 $= v\,(p)$		
μ  $= \varepsilon^\mu\,(p)$	 μ $= \varepsilon^{*\mu}\,(p)$	μ ν $= iD_F^{\mu\nu}\,(p)$	

and the gauge covariant derivative D_μ is

$$D_\mu = \partial_\mu + iee_f A_\mu\,. \tag{A.5}$$

The constant e is the (positive) elementary electric charge, and e_f is the quantum number associated with electric charge for the fermion under consideration. For electrons and other charged leptons, $e_f = -1$, while for quarks $e_f = e_q$ with $e_q = 2/3$ for the $q = u, c, t$ quarks and $e_q = -1/3$ for the $q = d, s, b$ quarks

This leads to the Feynman diagram building blocks shown in table A.1. The Dirac spinors u, v for the external fermions are further explained in appendix B.2. The fermion propagator is

$$iS_F\,(p) = i\frac{\not p + m}{p^2 - m^2 + i\varepsilon}\,. \tag{A.6}$$

The 4-vector $\varepsilon^\mu(p)$ specifies the polarization for an external photon with 4-momentum p , and is discussed further in appendix A.4.

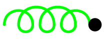
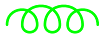
The gauge field A^μ carries redundant degrees of freedom, and a gauge is a set of constraints selected to eliminate this redundancy (see appendix A.5 for a brief discussion of this concept). Expressions relating to A^μ are in general different in different gauges. Covariant gauges are defined by the constraint $\partial_\mu A^\mu = 0$, and the photon propagator can then be expressed as

$$iD_F^{\mu\nu}\,(p) = i\frac{-\eta^{\mu\nu} + (1-\xi)\,p^\mu p^\nu/p^2}{p^2 + i\varepsilon}\,, \tag{A.7}$$

where ξ is an arbitrary parameter. Popular choices include the Feynman gauge, $\xi = 1$, and the Landau gauge, $\xi = 0$.

The $i\varepsilon$-term in the denominator of the propagators is included as a prescription for how to treat the poles at $p^2 = m^2$ or $p^2 = 0$ when integrating. The limit $\varepsilon \to 0$ should be taken at the end of a computation. The $i\varepsilon$-term is often dropped to save writing when doing calculations, but is implicitly assumed whenever relevant.

Table A.2: Lines in QCD Feynman diagrams

Incoming lines $p \longrightarrow$		Outgoing lines $p \longrightarrow$		Internal lines $p \longrightarrow$	
	$= u\,(p)$		$= \overline{u}\,(p)$	i  j	$= iS_F\,(p)\,\delta_{ij}$
	$= \overline{v}\,(p)$		$= v\,(p)$	a  b	$= i\delta_{ab}/(p^2 + i\varepsilon)$
μ 	$= \varepsilon^{\mu}\,(p)$	 μ	$= \varepsilon^{*\mu}\,(p)$	μ, a  ν, b	$= iD_F^{\mu\nu}\,(p)\,\delta_{ab}$

A.3 QCD

The Lagrangian for the quantum chromodynamics of a single quark flavor labeled q , with mass m_q and N_c color degrees of freedom, is

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \overline{\psi}_{q,i}\left(i\not{D}_{ij} - m_q\delta_{ij}\right)\psi_{q,j} - \frac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} \\ &= \overline{\psi}_{q,i}\left(i\not{\partial} - m_q\right)\delta_{ij}\psi_{q,j} - g_s\overline{\psi}_{q,i}T^a_{ij}\not{A}^a\psi_{q,j} - \frac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu}\,,\end{aligned} \tag{A.8}$$

where $\psi_{q,i}(x)$ is the spinor field of the quark q , with the index $i \in \{1, \ldots, N_c\}$ representing the quark color charge basis states, A^a_μ is the gauge field for the gluon, with the index $a \in \{1, \ldots, N_c^2 - 1\}$ representing the gluon color basis states, the gluon field strength tensor $F^a_{\mu\nu}$ is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu\,, \tag{A.9}$$

and the gauge covariant derivative D_μ is defined as

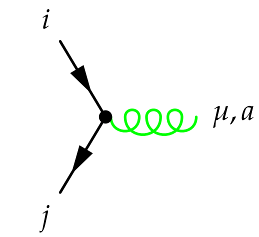
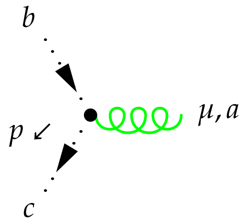
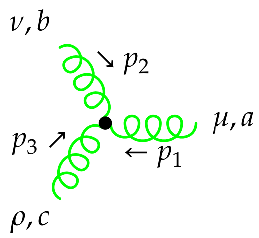
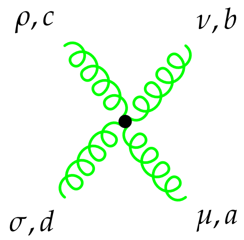
$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + ig_s T^a_{ij} A^a_\mu\,, \tag{A.10}$$

For the quarks constituting the hadrons seen in experiment, the number of color degrees of freedom is found to be $N_c = 3$. However, N_c is often kept general throughout theoretical calculations.

The factors f^{abc} and matrices T^a are discussed in appendix A.5.The constant g_s is the coupling constant of quantum chromodynamics.

This leads to the Feynman diagram building blocks shown in tables A.2 and A.3. In covariant gauges, defined by the condition $\partial_\mu A^{a\mu} = 0$, a set of unphysical ghost particles appear to compensate for unphysical degrees of freedom that are not eliminated in these gauges. The main text does not consider any processes to the orders where ghosts become relevant, but they are included here for completeness. The symbols $u, v, \varepsilon, S_F, D_F$ have the same meaning here as in the QED case, but as shown, the propagators acquire an extra color-conserving δ-factor in the QCD case. A color-conserving δ_{ij} must also be appended to the QED vertex factor when the fermions are quarks.

Table A.3: Vertices in QCD Feynman diagrams

	$= -ig_s T^a_{ij} \gamma_\mu$
	$= -g_s f^{abc} p^\mu$
	$= -g_s f^{abc} \Big[\ \eta^{\mu\nu} (p_1 - p_2)^\rho + \eta^{\nu\rho} (p_2 - p_3)^\mu + \eta^{\rho\mu} (p_3 - p_1)^\nu \Big]$
	$= -ig_s^2 \Big[\ f^{eab} f^{ecd} (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{eac} f^{edb} (\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma}) + f^{ead} f^{ebc} (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) \Big]$

A.4 Polarization

As mentioned above, an external photon or gluon is described by a polarization 4-vector $\varepsilon(p)$. This is the coefficient in a plane wave solution to the classical equation of motion for the gauge field A^μ [6, p. 96]:

$$A^\mu (x) \propto \varepsilon^\mu (p) \, e^{-ip\cdot x} + \varepsilon^{*\mu} (p) \, e^{ip\cdot x} \ . \tag{A.11}$$

The polarization therefore satisfies the momentum space equivalent of any equation satisfied by the field A^μ . In covariant gauges one has $\partial_\mu A^\mu = 0$, translating to the constraint $p\cdot\varepsilon(p) = 0$. For a general 4-momentum p (implying non-zero mass), this gives three polarization degrees of freedom, which can be represented by an orthonormal basis:

$$\varepsilon^\mu_r (p) \cdot \varepsilon^{*\mu}_{r'} (p) = -\delta_{rr'} \ , \quad r \in \{1, 2, 3\} \ . \tag{A.12}$$

Let the decomposition be such that $\varepsilon_3(p)$ denotes the polarization along the direction of travel: $\boldsymbol{\varepsilon}_3(p) \propto \boldsymbol{p}$, and $\varepsilon_3^0(p) = \boldsymbol{p} \cdot \boldsymbol{\varepsilon}_3(p)/p^0$. However, if the mass is zero, such as for gluons and photons, $p^2 = 0$ and $\varepsilon_3(p) \propto p$, and it is clear that equation (A.12) can no longer be satisfied when including $r = 3$. Thus, for massless particles longitudinal polarization is unphysical, and only two polarization degrees of freedom remain, $r \in \{1, 2\}$. Following [1, pp. 76-79], this can be enforced explicitly by adding a second constraint $\varepsilon(p) \cdot n = 0$, where n is any 4-vector satisfying $n \cdot p \neq 0$. This clearly also implies $n \cdot A = 0$, so this constraint defines a new class of gauges called physical gauges.

When calculating matrix elements, one frequently needs to find the polarization sum

$$\begin{aligned}\sum_r \varepsilon_r^\mu(p)\, \varepsilon_r^{*\nu}(p) &= A(p,n)\, \eta^{\mu\nu} + B(p,n)\, p^\mu p^\nu + C(p,n)\, n^\mu n^\nu \\ &\quad + D_1(p,n)\, p^\mu n^\nu + D_2(p,n)\, n^\mu p^\nu \,.\end{aligned} \tag{A.13}$$

This form of the right hand side could be written down by considering the only available independent variables and quantities carrying a Lorentz index. Applying the conditions $p \cdot \varepsilon_r(p) = 0$, $n \cdot \varepsilon_r(p) = 0$ and equation (A.12), the sum is found to be

$$\sum_r \varepsilon_r^\mu(p)\, \varepsilon_r^{*\nu}(p) = -\eta^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{n \cdot p} - n^2 \frac{p_\mu p_\nu}{(n \cdot p)^2} \,. \tag{A.14}$$

In physical gauges, the numerator of the gluon and photon propagator would look similar to this, and since these gauges eliminate all unphysical degrees of freedom, this would eliminate the need for ghost particles in QCD (at the expense of imposing a gauge condition that is not Lorentz covariant). On the other hand, not explicitly eliminating the unphysical degrees of freedom for *external* gluons (that is, neglecting n-dependent terms in equation (A.14) when applied to initial and final states) would introduce the need to consider additional diagrams with external ghost pairs in processes involving diagrams where external gluon pairs couple to triple gluon vertices. The middle ground when working in covariant gauges is to use the covariant propagator from equation (A.7) for internal gluons, and including diagrams with internal ghosts as necessary, while for external gluons using the polarization sum in equation (A.14) and thus avoiding the need for external ghosts.

A.5 Some results from $\mathrm{SU}(N_c)$

The fields appearing in the Lagrangians in equations (A.3) and (A.8) carry some redundancy in their description. This is to say that the fields may be transformed in certain ways without changing the value of the Lagrangian, and thus without changing the physics. This is called gauge symmetry, and is the origin of the name “gauge field” for the photon and gluon fiels A_μ, and of the concept of gauges introduced while discussing the propagator $D_F^{\mu\nu}$ and the polarization vectors.

Gauge symmetries are defined by symmetry groups. The gauge group of QED is U(1), while the gauge group of QCD is SU(N_c) for quarks with N_c color degrees of freedom. No properties of U(1), SU(N_c) or groups in general will be derived here, but some important results will be stated. For a physicist-friendly treatment of Lie groups in general, see e.g. [20]. For a discussion in the context of QCD, see [1].

That the gauge symmetry of QCD is SU(N_c) means that the QCD Lagrangian, equation (A.8), is invariant under a transformation of the quark field $\psi_q = (\psi_{q,1}, \dots, \psi_{q,N_c})$ in the fundamental representation of SU(N_c), provided that the gauge field $A_\mu = T^a A_\mu^a$ transforms in the adjoint representation of the same group (with an additional inhomogeneous term if the transformation is *local*, that is, varies continuously between spacetime points). These transformation properties are not explicitly applied in the main text, but their significance is that they explain how a gauge field with $N_c^2 - 1$ degrees of freedom must appear when requiring that a Lagrangian is invariant under a local SU(N_c) transformation of a matter field with N_c degrees of freedom. This explains the ranges of the color indices i, j, a . For real-world quarks, $N_c = 3$ such that $N_c^2 - 1 = 8$.

A general element of SU(N_c) may be expressed as $U(\theta) = \exp(i\theta^a T^a)$, where the θ^a are $N_c^2 - 1$ real parameters and the matrices T^a are known as the *generators* of SU(N_c), and are the same as the T^a appearing in equations (A.8) and (A.10). In the fundamental representation, the elements of SU(N_c) are unitary $N_c \times N_c$-matrices U, $UU^\dagger = U^\dagger U = \boldsymbol{I}$, with determinant $\det(U) = +1$. Hence, the generators are Hermitian and traceless $N_c \times N_c$ matrices $T^a = T^{a\dagger}$, $\mathrm{Tr}\,[T^a] = 0$. The most common representation of the generators of SU(3) is $T^a = \lambda^a/2$, where λ^a are the Gell-Mann matrices listed in equation (A.15).

$$
\begin{aligned}
&\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \\
&\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} ,
\end{aligned}
\tag{A.15}
$$

The algebra of the generators is defined by the structure constants f^{abc} :

$$
\left[T^a, T^b\right] = i f^{abc} T^c \,. \tag{A.16}
$$

For SU(3) with generators represented as above, the structure constants are antisymmetric in all indices, and up to permutation of indices the non-zero values are:

$$\begin{aligned} f^{123} &= 1\,, \\ f^{458} = f^{678} &= \frac{\sqrt{3}}{2}\,, \\ f^{147} = f^{165} = f^{246} = f^{345} = f^{257} &= \frac{1}{2}\,. \end{aligned} \tag{A.17}$$

For the fundamental representation of SU(N_c), the generators may be chosen such that the structure factors are antisymmetric in all indices. The normalization of the generators can then be written

$$T^a_{ij} T^{b\dagger}_{ji} = \mathrm{Tr}\left[T^a T^{b\dagger}\right] = T_F \delta^{ab}\,, \tag{A.18}$$

$$T^a_{ij} T^{a\dagger}_{jk} = \left(T^a T^{a\dagger}\right)_{ik} = C_F \delta_{ik}\,. \tag{A.19}$$

Clearly,

$$T^a_{ij} T^{a\dagger}_{ji} = \mathrm{Tr}\left[T^a T^{a\dagger}\right] = T_F\left(N_c^2 - 1\right) = C_F N_c\,. \tag{A.20}$$

Here, T_F is called the Dynkin index and may be chosen freely, and C_F is called the color factor. Since the generators are Hermitian, the daggers in the above identities may be removed without altering the results. For SU(3) with generators represented as above,

$$T_F = \frac{1}{2}\,, \tag{A.21}$$

$$C_F = \frac{4}{3}\,. \tag{A.22}$$

B

Properties of gamma matrices and Dirac spinors

This appendix reviews properties of the Dirac gamma matrices and Dirac spinors. All derivations are performed in a representation-independent manner, inspired by [21], but the presentation focuses on the properties needed in the main text.

B.1 Gamma matrices

The gamma matrices are any set of 4×4-matrices γ^μ , where μ is a Lorentz index, satisfying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\boldsymbol{I} \, . \tag{B.1}$$

The objects acted upon by these matrices are *spinors*, presented in appendix B.2, and the entries are referred to as $\gamma^\mu_{\alpha\beta}$ where α, β are called spinor indices.

More explicitly, Equation (B.1) can be written

$$\begin{aligned} \left(\gamma^0\right)^2 &= \boldsymbol{I} \, , \\ \left(\gamma^i\right)^2 &= -\boldsymbol{I} \, , \qquad i \in \{1, 2, 3\} \, , \\ \gamma^\mu\gamma^\nu &= -\gamma^\nu\gamma^\mu \, , \quad \mu \neq \nu \, . \end{aligned} \tag{B.2}$$

From this, it is clear that γ^0 can only have eigenvalues ± 1 , while the γ^i can only have eigenvalues $\pm i$.

A similarity transformation $\gamma^\mu \rightarrow \gamma'^\mu = S\gamma^\mu S^{-1}$ leaves equation (B.1) invariant and thus relates different representations of the gamma matrices. The most common representations are those where γ^0 is Hermitian and the γ^i anti-Hermitian, $\left(\gamma^0\right)^\dagger = \gamma^0$ and $\left(\gamma^i\right)^\dagger = -\gamma^i$. A convenient way of expressing this is

$$\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0 \, . \tag{B.3}$$

Such representations are related by *unitary* similarity transformations $\gamma^\mu \rightarrow \gamma'^\mu = U\gamma^\mu U^\dagger \, , \ U^\dagger = U^{-1}$.

The restriction in equation (B.3) is related to the origin of the gamma matrices as coefficients in the Dirac Hamiltonian

$$H = \gamma^0 \left(\boldsymbol{\gamma} \cdot \boldsymbol{p} + m \right) \ . \tag{B.4}$$

Equation (B.1) can be derived by requiring this Hamiltonian to satisfy the relativistic energy-momentum relation $H^2 = \boldsymbol{p}^2 + m^2$. The Hermiticity conditions, equation (B.3), are needed for H to be Hermitian. Only representations satisfying equation (B.3) are considered here.

It is convenient to define the matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{B.5}$$

It satisfies the following properties:

$$\begin{aligned} \left(\gamma^5\right)^\dagger &= \gamma^5 \ , \\ \left(\gamma^5\right)^2 &= \boldsymbol{I} \ , \\ \left\{\gamma^5, \gamma^\mu\right\} &= 0 \ . \end{aligned} \tag{B.6}$$

This is straightforward to show by applying equations (B.2) and (B.3). The γ^5 matrix is closely related to the concept of parity [5, pp. 235-238], but here it is only used as a convenient tool to prove trace identities.

Traces

It is often necessary to compute traces of products of gamma matrices. The trace of a matrix is the sum of its diagonal entries: $\mathrm{Tr}\left[A\right] = \sum_i A_{ii}$. The trace operation is obviously linear, and it is also invariant under cyclic permutations of the factors in its argument:

$$\mathrm{Tr}\left[A_1 A_2 \cdots A_{N-1} A_N\right] = \mathrm{Tr}\left[A_N A_1 A_2 \cdots A_{N-1}\right] \ . \tag{B.7}$$

This is known as the *cyclicity* of the trace.

The gamma matrices are traceless, as seen by

$$\begin{aligned} \mathrm{Tr}\left[\gamma^\mu\right] &= \mathrm{Tr}\left[\left(\gamma^5\right)^2 \gamma^\mu\right] \\ &= -\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^5\right] \\ &= -\mathrm{Tr}\left[\left(\gamma^5\right)^2 \gamma^\mu\right] \\ &= -\mathrm{Tr}\left[\gamma^\mu\right] \ . \end{aligned} \tag{B.8}$$

Here, the properties from equation (B.6) are used in the first two steps by inserting an identity in the form of $(\gamma^5)^2$ and then anticommuting. A cyclic permutation is performed in the third step.

By a similar argument, the trace of a product of any *odd* number of gamma matrices must vanish:

$$\begin{aligned}\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_N}\right] &= \mathrm{Tr}\left[\left(\gamma^5\right)^2\gamma^{\mu_1}\cdots\gamma^{\mu_N}\right] \\ &= (-1)^N\mathrm{Tr}\left[\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_N}\gamma^5\right] \\ &= (-1)^N\mathrm{Tr}\left[\left(\gamma^5\right)^2\gamma^{\mu_1}\cdots\gamma^{\mu_N}\right] \\ &= (-1)^N\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_N}\right] \ .\end{aligned} \tag{B.9}$$

The case of two matrices is straightforward:

$$\begin{aligned}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] &= \eta^{\mu\nu}\mathrm{Tr}\left[\boldsymbol{I}\right] \\ &= 4\eta^{\mu\nu} \ ,\end{aligned} \tag{B.10}$$

since by cyclicity, $\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = \mathrm{Tr}\left[\gamma^\nu\gamma^\mu\right] = \mathrm{Tr}\left[\{\gamma^\mu,\gamma^\nu\}/2\right]$.

In the general case, equation (B.1) can be applied repeatedly to reduce a trace of N gamma matrices to a sum of traces of $N-2$ gamma matrices by anticommuting one matrix through the rest of the product. By recursive application of this algorithm, the trace of any (even) number of gamma matrices may be found. For $N = 4$, the algorithm works as follows:

$$\begin{aligned}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] &= 2\eta^{\mu\nu}\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\right] - \mathrm{Tr}\left[\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\right] \\ &= 2\left(\eta^{\mu\nu}\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\right] - \eta^{\mu\rho}\mathrm{Tr}\left[\gamma^\nu\gamma^\sigma\right]\right) + \mathrm{Tr}\left[\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\right] \\ &= 2\left(\eta^{\mu\nu}\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\right] - \eta^{\mu\rho}\mathrm{Tr}\left[\gamma^\nu\gamma^\sigma\right] + \eta^{\mu\sigma}\mathrm{Tr}\left[\gamma^\nu\gamma^\rho\right]\right) \\ &\quad - \mathrm{Tr}\left[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\right] \ .\end{aligned} \tag{B.11}$$

The last term on the right hand side is equal to the expression on the left hand side by cyclicity. Substituting equation (B.10) gives the result:

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) \ . \tag{B.12}$$

Similarly,

$$\begin{aligned}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\delta\gamma^\lambda\right] = 4\Big(&\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\delta\lambda} - \eta^{\mu\nu}\eta^{\rho\delta}\eta^{\sigma\lambda} + \eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\sigma\delta} \\ &- \eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\delta\lambda} + \eta^{\mu\rho}\eta^{\nu\delta}\eta^{\sigma\lambda} - \eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\delta} \\ &+ \eta^{\mu\sigma}\eta^{\nu\rho}\eta^{\delta\lambda} - \eta^{\mu\sigma}\eta^{\nu\delta}\eta^{\rho\lambda} + \eta^{\mu\sigma}\eta^{\nu\lambda}\eta^{\rho\delta} \\ &- \eta^{\mu\delta}\eta^{\nu\rho}\eta^{\sigma\lambda} + \eta^{\mu\delta}\eta^{\nu\sigma}\eta^{\rho\lambda} - \eta^{\mu\delta}\eta^{\nu\lambda}\eta^{\rho\sigma} \\ &+ \eta^{\mu\lambda}\eta^{\nu\rho}\eta^{\sigma\delta} - \eta^{\mu\lambda}\eta^{\nu\sigma}\eta^{\rho\delta} + \eta^{\mu\lambda}\eta^{\nu\delta}\eta^{\rho\sigma}\Big) \ .\end{aligned} \tag{B.13}$$

Generalizing to d spacetime dimensions, it is still possible define an algebra satisfying equation (B.1). When doing dimensional regularization as discussed in appendix C, the intent is always to take the limit $d \to 4$ in the end, and one can therefore assume $\mathrm{Tr}\left[\boldsymbol{I}\right] = 4$ [1, p. 433]. Thus, equations (B.10), (B.12) and (B.13) may be used for dimensional regularization. The only important difference is that the contraction of the metric tensor with itself is changed, $\eta^\mu_\mu = d = 4 - 2\varepsilon$, and this must be kept in mind when contracting the traces with other quantities.

Contractions

The Lorentz index on the gamma matrix may be raised and lowered by the metric tensor in the same way as for vectors:

$$\eta_{\mu\nu}\gamma^{\nu} = \gamma_{\mu} \, . \tag{B.14}$$

Thus, contracted gamma matrix products are straightforward to simplify using the anticommutation relations. Some examples are shown here, calculated in $d = 4 - 2\varepsilon$ dimensions:

$$\begin{aligned} \gamma^{\mu}\gamma_{\mu} &= \frac{1}{2}\left\{\gamma^{\mu}, \gamma_{\mu}\right\} = \eta^{\mu}_{\mu}\boldsymbol{I} \\ &= d\boldsymbol{I} = (4 - 2\varepsilon)\,\boldsymbol{I} \, , \end{aligned} \tag{B.15}$$

$$\begin{aligned} \gamma^{\mu}\gamma^{\rho}\gamma_{\mu} &= \left(2\eta^{\mu\rho} - \gamma^{\rho}\gamma^{\mu}\right)\gamma_{\mu} \\ &= (2 - d)\,\gamma^{\rho} = -2\,(1 - \varepsilon)\,\gamma^{\rho} \, , \end{aligned} \tag{B.16}$$

$$\begin{aligned} \gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma_{\mu} &= \left(2\eta^{\mu\rho} - \gamma^{\rho}\gamma^{\mu}\right)\gamma^{\sigma}\gamma_{\mu} \\ &= 2\gamma^{\sigma}\gamma^{\rho} + 2\,(1 - \varepsilon)\,\gamma^{\rho}\gamma^{\sigma} \\ &= 4\eta^{\sigma\rho} - 2\varepsilon\gamma^{\rho}\gamma^{\sigma} \, . \end{aligned} \tag{B.17}$$

$$\begin{aligned} \gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\lambda}\gamma_{\mu} &= \left(2\eta^{\mu\rho} - \gamma^{\rho}\gamma^{\mu}\right)\gamma^{\sigma}\gamma^{\lambda}\gamma_{\mu} \\ &= 2\gamma^{\sigma}\gamma^{\lambda}\gamma^{\rho} - \gamma^{\rho}\left(4\eta^{\lambda\sigma} - 2\varepsilon\gamma^{\sigma}\gamma^{\lambda}\right) \\ &= 2\left(\gamma^{\sigma}\gamma^{\lambda} - 2\eta^{\lambda\sigma}\right)\gamma^{\rho} + 2\varepsilon\gamma^{\rho}\gamma^{\sigma}\gamma^{\lambda} \\ &= -2\gamma^{\lambda}\gamma^{\sigma}\gamma^{\rho} + 2\varepsilon\gamma^{\rho}\gamma^{\sigma}\gamma^{\lambda} \, . \end{aligned} \tag{B.18}$$

Combining these relations, one may even resolve “intertwined” contractions like

$$\begin{aligned} \gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\tau}\gamma_{\mu}\gamma^{\delta}\gamma_{\sigma}\gamma^{\lambda} = &- 8\eta^{\rho\delta}\gamma^{\tau}\gamma^{\lambda} \\ &+ 4\varepsilon\left(2\eta^{\rho\tau}\gamma^{\delta}\gamma^{\lambda} + \gamma^{\rho}\gamma^{\delta}\gamma^{\tau}\gamma^{\lambda}\right) \\ &- 4\varepsilon^{2}\gamma^{\rho}\gamma^{\tau}\gamma^{\delta}\gamma^{\lambda} \, . \end{aligned} \tag{B.19}$$

The trace of this expression is

$$\begin{aligned} \operatorname{Tr}\left[\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\tau}\gamma_{\mu}\gamma^{\delta}\gamma_{\sigma}\gamma^{\lambda}\right] = 16\Big[&\left(-2 + \varepsilon\,(1 + \varepsilon)\right)\eta^{\rho\delta}\eta^{\tau\lambda} \\ &+ \varepsilon\,(1 - \varepsilon)\left(\eta^{\rho\tau}\eta^{\delta\lambda} + \eta^{\rho\lambda}\eta^{\delta\tau}\right)\Big] \, . \end{aligned} \tag{B.20}$$

The contraction of a the gamma matrices with a 4-vector is expressed by the so-called Feynman slash:

$$\begin{aligned} \not{a} &= a_{\mu}\gamma^{\mu} \\ &= a^{0}\gamma^{0} - \boldsymbol{a}\cdot\boldsymbol{\gamma} \, . \end{aligned} \tag{B.21}$$

Note that slashed vectors are *matrices*, and do not in general commute with other slashed vectors or gamma matrices.

Slashed vectors may appear in contracted products of gamma matrices like those shown above:

$$\begin{aligned}\gamma^\mu \not{a} \gamma_\mu &= a_\rho \gamma^\mu \gamma^\rho \gamma_\mu \\ &= -2\,(1-\varepsilon)\,\not{a}\ ,\end{aligned} \tag{B.22}$$

$$\gamma^\mu \not{a} \not{b} \gamma_\mu = 4\,(a \cdot b)\,\boldsymbol{I} - 2\varepsilon \not{a} \not{b}\ . \tag{B.23}$$

$$\gamma^\mu \not{a} \not{b} \not{c} \gamma_\mu = -2 \not{c} \not{b} \not{a} + 2\varepsilon \not{a} \not{b} \not{c}\ . \tag{B.24}$$

Products of slashed vectors may be rewritten using the anticommutation relations:

$$\begin{aligned}\not{a} \not{b} + \not{b} \not{a} &= a_\mu b_\nu \left\{\gamma^\mu, \gamma^\nu\right\} \\ &= 2\,(a \cdot b)\,\boldsymbol{I}\ ,\end{aligned} \tag{B.25}$$

$$\not{a} \not{a} = a^2 \boldsymbol{I}\ . \tag{B.26}$$

For real-valued vectors, equation (B.3) may be applied directly to the corresponding slashed vector:

$$\begin{aligned}\not{a}^\dagger &= \gamma^0 \not{a} \gamma^0 \\ &= a^0 \gamma^0 + \boldsymbol{a} \cdot \boldsymbol{\gamma}\ .\end{aligned} \tag{B.27}$$

Combined with equation (B.21), this yields

$$\not{a}^\dagger + \not{a} = 2 a^0 \gamma^0\ . \tag{B.28}$$

B.2 Dirac spinors

The modern form of the Dirac equation is found by substituting the Dirac Hamiltonian (equation (B.4)) into the Schrödinger equation $i\partial_t \psi = H\psi$, using the position space momentum operator $\boldsymbol{p} = -i\boldsymbol{\nabla}$, and multiplying with γ^0 to set the temporal and spatial derivatives on equal footing. The result is

$$(i\not{\partial} - m)\,\psi = 0\ . \tag{B.29}$$

where $\not{\partial} = \gamma^\mu \partial_\mu$, and ψ is a *spinor*, which may be viewed as a complex vector with components ψ_α where α is a spinor index.

A Dirac spinor $u(p)$ describes a free fermion with mass m and 4-momentum $p = (p^0, \boldsymbol{p})$, and is related to a plane-wave solution of the Dirac equation:

$$\psi\,(x) = u\,(p)\,e^{-ip\cdot x}\ . \tag{B.30}$$

Substituting this into equation (B.29) and dividing by $\exp\{-ip \cdot x\}$ gives the Dirac equation in momentum space:

$$(\not{p} - m)\,u\,(p) = 0\ . \tag{B.31}$$

Multiplying with $\not{p} + m$ shows that in order to find a non-trivial solution for $u(p)$, the 4-momentum must satisfy the relativistic energy-momentum relation:

$$(\not{p} + m)\,(\not{p} - m)\,u\,(p) = \left(p^2 - m^2\right) u\,(p) = 0\,, \tag{B.32}$$

which is satisfied if

$$\left(p^0\right)^2 = \boldsymbol{p}^2 + m^2 \equiv E^2\,. \tag{B.33}$$

A 4-momentum satisfying this constraint is written $p = (E, \boldsymbol{p})$.

There is an ambiguity in the definition of $E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$. Hence, for a given momentum $\boldsymbol{p}$ there are two sets of solutions to equation (B.31) – one with $E > 0$ and one with $E < 0$. This is resolved by interpreting a particle with 4-momentum p and energy $E < 0$ as an antiparticle with 4-momentum $-p$. To accomplish this transparently, another set of Dirac spinors is introduced: $v(p) \propto u(-p)$, describing a free antifermion with 4-momentum p . The full set of Dirac spinors thus describes the plane waves

$$\psi\,(x) = \begin{cases} u\,(p)\,e^{-ip\cdot x}\,, \\ v\,(p)\,e^{ip\cdot x}\,, \end{cases} \tag{B.34}$$

now only allowing solutions with $E > 0$. Substituting into equation (B.29) gives the two momentum-space Dirac equations for particle and antiparticle spinors:

$$(\not{p} - m)\,u\,(p) = 0\,, \tag{B.35}$$

$$(\not{p} + m)\,v\,(p) = 0\,. \tag{B.36}$$

In order to discover the properties of these spinors, the first step is to find expressions for them in the case that $\boldsymbol{p} = 0$, such that $E = m$. Equations (B.35) and (B.36) then reduce to

$$\left(\gamma^0 - 1\right) u\,(m, 0) = 0\,, \tag{B.37}$$

$$\left(\gamma^0 + 1\right) v\,(m, 0) = 0\,. \tag{B.38}$$

Hence, these spinors are eigenvectors of γ^0 , with eigenvalue $+1$ for $u(m, 0)$ and -1 for $v(m, 0)$. It has been shown in appendix B.1 that ± 1 are the only eigenvalues of γ^0 , and that it is traceless, hence there must be two linearly independent eigenvectors for each eigenvalue. Let ξ^s and χ^s be two sets of orthonormal eigenvectors, one for each eigenvalue, indexed by $s = \pm 1$:

$$\gamma^0 \xi^s = \xi^s\,, \qquad \left(\xi^s\right)^\dagger \xi^{s'} = \delta^{ss'}\,, \tag{B.39}$$

$$\gamma^0 \chi^s = -\chi^s\,, \qquad \left(\chi^s\right)^\dagger \chi^{s'} = \delta^{ss'}\,. \tag{B.40}$$

Since γ^0 is Hermitian, the eigenspaces are orthogonal:

$$\left(\xi^s\right)^\dagger \chi^{s'} = 0\,. \tag{B.41}$$

The orthogonal decomposition within each eigenspace can be made explicit, but this is not necessary here.

The zero-momentum Dirac spinors must be proportional to these eigenvectors:

$$u^s\,(m,0) \propto \xi^s\,, \tag{B.42}$$

$$v^s\,(m,0) \propto \chi^{-s}\,. \tag{B.43}$$

The minus sign in equation (B.43) is conventional. The index s on u and v is then interpreted as the spin state of the fermions.

For general momenta, it is straightforward to show that the spinors satisfy equations (B.35), (B.36), (B.42) and (B.43) if they are defined as

$$u^s\,(p) \propto (\not{p} + m)\,\xi^s\,, \tag{B.44}$$

$$v^s\,(p) \propto (-\not{p} + m)\,\chi^{-s}\,, \tag{B.45}$$

Normalization

To determine a normalization for the spinors, consider the expression $(\xi^s)^\dagger \not{p} \xi^{s'}$. Since $\gamma^0 \xi^s = \xi^s$, this is equal to $(\xi^s)^\dagger \gamma^0 \not{p} \gamma^0 \xi^{s'} = (\xi^s)^\dagger \not{p}^\dagger \xi^{s'}$. Thus, only the Hermitian part of $\not{p}$, the $E\gamma^0$-term, can contribute. The same is true for χ^s. Hence,

$$\begin{aligned}(\xi^s)^\dagger \not{p} \xi^{s'} &= E\,(\xi^s)^\dagger\,\gamma^0 \xi^{s'} \\ &= E\delta^{ss'}\,,\end{aligned} \tag{B.46}$$

$$\begin{aligned}(\chi^s)^\dagger \not{p} \chi^{s'} &= E\,(\chi^s)^\dagger\,\gamma^0 \chi^{s'} \\ &= -E\delta^{ss'}\,.\end{aligned} \tag{B.47}$$

Equation (B.28) gives $\not{p}^\dagger + \not{p} = 2E\gamma^0$, such that

$$\begin{aligned}(\pm\not{p} + m)^\dagger\,(\pm\not{p} + m) &= \left(2E\gamma^0 - (\not{p} \mp m)\right)(\not{p} \pm m) \\ &= 2E\gamma^0\,(\not{p} \pm m)\;.\end{aligned} \tag{B.48}$$

Combining these results gives

$$\begin{aligned}u^s\,(p)^\dagger\,u^{s'}\,(p) &\propto 2E\,(\xi^s)^\dagger\,\gamma^0\,(\not{p} + m)\,\xi^{s'} \\ &= 2E\,(E + m)\,\delta^{ss'}\,,\end{aligned} \tag{B.49}$$

$$\begin{aligned}v^s\,(p)^\dagger\,v^{s'}\,(p) &\propto 2E\,(\chi^{-s})^\dagger\,\gamma^0\,(\not{p} - m)\,\chi^{-s'} \\ &= 2E\,(E + m)\,\delta^{ss'}\,.\end{aligned} \tag{B.50}$$

The spinors are then defined as

$$u^s(p) = \frac{(\not{p} + m)}{\sqrt{E+m}}\xi^s , \tag{B.51}$$

$$v^s(p) = \frac{(-\not{p} + m)}{\sqrt{E+m}}\chi^{-s} , \tag{B.52}$$

such that the normalization reads

$$u^s(p)^\dagger u^{s'}(p) = 2E\delta^{ss'} , \tag{B.53}$$

$$v^s(p)^\dagger v^{s'}(p) = 2E\delta^{ss'} . \tag{B.54}$$

This also shows that different spin states are orthogonal for any momentum, when the spinor is treated as a complex vector.

The Dirac adjoint

Frequently when working with Dirac spinors, the Dirac adjoint is used rather than the Hermitian adjoint. The Dirac adjoint of a spinor ψ is defined as

$$\overline{\psi} = \psi^\dagger\gamma^0 . \tag{B.55}$$

The rationale for introducing the Dirac adjoint is that it can be used to construct quantities that are well-behaved under Lorentz transformations. Examples are the inner product $\overline{\varphi}\psi$, which is a Lorentz scalar, and the current $\overline{\varphi}\gamma^\mu\psi$, which is a Lorentz 4-vector. Corresponding quantities defined using $\varphi^\dagger$ do not behave as nicely. This is obviously a consequence of the transformation properties of spinors under Lorentz transformations, but these properties will not be discussed here.

Many useful quantities can be defined by sandwiching a matrix expression between a spinor and a Dirac adjoint, in general expressed as $\overline{\varphi}\,\Gamma\psi$ for spinors ψ and φ and a matrix Γ . When squaring scattering amplitudes it is often necessary to find the complex conjugate of such expressions. This can be expressed as

$$\begin{aligned}
[\overline{\varphi}\,\Gamma\psi]^* &= \left[\varphi^\dagger\gamma^0\Gamma\psi\right]^\dagger \\
&= \psi^\dagger\Gamma^\dagger\gamma^0\varphi \\
&= \psi^\dagger\gamma^0\gamma^0\Gamma^\dagger\gamma^0\varphi \\
&= \overline{\psi}\,\overline{\Gamma}\varphi ,
\end{aligned} \tag{B.56}$$

where $\overline{\Gamma}$ is defined for any matrix as

$$\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 . \tag{B.57}$$

If Γ can be expressed as a product of gamma matrices, $\Gamma = \gamma^\mu \cdots \gamma^\nu$, one finds

$$\begin{aligned}\overline{\Gamma} &= \gamma^0 \left(\gamma^\nu\right)^\dagger \cdots \left(\gamma^\mu\right)^\dagger \gamma^0 \\ &= \gamma^0 \left(\gamma^\nu\right)^\dagger \gamma^0\gamma^0 \cdots \gamma^0\gamma^0 \left(\gamma^\mu\right)^\dagger \gamma^0 \\ &= \gamma^\nu \cdots \gamma^\mu \,.\end{aligned} \tag{B.58}$$

This leads to the following convenient identity:

$$\left[\overline{\varphi}\gamma^\mu \cdots \gamma^\nu \psi\right]^* = \overline{\psi}\gamma^\nu \cdots \gamma^\mu \varphi \,. \tag{B.59}$$

It is also useful to find the adjoint of the Dirac equations in momentum space, equations (B.35) and (B.36), which is straightforward using equation (B.27):

$$\overline{u}\,(p)\,(\not{p} - m) = 0 \,, \tag{B.60}$$

$$\overline{v}\,(p)\,(\not{p} + m) = 0 \,. \tag{B.61}$$

Inner products

As mentioned in the previous section, one can define a Lorentz-invariant inner product of two spinors ψ and φ as $\overline{\varphi}\psi = (\overline{\psi}\varphi)^*$. This is similar to the usual inner product on a complex vector space, but uses the Dirac adjoint in place of the Hermitian adjoint.

To find the various inner products of $u^s(p)$ and $v^s(p)$, consider the expression

$$\begin{aligned}\left(\pm\not{p} + m\right)^\dagger \gamma^0 \left(\pm\not{p} + m\right) &= \gamma^0 \left(\not{p} \pm m\right) \left(\not{p} \pm m\right) \\ &= \gamma^0 \left(p^2 \pm 2m\not{p} + m^2\right) \\ &= 2m\gamma^0 \left(\pm\not{p} + m\right) \,,\end{aligned} \tag{B.62}$$

Comparing with equation (B.48), it is clear that replacing $u^s(p)^\dagger$ and $v^s(p)^\dagger$ with $\overline{u}^s(p)$ and $\overline{v}^s(p)$ in equations (B.53) and (B.54) amounts to replacing a factor of E with a factor of m and inserting a minus sign for the v spinors:

$$\overline{u}^s\,(p)\,u^{s'}\,(p) = 2m\delta^{ss'} \,, \tag{B.63}$$

$$\overline{v}^s\,(p)\,v^{s'}\,(p) = -2m\delta^{ss'} \,. \tag{B.64}$$

Thus, different spin states are orthogonal also with respect to the spinor inner product. Particle and antiparticle states are also orthogonal, since

$$\left(\pm\not{p} + m\right)^\dagger \gamma^0 \left(\mp\not{p} + m\right) = \gamma^0 \left(\not{p} \pm m\right) \left(\not{p} \mp m\right) = 0 \,. \tag{B.65}$$

such that

$$\overline{v}^s\,(p)\,u^{s'}\,(p) = \overline{u}^s\,(p)\,v^{s'}\,(p) = 0 \,. \tag{B.66}$$

Completeness relations

Consider the matrices

$$P_u(p) = \sum_s u^s(p)\,\overline{u}^s(p) \;, \tag{B.67}$$

$$P_v(p) = \sum_s v^s(p)\,\overline{v}^s(p) \;. \tag{B.68}$$

The four u- and v-spinors for a given momentum p form a complete basis for the spinor space. A general spinor can therefore be written

$$\psi = \sum_r a_r u^r(p) + \sum_t b_t v^t(p) \;. \tag{B.69}$$

Applying $P_u(p)$ to ψ and using equations (B.63) and (B.64) gives

$$\begin{aligned} &P_u(p)\left(\sum_r a_r u^r(p) + \sum_t b_t v^t(p)\right) \\ &= 2m\sum_s u^s(p)\left(\sum_r a_r \delta^{sr} + \sum_t b_t \cdot 0\right) \\ &= 2m\left(\sum_r a_r u^r(p)\right) \;. \end{aligned} \tag{B.70}$$

Hence, $P_u(p)/2m$ must be the projection operator into the subspace spanned by $u^s(p)$. Similarly, $-P_v(p)/2m$ is the projection operator into the subspace spanned by $v^s(p)$. Comparing with equations (B.35) and (B.36), it is therefore clear that

$$\sum_s u^s(p)\,\overline{u}^s(p) = \not{p} + m \;, \tag{B.71}$$

$$\sum_s v^s(p)\,\overline{v}^s(p) = \not{p} - m \;. \tag{B.72}$$

These are the completeness relations for Dirac spinors.

C

Dimensional regularization

To regularize divergent integrals in quantum field theory, an approach that is often used is to compute the integrals in an arbitrary d-dimensional spacetime with 1 timelike dimension and $d-1$ spacelike dimensions. Then, setting $d = 4 - 2\varepsilon$, the divergences of the integrals in 4-dimensional spacetime manifest as poles when $\varepsilon \to 0$.

Certain quantities need to be reinterpreted in order to make sense when generalizing the number of dimensions. Most obviously, the contraction of the metric tensor in Minkowski space with itself becomes

$$\eta^{\mu\nu}\eta_{\mu\nu} = \eta^{\mu}_{\mu} = d = 4 - 2\varepsilon \,, \tag{C.1}$$

instead of the usual $\eta^{\mu}_{\mu} = 4$.

In addition, measures must be taken to ensure consistency in the dimension of the Lagrangian and the action. The action is the integral of the Lagrangian over all spacetime:

$$S = \int d^d x \mathcal{L} \,, \tag{C.2}$$

and it should have the same dimensions as Planck's constant, which is set to $\hbar = 1$ in this text. Since the dimension of $d^d x$ is obviously $[d^d x] = L^d$, where L denotes a length dimension, the dimension of $\mathcal{L}$ must be $[\mathcal{L}] = L^{-d} = M^d$. Here, M is a mass dimension, and the relation $M = L^{-1}$ is a consequence of setting $\hbar = c = 1$.

From this one can deduce the dimensions of the fields and coupling constants in equations (A.3) and (A.8). In partuclar, one finds that $[g_s] = [e] = M^{2-(d/2)} = M^{\varepsilon}$. However, being physical parameters, the coupling constants should be defined independently of the spacetime dimension. An arbitrary mass scale μ is therefore introduced, and when doing dimensional regularization one performs the replacement

$$\begin{aligned} e &\to \mu^{\varepsilon} e \,, \\ g_s &\to \mu^{\varepsilon} g_s \,, \end{aligned} \tag{C.3}$$

leaving the values of e and g_s unchanged.

In d dimensions, d-dimensional integration measures such as $d^d p/(2\pi)^d$ must be used in place of the usual 4-dimensional ones. For integrals of the form

$$\int \frac{d^d p}{(2\pi)^d} \frac{f(p)}{\left(p^2 - A + i\varepsilon\right)^n} = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \int \frac{d^{d-1}\boldsymbol{p}}{(2\pi)^{d-1}} \frac{f(E, \boldsymbol{p})}{\left(E^2 - \boldsymbol{p}^2 - A + i\varepsilon\right)^n} , \quad \text{(C.4)}$$

it is often beneficial to avoid the poles by rotating from Minkowski to Euclidean space, defining the Euclidean vector $p_E = (p_{0,E}, \boldsymbol{p})$, where $p_{0,E} = -iE$ and E is the time-component of the 4-vector $p = (E, \boldsymbol{p})$. Thus, $p_E^2 \equiv p_{0,E}^2 + \boldsymbol{p}^2 = -p^2$. The term $i\varepsilon$ in equation (C.4) entails an analytic continuation of the E-integral into the complex plane to avoid the poles at $E = \pm\sqrt{\boldsymbol{p}^2 + A}$, and a closed integration contour with no poles inside is obtained by going *up* the real axis, *down* the imaginary axis, and closing the contour in the first and third quadrants (assuming that $f(E, \boldsymbol{p})$ has no poles in the interior of this curve). Thus,

$$\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{f(E, \boldsymbol{p})}{\left(E^2 - \boldsymbol{p}^2 - A + i\varepsilon\right)^n} - \int_{-i\infty}^{+i\infty} \frac{dE}{2\pi} \frac{f(E, \boldsymbol{p})}{\left(E^2 - \boldsymbol{p}^2 - A + i\varepsilon\right)^n} = 0 . \quad \text{(C.5)}$$

Making the substitution $p_{0,E} = -iE$ gives

$$\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{f(E, \boldsymbol{p})}{\left(E^2 - \boldsymbol{p}^2 - A + i\varepsilon\right)^n} = i \int_{-\infty}^{+\infty} \frac{dp_{0,E}}{2\pi} \frac{f\left(ip_{0,E}, \boldsymbol{p}\right)}{\left(-p_{0,E}^2 - \boldsymbol{p}^2 - A + i\varepsilon\right)^n} . \quad \text{(C.6)}$$

Hence,

$$\int \frac{d^d p}{(2\pi)^d} \frac{f(E, \boldsymbol{p})}{\left(p^2 - A + i\varepsilon\right)^n} = i \int \frac{d^d p_E}{(2\pi)^d} \frac{f\left(ip_{0,E}, \boldsymbol{p}\right)}{\left(-p_E^2 - A + i\varepsilon\right)^n} . \quad \text{(C.7)}$$

Note that the $i\varepsilon$-term is no longer significant if $A > 0$. On the other hand, if $A < 0$ the rotation is not very helpful, since poles remain in the integrand. The solution in this case is to rotate the spatial axes $\boldsymbol{p}$ in their respective complex planes instead, defining the euclidean vector $p_E = (E, \boldsymbol{p}_E)$ with $\boldsymbol{p}_E = -i\boldsymbol{p}$, such that $p_E^2 = p^2$. The poles lie on opposite sides of the axes in the p_i-plane compared to the E-plane, so the integration contour must now go *up* the imaginary axis and close in the second and fourth quadrants. The equivalent of equation (C.6) therefore acquires a negative sign, so the rotation introduces a factor of $-i$ per axis. The final result is

$$\int \frac{d^d p}{(2\pi)^d} \frac{f(E, \boldsymbol{p})}{\left(p^2 - A + i\varepsilon\right)^n} = (-i)^{d-1} \int \frac{d^d p_E}{(2\pi)^d} \frac{f(E, i\boldsymbol{p}_E)}{\left(p_E^2 - A + i\varepsilon\right)^n} . \quad \text{(C.8)}$$

Here the $i\varepsilon$-term is insignificant as long as $A < 0$.

The main benefit from rotating to Euclidean space is that the integrands are often isotropic there, enabling further simplification:

$$\int \frac{d^d p_E}{(2\pi)^d} f\left(p_E^2\right) = \frac{\Omega_d}{(2\pi)^d} \int_0^\infty d\tilde{p}_E\, \tilde{p}_E^{d-1} f\left(\tilde{p}_E^2\right) \tag{C.9}$$

where $\tilde{p}_E = \sqrt{p_E^2}$ is the Euclidean length of p_E , and Ω_d is the volume of the unit sphere S^{d-1} in d dimensions.[1] The value of Ω_d is stated here without derivation:

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \,. \tag{C.10}$$

A proof can be found in e.g. [22].

C.1 Important identities

A number of mathematical identities come in handy when working in d dimensions. Some of the most important are given here without proof.

The Euler Γ- and β-functions

The Euler Γ-function is defined as

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \,, \quad \Re\{z\} > 0 \,, \tag{C.11}$$

and extended to the whole complex plane by analytic continuation. It satisfies the following identities:

$$\Gamma(z+1) = z\Gamma(z) \,, \tag{C.12}$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) . \tag{C.13}$$

For integer n ,

$$\Gamma(n) = (n-1)! \,. \tag{C.14}$$

The first few terms in series expansion of $\Gamma(z)$ around $z = 1$ are

$$\Gamma(1+\varepsilon) = 1 - \gamma_E \varepsilon + \left(\frac{\pi^2}{12} + \frac{1}{2}\gamma_E^2\right) \varepsilon + \mathcal{O}\left(\varepsilon^2\right) \,, \tag{C.15}$$

where $\gamma_E = 0.577216\ldots$ is the Euler-Mascheroni constant.

The Euler β-function is defined as

$$\begin{aligned} \beta(x,y) &= \int_0^1 t^{x-1} (1-t)^{y-1} dt \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \,, \end{aligned} \qquad \Re\{x\}, \Re\{y\} > 0 \,. \tag{C.16}$$

[1] Note that the unit sphere is the $(d-1)$-dimensional surface of the unit ball. Also, note that *volume* is used in a general sense here. For example, $\Omega_3 = 4\pi$ is what would normally be called the *surface area* of the unit sphere S^2 .

Feynman parameter integrals

The general Feynman parameter integral can be written [1, p. 429]

$$\frac{1}{A_1^{n_1}\cdots A_k^{n_k}} = \frac{\Gamma\left(n_1+\cdots+n_k\right)}{\Gamma\left(n_1\right)\cdots\Gamma\left(n_k\right)}\int_0^1 d\alpha_1\cdots d\alpha_k \frac{\alpha_1^{n_1-1}\cdots\alpha_k^{n_k-1}\delta\left(1-\sum_i\alpha_i\right)}{\left(\alpha_1 A_1+\cdots+\alpha_k A_k\right)^{n_1+\cdots+n_k}}. \tag{C.17}$$

The special cases used in the main text are those with two or three factors in the denominator:

$$\frac{1}{AB} = \int_0^1 \frac{d\alpha}{\left(\alpha A+(1-\alpha)\,B\right)^2}\,, \tag{C.18}$$

$$\frac{1}{ABC} = \int_0^1 d\alpha\int_0^{1-\alpha} d\beta\frac{2}{\left(\alpha A+\beta B+(1-\alpha-\beta)\,C\right)^3}\,. \tag{C.19}$$

C.2 Phase space in d dimensions

The differential phase space of n outgoing particles in the usual 4 spacetime dimensions, with 4-momenta $p_i,\ i=1,\ldots,N$, is

$$d\Phi_n = (2\pi)^4\delta^{(4)}\left(Q-\sum_{i=1}^N p_i\right)\prod_{i=1}^N\frac{d^4p_i}{(2\pi)^4}\,2\pi\delta\left({p_i}^2-m_i^2\right)\Theta\left(p_i^0\right)\,. \tag{C.20}$$

Here, Q is the total 4-momentum and Θ is the Heaviside step function.

This expression is readily generalized to d spacetime dimensions:

$$d\Phi_n^{(d)} = (2\pi)^d\delta^{(d)}\left(Q-\sum_{i=1}^N p_i\right)\prod_{i=1}^N\frac{d^dp_i}{(2\pi)^d}\,2\pi\delta\left({p_i}^2-m_i^2\right)\Theta\left(p_i^0\right)\,. \tag{C.21}$$

Using the relation

$$\delta(g(x)) = \sum_i\frac{\delta(x-x_i)}{|g'(x_i)|}\,, \tag{C.22}$$

where the x_i are the roots of $g(x)$, gives

$$\delta\left({p_i}^2-m_i^2\right)\Theta\left(p_i^0\right) = \frac{\delta\left(p_i^0-E_i\left(\boldsymbol{p}_i^2\right)\right)}{2E_i\left(\boldsymbol{p}_i^2\right)}\,, \tag{C.23}$$

with

$$E_i\left(\boldsymbol{p}^2\right) = \sqrt{\boldsymbol{p}^2+m_i^2}\,. \tag{C.24}$$

Thus, upon substituting $p_i^0 = E_i(\boldsymbol{p}_i^2)$ in the integrand, the following is an equivalent phase space measure:

$$d\Phi_n^{(d)} = (2\pi)^d\delta^{(d)}\left(Q-\sum_{i=1}^N p_i\right)\prod_{i=1}^N\frac{d^{d-1}\boldsymbol{p}_i}{(2\pi)^{d-1}2E_i\left(\boldsymbol{p}_i^2\right)}\,. \tag{C.25}$$

2 particles

In the case of two outgoing particles, the phase space integral of a function $f(\boldsymbol{p}_1, \boldsymbol{p}_2)$ can be written

$$\begin{aligned} I &= \int \frac{d^{d-1}\boldsymbol{p}_1 d^{d-1}\boldsymbol{p}_2}{4(2\pi)^{d-2} E_1\left(\boldsymbol{p}_1^2\right) E_2\left(\boldsymbol{p}_2^2\right)} \\ &\quad \times \delta\left(Q^0 - E_1\left(\boldsymbol{p}_1^2\right) - E_2\left(\boldsymbol{p}_2^2\right)\right) \delta^{(d-1)}\left(\boldsymbol{Q} - \boldsymbol{p}_1 - \boldsymbol{p}_2\right) f\left(\boldsymbol{p}_1, \boldsymbol{p}_2\right) \\ &= \int \frac{d^{d-1}\boldsymbol{p}_1}{4(2\pi)^{d-2} E_1\left(\boldsymbol{p}_1^2\right) E_2\left((\boldsymbol{Q} - \boldsymbol{p}_1)^2\right)} \\ &\quad \times \delta\left(Q^0 - E_1\left(\boldsymbol{p}_1^2\right) - E_2\left((\boldsymbol{Q} - \boldsymbol{p}_1)^2\right)\right) f\left(\boldsymbol{p}_1, \boldsymbol{Q} - \boldsymbol{p}_1\right) . \end{aligned} \tag{C.26}$$

In the COM frame, $\boldsymbol{Q} = 0$ and $Q^0 = \sqrt{Q^2} = \sqrt{s}$. To proceed, assumptions must be made on the form of $f(\boldsymbol{p}_1, -\boldsymbol{p}_1)$. In general, even a scalar integrand such as a matrix element may have a directional dependence on $\boldsymbol{p}_1$ through its dependence on other vectors, e.g. the momenta of incoming particles. If it is isotropic, however, one may define $\tilde{p} = \sqrt{\boldsymbol{p}_1^2}$ and write, using equation (C.9),

$$I = \int_0^\infty \frac{\Omega_{d-1} \tilde{p}^{d-2} d\tilde{p}}{4(2\pi)^{d-2} E_1\left(\tilde{p}^2\right) E_2\left(\tilde{p}^2\right)} \delta\left(\sqrt{s} - E_1\left(\tilde{p}^2\right) - E_2\left(\tilde{p}^2\right)\right) f\left(\tilde{p}\right) . \tag{C.27}$$

Let $g(\tilde{p})$ be the argument of the δ-function, such that

$$g(\tilde{p}) = \sqrt{s} - E_1\left(\tilde{p}^2\right) - E_2\left(\tilde{p}^2\right) , \tag{C.28}$$

$$\begin{aligned} g'(\tilde{p}) &= -\frac{\tilde{p}}{E_1\left(\tilde{p}^2\right)} - \frac{\tilde{p}}{E_2\left(\tilde{p}^2\right)} \\ &= -\tilde{p}\left(\frac{E_1\left(\tilde{p}^2\right) + E_2\left(\tilde{p}^2\right)}{E_1\left(\tilde{p}^2\right) E_2\left(\tilde{p}^2\right)}\right) . \end{aligned} \tag{C.29}$$

The solutions to $g(\tilde{p}) = 0$ are

$$\tilde{p}_\pm = \pm \frac{1}{2\sqrt{s}} \sqrt{s^2 + 2s(m_1 + m_2)^2 + (m_1 - m_2)^2} . \tag{C.30}$$

Only the positive solution contributes in the integral, which becomes

$$I = \frac{\Omega_{d-1} \tilde{p}_+^{d-3}}{4(2\pi)^{d-2}\sqrt{s}} f\left(\tilde{p}_+\right) . \tag{C.31}$$

Substituting $d = 4 - 2\varepsilon$, the spherical volume can be written

$$\Omega_{3-2\varepsilon} = \frac{2\pi^{(3/2)-\varepsilon}}{\Gamma\left((3/2) - \varepsilon\right)} . \tag{C.32}$$

Using equations (C.12) and (C.13), one can rewrite

$$\begin{aligned} \Gamma\left(\frac{3}{2}-\varepsilon\right) &= 2^{2(\varepsilon-1)}\sqrt{\pi}\,\frac{\Gamma\,(3-2\varepsilon)}{\Gamma\,(2-\varepsilon)} \\ &= 2^{2\varepsilon-1}\sqrt{\pi}\,\frac{\Gamma\,(2-2\varepsilon)}{\Gamma\,(1-\varepsilon)}\,. \end{aligned} \tag{C.33}$$

Thus,

$$I = \frac{1}{4\pi}\left(\frac{\pi}{\tilde{p}^2}\right)^{\varepsilon}\frac{\tilde{p}}{\sqrt{s}}\,\frac{\Gamma\,(1-\varepsilon)}{\Gamma\,(2-2\varepsilon)}\,f\,(\tilde{p})\ , \tag{C.34}$$

where the subscript $+$ is dropped and $\tilde{p}$ is understood to be subject to the constraints from 4-momentum conservation.

From this expression the fully integrated two-particle phase space factor in $d = 4-2\varepsilon$ dimensions can be isolated. It is valid in the COM frame for integrands that are isotropic in the momentum, and reads:

$$\Phi_2^{(d)} = \frac{1}{4\pi}\left(\frac{\pi}{\tilde{p}^2}\right)^{\varepsilon}\frac{\tilde{p}}{\sqrt{s}}\,\frac{\Gamma\,(1-\varepsilon)}{\Gamma\,(2-2\varepsilon)}\,. \tag{C.35}$$

3 particles

In the case of three outgoing particles, it is most convenient to give the phase space integral in terms of the energy fractions in the COM frame, defined as

$$x_i = \frac{2p_i\cdot Q}{Q^2}\,. \tag{C.36}$$

The phase space integral is not derived here, but cited from [1, p. 434]. The result is

$$\begin{aligned} \int d\Phi_3^{(d)}\,f\,(p_1,p_2,p_3) &= \frac{Q^2}{2\,(4\pi)^3}\left(\frac{4\pi}{Q^2}\right)^{2\varepsilon}\frac{1}{\Gamma\,(2-2\varepsilon)} \\ &\quad\times\int_0^1 dx_1\int_{1-x_1}^1 dx_2\,\frac{f\,(x_1,x_2,x_3)}{\left[(1-x_1)\,(1-x_2)\,(1-x_3)\right]^{\varepsilon}} \end{aligned} \tag{C.37}$$

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