2 SNe, SNRs, accretion, BHs

2.1 Core collapse supernovae

Iron burning In the last possible fusion reaction, ab α particle is added to ^{24}Cr. Inserting into Eq. (1.65) for the reduced mass $\mu = 4\times 52/(4+52)m_N$ and $Z_1^2 Z_2^2 = (2\times 24)^2$, we find that iron burning requires temperatures in the MeV range. Thus electrons are relativistic, and photons can produce electron-positron pairs that in turn can produce neutrinos, $\gamma\gamma \leftrightarrow e^+e^- \to \bar{\nu}\nu$. The latter can escape from the core, leading to efficient cooling to $T \sim m_e \sim 0.5\,\text{MeV}$.

SNe II Type II or core collapse supernovae occur at the end of the fusion process in very massive stars, $M \gtrsim (5-8)M_\odot$. Theses stars develop an onion-like structure with a degenerate Fe core. After the core is completely fused to iron, no further processes releasing energy are possible. Instead, photo-disintegration destroys the heavy nuclei, e.g. via $\gamma+{}^{56}\text{Fe} \to {}^4\text{He}+4n$, and removes the thermal energy necessary to provide pressure support. In the following collapse of the star, the density increases and the free electrons are forced together with protons to form neutrons via inverse beta decay, $e^- + p \to n + \nu_e$: A proto-neutron star forms. When the core density reaches nuclear density, the equation of state of nuclear matter stiffens and infalling material is "reflected," a shock wave propagates outwards heated by neutrino emission from the proto-neutron star. If the SN is successful, a neutron star is left over; otherwise a black hole remains.

The released gravitational binding energy,

$$\Delta E = \left[-\frac{-GM^2}{R}\right]_{\text{star}} - \left[-\frac{-GM^2}{R}\right]_{\text{NS}} \sim 5\times 10^{53}\text{erg}\left(\frac{10\text{km}}{\text{R}}\right)\left(\frac{\text{M}_{\text{NS}}}{1.4\text{M}_\odot}\right) \tag{2.1}$$

is emitted mainly via neutrinos (99%). Only 1% is transferred into kinetic energy of the exploding star and only 0.01% goes into photons.

Ex.: Neutrinos from type II supernova.
The proto-neutron formed during the core collapse of a massive star emits copiously neutrinos. Its mass is $\approx 1.4M_\odot$, and its radius $\approx$ 15 km. Estimate the total (gravitational potential) energy E_b released. Apply the virial theorem to a nucleon N at the surface of the proto-neutron star and estimate its kinetic energy E_N. Estimate the number N_ν of neutrinos emitted and the duration of the neutrino signal (random walk) using $E_\nu = E_N$, $E_b = N_\nu E_\nu$ and $\sigma_\nu = 10^{-43}\text{cm}^2(E_\nu/\text{MeV})^2$. For the case of SN1987A in the Large Magellanic Cloud at a distance of 50 kpc, how many neutrinos were observed (using the same σ_ν) in a detector with 10^{32} protons?
The gravitational energy released by the collapse is $E_b \approx 3GM^2/(5R) \approx 2.1\times 10^{53}$ erg. The mean kinetic energy E_N of a nucleon is $E_N = E_{\text{pot}}/2 \approx Gm_N M/(2R) \approx 1\times 10^{-4}$ erg or 64 MeV. The number of emitted neutrinos follows as $N_\nu = E_N/E_b \approx 2\times 10^{57}$.
The number of steps in a random walk with step size ℓ_{int} needed to reach the distance R is $N = R^2/\ell_{\text{int}}^2$. Hence the duration τ of the neutrino signal is $\tau \approx N\ell_{\text{int}}/c = R^2/(c\ell_{\text{int}})$. The step size ℓ_{int} is found as

$\ell_{\rm int} = 1/(n\sigma) \approx 1$ m for $E_\nu = 30$ MeV and $n \approx 10^{-38}/{\rm cm}^3$. Thus $\tau \approx 1$ s.
The neutrino flux at Earth is $\phi_\nu = N_\nu/(4\pi D^2)$ with $D = 50$ kpc. The event number $N_{\rm ev}$ in a detector with N_p targets each with cross section σ is $N_{\rm ev} = N_p\sigma\phi_\nu \approx 54$. However, from all 2×3 neutrino types, only $\bar\nu_e$ have the quoted large cross section with protons. Thus $N_{\rm ev}(\bar\nu_e) \approx N_{\rm ev}/6$. Our simplified picture agrees roughly with reality: From SN1987A were 11-12 and 8 neutrino events with energies $20-40$ MeV during ~ 10 s observed in the two water Cherenkov detectors operating at that time.

Cold ideal npe gas (Inverse) beta-decay leads an equilibrium distribution of neutrons, protons, and electrons in a (proto-) neutron star. The chemical potential are connected by $\mu_e + \mu_p = \mu_n + \mu_\nu$ with $\mu_n u = 0$, since neutrinos can escape freely and thus their number density is negligible.

For a cold degenerate gas, we can neglect the temperature and the Fermi-Dirac distribution function becomes a step function. Thus all levels up-to the Fermi momentum p_F are filled, and the number density of species with two spin degrees of freedom is

$$n = \frac{2}{h^3}\int_0^{p_F} 4\pi p^2 {\rm d}p = \frac{8\pi}{3h^3}\,p_F^3 = \frac{1}{3\pi^2\lambda^3}\,x^3 \tag{2.2}$$

with $x \equiv p_F/(mc)$ and $\lambda = \hbar/(mc)$. Since the Fermi energy equals the chemical potential, we have

$$m_e(1+x_e^2)^{1/2} + m_p(1+x_p^2)^{1/2} = m_n(1+x_n^2)^{1/2}\,. \tag{2.3}$$

Charge neutrality implies that $n_e = n_p$ or $m_e x_e = m_p x_p$. Thus we can eliminate x_e and find the ratio n_p/n_n. Squaring twice (2.3) and using $Q, m_e \ll m_n$, we obtain

$$\frac{n_p}{n_n} = \left(\frac{m_p x_p}{m_n x_n}\right)^3 \approx \frac{1}{8}\left\{\frac{1+\frac{4Q}{m_n x_n^2} + 4\frac{Q^2-m_e^2}{m_n^2 x_n^4}}{1+1/x_n^2}\right\}^{3/2} . \tag{2.4}$$

Ex.: a.) Derive the limiting value $n_p/n_n = 1/8$ directly. b) What happens in the opposite limit, $x_n \to 0$?
a) For large x_n, all three species are relativistic and Eq. (2.3) becomes $m_e x_e + m_p x_p = m_n x_n$. Charge neutrality implies then immediately that

$$n_p/n_n = \left(\frac{m_p x_p}{m_n x_n}\right)^3 = (1/2)^3 = 1/8\,.$$

b) The limit $x_n \to 0$ corresponds to a dilute, classical gas. Thus Eq. 2.2 should be replaced by the corresponding expression for a Boltzmann gas and one obtains a Saha-like expression.

2.2 Supernova remnants

A supernova explosions acts as a point-like injection of energy E in in the interstellar medium (ISM) . We assume that the ISM has a constant density ρ. The interstellar medium has low temperature, so we can neglect the gas pressure. Thus the problem is fully described by two parameters, E and ρ. Using only these two quantities, one cannot form a combination $E^\alpha\rho^\beta$ with dimension of a length or time. This means that the problem does not contain a typical time- or length-scale[1] and the solution has to be thus self-similar.

[1]Contrast this to stellar physics, where M, R and G can be combined e.g. to $\tau_{\rm ff}$.

If $R_{\rm sh}$ denotes the position of the shock, then dimensional analysis shows that

$$R_{\rm sh}(t) = \alpha \left(\frac{Et^2}{\rho}\right)^{1/5}, \tag{2.5}$$

where α is dimensionless constant. The shock velocity follows differentiating $R_{\rm sh}$ as

$$v_{\rm sh}(t) = \dot{R}_{\rm sh}(t) = \frac{2\alpha}{5}\left(\frac{E}{\rho t^3}\right)^{1/5} = \frac{2}{5}\frac{R_{\rm sh}}{t}\,. \tag{2.6}$$

2.3 Accretion

Bondi accretion We consider the spherically symmetric accretion of gas in the gravitational field of a mass M. The continuity equation becomes then in the stationary case

$$\nabla\cdot(\rho\mathbf{v}) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\rho v) = 0 \tag{2.7}$$

with $\mathbf{v} = v\mathbf{e}_r$. Thus $r^2\rho u = $ const. or the mass flow $\dot{M}$ through a shell at radius r is independent of r,

$$\dot{M} = 4\pi r^2\rho v\,. \tag{2.8}$$

The Euler equation expresses the momentum change of a fluid element due to the pressure gradient and the gravitational force,

$$v\,\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2}\,. \tag{2.9}$$

We want to combine these two equations using the definition of the sound speed $v_s^2 = \mathrm{d}P/\mathrm{d}\rho$. We rewrite the first term on the RHS as

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = v_s^2\,\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\,. \tag{2.10}$$

Rewriting also $v' = \mathrm{d}v/\mathrm{d}r$ as a logarithmic derivatives, Eq. (2.9) becomes

$$v^2\,\frac{\mathrm{d}\ln v}{\mathrm{d}r} = -v_s^2\,\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{GM}{r^2}\,. \tag{2.11}$$

Next we perform the differentiation in $(r^2\rho u)' = 0$

$$\frac{2}{r} + \frac{\rho'}{\rho} + \frac{v'}{v} = 0 \tag{2.12}$$

or

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} = -\frac{2}{r} - \frac{\mathrm{d}\ln v}{\mathrm{d}r}\,. \tag{2.13}$$

Now we can eliminate $\mathrm{d}\ln\rho/\mathrm{d}r$ and obtain

$$v^2\,\frac{\mathrm{d}\ln v}{\mathrm{d}r} = -v_s^2\left(\frac{2}{r} - \frac{\mathrm{d}\ln v}{\mathrm{d}r}\right) - \frac{GM}{r^2} \tag{2.14}$$

or

$$(v^2 - v_s^2)\,\frac{\mathrm{d}\ln v}{\mathrm{d}r} = \frac{2v_s^2}{r}\left(1 - \frac{GM}{2v_s^2 r}\right) . \tag{2.15}$$

It is clear that the radius ("the sonic point")

$$r_s = \frac{GM}{2v_s^2} \tag{2.16}$$

has a special meaning: Since at $r = r_s$ the RHS vanishes, the velocity v becomes at $r = r_s$ either equal to the sound speed or it has there a maximum. Physically, we have to require that v is monotonically increasing[2] for decreasing r and thus the former is true. Choosing by this requirement the right boundary conditions for our equations selects the allowed value of $\dot{M}$, similar as only certain energies E are allowed in QM.

For a given temperature of the infalling gas, we know the sound speed v_s and thus r_s. Then the mass inflow $\dot{M}$ is determined via Eq. (2.8). However, in general we want to connect $\dot{M}$ to the measured density at large distances, ρ_∞. We obtain the relation between ρ_∞ and ρ_s integrating (2.11),

$$\left.\frac{v^2}{2}\right| = -v_s^2\ \ln\rho| - \left.\frac{GM}{r}\right| \tag{2.17}$$

or

$$v^2 = 2v_s^2\left[\ln\frac{\rho_s}{\rho} - \frac{3}{2}\right] + \frac{2GM}{r} . \tag{2.18}$$

For $r \to 0$, the velocity is given by $v^2 \to 2GM/r$, i.e. the gas is in free fall. From the opposite limit, $r \to \infty$, we find with $v \to 0$ $\ln\rho_s/\rho_\infty = 3/2$ or $\rho_s = \rho_\infty \mathrm{e}^{3/2} \approx 4.48\rho_\infty$.

Ex.: Find the mass collected by a star with $M = M_\odot$ in a hydrogen cloud with temperature $T = 200\,\mathrm{K}$ and density $n_H = 1\,\mathrm{cm}^{-3}$.
With $r_s = GM_\odot/2v_s^2 \approx 4\times 10^{11}$ cm we obtain

$$\dot{M} = 4\pi r_s^2 \rho_s v_s \sim 3\times 10^{21}\mathrm{g/yr} .$$

Thus such a star accretes 1% of its mass during the age of the universe.

Accretion disks Accretion disks are naturally formed when angular momentum conservation forces infalling gas into a two-dimensional pancake-like structure. If such a disk is dense enough, viscous processes can consume angular momentum and heat the disk up.

Consider a mass element $\mathrm{d}M$ falling from a Keplerian orbit at $r + \mathrm{d}r$ to r due to viscous interactions. Half of the gain in potential energy is used to increase the kinetic energy of the mass element, while the other half is radiated away and heats up the environment,

$$\mathrm{d}E_{\mathrm{heat}} = \frac{1}{2}\left(\frac{GM\mathrm{d}M}{r} - \frac{GM\mathrm{d}M}{r+\mathrm{d}r}\right) . \tag{2.19}$$

The luminosity of the accretion disk emitted by the shell at radius r is thus

$$L = \frac{\mathrm{d}E_{\mathrm{heat}}}{\mathrm{d}t} = 2(2\pi r)\mathrm{d}r\sigma T^4 = \frac{1}{2}\frac{GM\dot{M}}{r}\left[1 - (1 - \mathrm{d}r/r + \ldots)\right] = \frac{1}{2}\frac{GM\dot{M}\mathrm{d}r}{r^2} , \tag{2.20}$$

[2] Choosing v monotonically increasing for increasing r leads to an outflow of mass appropriate for modeling a stellar wind

where we used the Stefan-Boltzmann law and the factor two accounts for the "top" and "bottom" side of the disk. The temperature profile follows as

$$T = \left(\frac{GM\dot{M}}{8\pi\sigma r^3}\right)^{1/4} \propto r^{-3/4}\,. \tag{2.21}$$

The total luminosity can be obtained integrating Eq. (2.20) from the inner to the outer edge of the disc,

$$L_{\rm tot} = \frac{1}{2}GM\dot{M}\int_{r_{\rm in}}^{r_{\rm out}}\frac{{\rm d}r}{r} = \frac{1}{2}GM\dot{M}\left(\frac{1}{r_{\rm in}} - \frac{1}{r_{\rm out}}\right) \approx \frac{1}{2}\,\frac{GM\dot{M}}{r_{\rm in}} \qquad \text{for} \quad r_{\rm in} \ll r_{\rm out}\,. \tag{2.22}$$

For the efficiency, we compare $L_{\rm tot}$ with the total rest mass of the accreted material, $\dot{M}c^2$. Expressing $r_{\rm in}$ by the Schwarzschild radius, $r_{\rm in} = xR_s = 2xGM/c^2$, we see that the efficiency is of order $\eta = L_{\rm tot}/(\dot{M}c^2) = 1/(4x)$. Hence the efficiency to release energy by accretion can be very large, if the disk extends close to the event horizon of the black hole.

We can obtain the spectral distribution of the emitted photon energies by integrating over the radial distribution,

$$L_\nu = 2\pi\int_{r_{\rm in}}^{r_{\rm out}}{\rm d}r r(\pi B_\nu) \propto \nu^3\int_{r_{\rm in}}^{r_{\rm out}}\frac{{\rm d}r\, r}{\exp[h\nu/(kT)] - 1}\,. \tag{2.23}$$

Substituting $x = h\nu r^{3/4}/k$, it follows

$$L_\nu \propto \nu^{1/3}\int_{x_{\rm in}}^{x_{\rm out}}{\rm d}x\,\frac{x^{5/3}}{{\rm e}^x - 1}\,. \tag{2.24}$$

Hence the spectral luminosity is $L_\nu \propto \nu^{1/3}$ in the frequency range where the integral is dominated by intermediate r values, $r_{\rm in} \ll r \ll r_{\rm out}$. At low frequency, the spectral luminosity is dominated by the Rayleigh-Jeans tail of the outer edge of the disk, $L_\nu \propto \nu^2$, while for frequencies above $kT(r_{\rm in})/h$ the spectrum is exponentially suppressed.

2.4 Basics of general relativity and black holes

Basic properties of gravitation

1. In classical mechanics, the equality of gravitating mass $m_g = F/g$ and inertial mass $m_i = F/a$ is a puzzle noticed already by Newton. This equality is called the "equivalence principle". Knowing more forces, this puzzle becomes even stronger. Contrast the acceleration of a particle in a gravitational field to the one in a Coulomb field: In the latter, two independent properties, namely its charge q giving the strength of the electric force acting on it and its mass m_i,i.e. the inertia of the particle, are needed. The equivalence of gravitating and inertial mass has been tested starting from Bessel, comparing e.g. the period of a pendulum of different materials,

$$P = 2\pi\sqrt{\frac{m_i l}{m_g g}}\,. \tag{2.25}$$

While $m_i = m_g$ can be achieved for one material by a convenient choice of units, there should be deviations for test bodies with differing compositions. Current limits for departures from the equivalence principle are $\Delta a_i/a < 10^{-12}$.

2. Newton's law postulates as the Coulomb law an instantaneous interaction. This is in contradiction to special relativity. Thus, as interactions with electromagnetic fields replace the Coulomb law, a corresponding description should be found for gravity. Moreover, the equivalence of mass and energy found in special relativity requires that, in a loose sense, energy not only mass should couple to gravity: Imagine a particle-antiparticle pair falling down a gravitational potential wall, gaining energy and finally annihilating into two photons moving the gravitational potential wall outwards. If the two photons would not loose energy climbing up the gravitational potential wall, a perpetum mobile could be constructed. If all forms of energy act as sources of gravity, then the gravitational field itself is gravitating. Thus the theory is non-linear and its mathematical structure is much more complicated than Maxwell's equations.

3. Gravity can be switched-off locally, just by cutting the rope of an elevator. Inside a freely falling elevator, one does not feel the effect of gravity.

Motivated by 2., Einstein used 1., the principle of equivalence, and 3. to derive general relativity, a theory that describes the effect of gravity as deformation of the space-time known from special relativity.

2.5 Schwarzschild metric

Because Einstein's theory has a rather more complicated mathematical structure than Newton's, no analytical solution to the two-body problem is known. Instead, we are looking first for the effect of a finite point-mass on the surrounding space-time and solve then for the motion of a test particle in this space-time.

2.5.1 Heuristic derivation

Consider a freely falling elevator in the gravitational field of a radial-symmetric mass distribution with total mass M. Since the elevator is freely falling, no effects of gravity are felt inside and the space-time coordinates from $r = \infty$ should be valid inside. Let us call these coordinates K_∞ with x_∞ (parallel to movement), y_∞, z_∞ (transverse) and t_∞. The elevator has the velocity v at the distance r from the mass M , measured in the coordinate system $K = (r, \phi, \vartheta, t)$ in which the mass M is at $r = 0$ at rest.

We assume that special relativity can be used for the transformation between K at rest and K_∞ moving with $v = \beta c$, as long as the gravitational field is weak. We shall see shortly what "weak" means in this context. For the moment, we presume that the effects of gravity are small, if the velocity of the elevator that was at rest at $r = \infty$ is still small, $v \ll c$. Then

$$\mathrm{d}t_\infty = \mathrm{d}t\sqrt{1-\beta^2} \tag{2.26}$$

$$\mathrm{d}x_\infty = \frac{\mathrm{d}r}{\sqrt{1-\beta^2}} \tag{2.27}$$

$$\mathrm{d}y_\infty = r\mathrm{d}\vartheta \tag{2.28}$$

$$\mathrm{d}z_\infty = r\sin\vartheta\mathrm{d}\phi\,. \tag{2.29}$$

Thus the line-element of special relativity, i.e. the infinitesimal distance between two space-time events,

$$\mathrm{d}s^2 = \mathrm{d}x_\infty^2 + \mathrm{d}y_\infty^2 + \mathrm{d}z_\infty^2 - c^2\mathrm{d}t_\infty^2 \tag{2.30}$$

becomes

$$ds^2 = \frac{dr^2}{1-\beta^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2) - (1-\beta^2)dt^2 \,. \tag{2.31}$$

Next, we want to relate the factor $1-\beta^2$ to the quantities M and r. Consider the energy of the elevator with rest mass m,

$$(\gamma - 1)mc^2 - \frac{G\gamma mM}{r} = 0 \,, \tag{2.32}$$

where the first term is the kinetic energy and the second the Newtonian expression for the potential energy. According to 2), we made here the crucial assumption that gravity couples not only to the mass of the elevator but to its total energy. Dividing by γmc^2 gives

$$\left(1 - \frac{1}{\gamma}\right) - \frac{GM}{rc^2} = 0 \,. \tag{2.33}$$

Remembering the definition $\gamma = 1/\sqrt{1-\beta^2}$ and introducing $\alpha = GM/c^2$, we have

$$\sqrt{1-\beta^2} = 1 - \frac{\alpha}{r} \tag{2.34}$$

or

$$1 - \beta^2 = 1 - \frac{2\alpha}{r} + \frac{\alpha^2}{r^2} \approx 1 - \frac{2\alpha}{r} \,. \tag{2.35}$$

In the last step, we neglected the term $(\alpha/r)^2$, since we attempt only an approximation for large distances, where gravity is still weak. Inserting this expression into Eq. (2.31), we obtain the metric describing the gravitational field produced by a radial symmetric mass distribution,

$$ds^2 = \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2) - c^2 dt^2 \left(1 - \frac{2GM}{rc^2}\right) \,. \tag{2.36}$$

Surprisingly, this agrees with the exact result found by Karl Schwarzschild 1916. The geometry of this space-time is contained in the coefficients in front the infinitesimal displacements dx^i,

$$ds^2 = \sum_{i,j=1}^{4} g_{ij} dx^i dx^j \,. \tag{2.37}$$

In case of euclidean space, we can find a coordinates such that the matrix g_{ij} becomes diagonal with $g_{ij} = (1, \dots, 1)$, in case of Minkowski space such that the matrix becomes $g_{ij} = (1, 1, 1, -1)$

2.5.2 Interpretation and consequences

Gravitational redshift The time measured by an observer is the proper-time $d\tau = -c ds$. If we choose two static observers at the position r and r' in the Schwarschild metric, then with $dr = d\phi = d\vartheta = 0$, we find

$$\frac{d\tau(r)}{d\tau(r')} = \frac{\sqrt{-g_{44}(r)}}{\sqrt{-g_{44}(r')}} \,. \tag{2.38}$$

The time-intervals $d\tau_\infty$ and $d\tau(r)$ are different and thus the time measured by clocks at different distances r from the mass M will differ too. Since frequencies are inversely proportional to time, the frequency or energy of a photon traveling from r to r' will be affected by the gravitational field as

$$\frac{\nu(r')}{\nu(r)} = \sqrt{\frac{1-\frac{2GM}{rc^2}}{1-\frac{2GM}{r'c^2}}}\,. \tag{2.39}$$

An observer at $r' \to \infty$ will receive photons with frequency

$$\nu_\infty = \sqrt{1-\frac{2GM}{rc^2}}\nu(r) \approx \left(1-\frac{|V_N|}{c^2}\right)\nu(r)\,, \tag{2.40}$$

where the last approximation is only valid for weak gravitational field, $2GM/(rc^2) \equiv V_N/c^2 \ll 1$. Thus the frequency of a photon is redshifted in a gravitational field, The size of this effect is of order V_N/c^2, where V_N is the Newtonian gravitational potential. The condition $V_N/c^2 \ll 1$ indicates that gravitational fields are weak and that Newtonian gravity is a sufficient approximation.

Schwarzschild radius What is the meaning of $r = R_S \equiv 2\alpha$? At

$$R_S = \frac{2GM}{c^2} = 3\,\mathrm{km}\,\frac{M}{M_\odot} \tag{2.41}$$

the coordinate system (2.36) becomes ill-defined. However, this does not mean necessarily that at $r = R_S$ physical quantities like tidal forces become infinite. Instead $r = R_S$ is an event horizon: We cannot obtain any information about what is going on inside R_S, if the gravitating mass is concentrated[3] inside a radius smaller than R_s. An object smaller than its Schwarzschild radius is called a black hole. The black hole is fully characterized by its mass M (and possibly its angular momentum L and electric charge q). To understand this better, we consider next what happens to a photon crossing the event horizon at $r = R_S$ as seen from an observer at $r = \infty$.

Approaching a black hole Light rays are characterized by $\mathrm{d}s^2 = 0$. Choosing a light ray in radial direction with $\mathrm{d}\phi = \mathrm{d}\vartheta = 0$, the metric (2.36) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \left(1-\frac{2\alpha}{r}\right)c\,. \tag{2.42}$$

Thus light traveling towards the star, as seen from the outside[4], will travel slower and slower as it comes closer to the Schwarzschild radius $r = 2\alpha$.

In fact, for an observer at infinity the signal will reach $r = 2\alpha$ only asymptotically for $t \to \infty$.

The last result can be derived immediately for light-rays. Choosing a light-ray in radial direction with $\mathrm{d}\phi = \mathrm{d}\vartheta = 0$, the metric (2.36) simplifies with $\mathrm{d}s^2 = 0$ to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 1-\frac{2M}{r}\,. \tag{2.43}$$

[3]Recall that in Newtonian gravity only the enclosed mass $M(r)$ contributes to the gravitational potential outside r for a spherically symmetric system. Thus, e.g. the Sun is not a black hole, since for all r the enclosed mass is $M(r) < rc^2/2G$.

[4]Any observer measures in the moment when the photon passes his position that the photon travels with the speed of light

Thus light traveling towards the star, as seen from the outside, will travel slower and slower as it comes closer to the Schwarzschild radius $r = 2\alpha$. Integrating gives

$$t = \int \mathrm{d}r \left(1 - \frac{2M}{r}\right)^{-1} = t' - 2M \ln(1 - 2M/r) \to \infty \qquad \text{for} \quad r \to 2M \,, \tag{2.44}$$

where t' is the start time at $r \to \infty$. Since the coordinate time t agrees with the proper time for an observer at infinity, a photon reaches the Schwarzschild radius $r = 2M$ only asymptotically for $t \to \infty$ for such an observer. Similarly, the communication with a freely falling space ship becomes impossible as it reaches $r = R_s$. A more detailed analysis shows that indeed no signal can cross the surface at $r = R_s$.

Perihelion precession Ellipses are solutions only for a potential $V(r) \propto 1/r$. General relativity generate corrections to the Newtonian $1/r$ potential, and as a result the perihelion of ellipses describing the motion of e.g. planets process. This effect is largest for Mercury, where $\Delta\phi/\Delta t \approx 43''/\mathrm{yr}$. This was the main known discrepancy of the planetary motions in the solar system with Newtonian gravity at the time, when Einstein and others worked on relativistic theories of gravity.

Light deflection The factors $(1 - 2\alpha/r)$ in the line-element $\mathrm{d}s$ will lead to the bending of light in gravitational fields. The measurement of the deflection of light by the Sun during the solar eclipse 1919 was the first crucial test for general relativity.

We shall discuss this effect in the limit of small deflections. We can view the spatial part of the Schwarschild metric as an refractive index n,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1 - \frac{2\alpha}{r}\right) c \stackrel{!}{=} \frac{c}{n} \,. \tag{2.45}$$

The space-dependence of the refractive index leads according to Hyggen's principle to the deflection of the light-rays, see Fig. 2.1: Two light-rays separated by $\mathrm{d}b$ have the velocity difference Δv. If the light-rays move initially along the z axis with the impact parameter b to the Sun, then the resulting infinitesimal deflection is

$$\mathrm{d}\vartheta = \frac{\Delta v \delta t}{\Delta b} = \frac{\partial(n^{-1})}{\partial b} c\delta t = \frac{\partial(n^{-1})}{\partial b} \mathrm{d}z \,. \tag{2.46}$$

Integrating with

$$n^{-1} = 1 - \frac{2GM}{rc^2} = 1 - \frac{2GM}{(z^2 + b^2)^{1/2} c^2} \tag{2.47}$$

and thus

$$\mathrm{d}\vartheta = \frac{2GM}{(z^2 + b^2)^{3/2} c^2} \, b\mathrm{d}z \tag{2.48}$$

we obtain

$$\vartheta = \int \mathrm{d}\vartheta = \frac{2GM}{c^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{(z^2 + b^2)^{3/2}} = \frac{4GM}{bc^2} \,. \tag{2.49}$$

Thus the deflection angle is determined by the ratio of the Schwarzschild radius and the impact parameter of the light-ray, $\vartheta = 2R_S/b$. Numerically, one obtains for a light-ray grazing the Sun $\vartheta = 1.75''$.

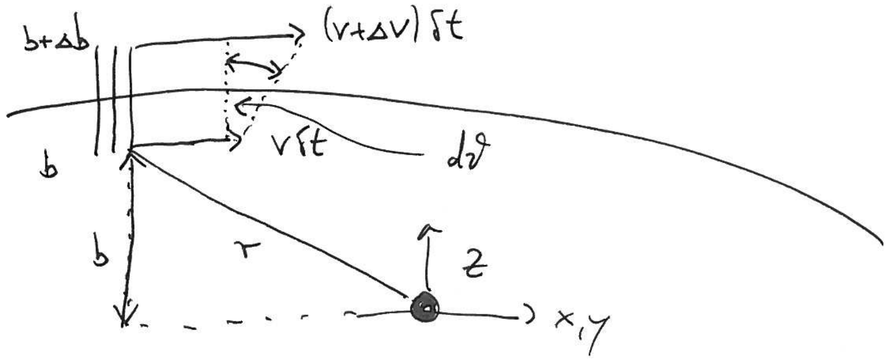

Figure 2.1: The difference velocity $v(r) = c/n(r)$ of light-rays leads according to Hyggen's principle to the deflection of light.

3 Evolution of stellar clusters

A look at the night-sky with binoculars or a small telescope shows that stars are not uniformly distributed on the sky. They are concentrated in groups of stars like the Plejades and a thin band. The thin band with a milky appearance is the disc of our own galaxy, the Milky Way, that we will discuss in the next chapter. In this chapter, we discuss cluster of stars and in particular their evolution.

The crucial points in our derivation of the virial theorem $2\langle U_{\rm kin}\rangle = -\langle U_{\rm pot}\rangle$ for a star was the assumption of a gravitationally bound system in equilibrium. Thus it holds also for any other system like a cluster of stars or galaxies, if this system fulfills the two conditions to be i) gravitationally bound, and ii) in equilibrium. The first condition can be checked directly by the virial theorem, while the second needs a more detailed discussion. In particular, we will see that a gravitationally bound system evolves and thus can be only approximately "in equilibrium." In this context, we define a system as "in equilibrium" or "dynamically relaxed", if the interchange of energy between the members of a cluster is fast compared to the evolution of the cluster.

3.1 Time-scales of cluster evolution

Rms and escape velocity from the virial theorem: The total kinetic energy of cluster with mass $M = Nm$ is

$$\langle E_{\rm kin}\rangle = \frac{1}{2}M\langle v^2\rangle \tag{3.1}$$

where $v_{\rm rms} \equiv \langle v^2\rangle^{1/2}$ is the root mean square (rms) velocity. Applying the virial theorem and using $\langle E_{\rm pot}\rangle = 3GM^2/5D$, it follows

$$\langle v^2\rangle = \frac{3GM}{5D}\,. \tag{3.2}$$

Ex.: Find the typical rms velocity of stars in a spherical cluster with size $D = 5$ pc that consists of 10^6 stars with average mass $m = 0.5M_\odot$.

$$\langle v^2\rangle = \frac{3GM}{5D} = 2.5\times 10^{12}{\rm cm}^2/{\rm s}^2 \tag{3.3}$$

or $v_{\rm rms} \approx 16$km/s. This value can be compared with observations; if the observed velocities are significantly higher, the cluster cannot be gravitationally bound or its total mass has to be higher.

Crossing time: The crossing time $t_{\rm cr}$ is the typical time required for a star in the cluster to travel the characteristic size D of the cluster (typically taken to be the half-mass radius). Thus, $t_{\rm cr} \sim D/v$ or $t_{\rm cr} \sim 3\times 10^5$ yr for the values of our example.

Relaxation time: The relaxation time $t_{\rm rel}$ is the typical time in which the star's velocity changed an amount comparable to its original velocity by gravitational encounters. Thus one might think of it as the time-scale after which the velocity distribution of stars bound in a cluster has reached an equilibrium distribution by the exchange energy and momentum with each other. But since part of the stars will have velocities $v > v_{\rm esc}$ and escape, the velocity distribution of a cluster is not stationary: High-velocity stars escape, the cluster contracts and its core is heated-up.

The depth τ for collisions of stars with each-other is

$$\tau = \frac{1}{\sigma n l} = \frac{1}{\sigma n v t} \tag{3.4}$$

when n denotes the density of stars and $l = vt$ the path traveled by a singe star. Let us call the relaxation time $t_{\rm rel}$ the time for which $\tau = 1$. Thus $t_{\rm rel} = 1/(\sigma n v)$.

What should we use for R (as function of v!) in $\sigma = \pi R^2$? Stars are certainly gravitationally interacting with each other, if they are bound to each other. Therefore we can estimate the effective interaction range R from $T = V$, $mv^2/2 = Gm^2/R$ or $R = 2Gm/v^2$. Then

$$t_{\rm rel} = \frac{1}{\pi R^2 n v} = \frac{v^3}{4\pi n (Gm)^2} \,. \tag{3.5}$$

Inserting $1/n = (m/M)(4\pi/3)D^3$ (with $M = Nm$ as total mass of the cluster) gives

$$t_{\rm rel} = \frac{v^3 D^3}{3G^2 m M} \,. \tag{3.6}$$

If the cluster is dynamically relaxed and the virial theorem applies, then $v^2 = 3GM/(D5)$ and thus

$$t_{\rm rel} = \frac{D}{v}\frac{M}{m}\,\frac{v^4 D^2}{3G^2 M^2} \sim \frac{ND}{v} = N t_{\rm cross} \,. \tag{3.7}$$

Note that $t_{\rm rel} \gg t_{\rm cross}$, in striking contrast to an ordinary gas

We should take into account how much momentum per collision is exchanged: In a collision at small impact parameter b the momentum transfer is larger than in one at large b. Moreover, one cannot treat relaxation just as a two-body process, because of the infinite range of the gravitational force. Formalizing this, the relaxation time given by Eq. (3.7) becomes reduced by a logarithmic term, $\approx 12\ln(N/2)$, or

$$t_{\rm rel} \approx 0.1 t_{\rm cross} \frac{N}{\ln(N)} \,. \tag{3.8}$$

Thus, $t_{\rm rel} \sim 2 \times 10^9$ yr for the values of our example.

Evaporation time: The evaporation time for a cluster is the time required for the cluster to dissolve through the gradual loss of stars that gain sufficient velocity through encounters to escape its gravitational potential.

Assuming an isolated cluster with negligible stellar evolution, the evaporation time $t_{\rm ev}$ can be estimated by assuming that a constant fraction α of the stars in the cluster is evaporated every relaxation time. Thus, the rate of loss is $dN/dt = -\alpha N/t_{\rm rel} = -N/t_{\rm ev}$. The value of α can be determined by noting that the escape speed $v_{\rm esc}$ at a point x is related to the gravitational potential $E_{\rm pot}(x)$ at that point by $v_{\rm esc}^2 = -2E_{\rm pot}(x)$. (The total energy of a particle able to escape gas to be equal or larger than zero, i.e. $T+V \geq 0$ or $v_{\rm esc}^2 \geq 2GM/R$.) If the system is virialized (as we would expect after a relaxation time), then also $\langle v^2 \rangle = 3GM/(5R)$.

Thus, stars with speeds above twice the RMS speed will evaporate. Assuming a Maxwellian distribution of speeds, the fraction of stars with $v > 2v_{rms}$ is $\alpha = 7.4 \times 10^{-3}$. Therefore, the evaporation time is

$$t_{\rm ev} = \frac{t_{\rm rel}}{\alpha} \approx 136 t_{\rm rel} \,. \tag{3.9}$$

Stellar evolution and tidal interactions with the galaxy tend to shorten the evaporation time. Using a typical $t_{\rm rel}$ for a globular cluster, we see that $t_{\rm ev} \sim 10^{10}$ yr, which is comparable to the observed age of globular clusters.

3.2 Isothermal sphere

In an isothermal sphere, the temperature does not depends on the radius. Thus the density distribution $\rho(r)$ depends only on the gravitational potential $\phi(r)$. With $\rho(r) \propto \exp(-E/kT)$, the density is

$$\rho(r) = \rho_0 \exp\left\{-\beta[\phi(r) - \phi_0]\right\} \,. \tag{3.10}$$

The connection between the mass density and the gravitational potential is given by the Poisson equation,

$$\Delta\phi = 4\pi G\rho_0 \exp\left\{-\beta[\phi(r) - \phi_0]\right\} \tag{3.11}$$

We introduce a dimensionless radius $x = r/L_0$ with the help of the Jeans length,

$$L_0 = (4\pi G\rho_0\beta)^{-1/2} \tag{3.12}$$

Setting also

$$y = \beta(\phi - \phi_0) \tag{3.13}$$

and assuming spherical symmetry, we can transform the Poisson equation (3.11) into an ordinary differential equation for $y(x)$,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\phi\right) = 4\pi G\rho(r) \quad \to \quad \frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{e}^{-y} \,. \tag{3.14}$$

Thus we see that i) the Lane-Emden equation is a special case of the Poisson equation and ii) an isothermal sphere corresponds to a polytropic EoS with $n \to \infty$. From the solution $y = 2\ln x$, we obtain immediately

$$\rho(r) = \rho_0 \mathrm{e}^{-y} = \rho_0 \left(\frac{L_0}{r}\right)^2 \,. \tag{3.15}$$

This solution is rather pathological: The density diverges for small r, while the integrated mass $M(r)$ diverges for $r \to \infty$. Although the solution is thus unphysical, it is as an limiting case very useful. First, we note that $\rho(r) \propto r^{-2}$ gives rotation curves $v(r) \propto$ const. as observed in galaxies. Second, a real physical system has a finite size, connected e.g. to the distance between neighbors, while physical processes like annihilations or degeneracy pressure become important for large ρ. Thus the isothermal sphere is a useful description for intermediate distances.

Last but not least, the solution is important since it is an attractor. To see the exact meaning of this statement we rewrite the differential equation in a from similar to the equation of motion for a dynamical system. Changing variables to $q = \ln(x)$, we obtain

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{e}^{-3q}\frac{\mathrm{d}}{\mathrm{d}q}\left(\mathrm{e}^q\frac{\mathrm{d}y}{\mathrm{d}q}\right) = \mathrm{e}^{-y} \tag{3.16}$$

and setting $z = 2q - y$

$$\frac{\mathrm{d}^2z}{\mathrm{d}q^2} + \frac{\mathrm{d}z}{\mathrm{d}q} = -(\mathrm{e}^z - 2) = -\frac{\partial V}{\partial z}\,. \tag{3.17}$$

We can now give all three terms of this equation a simple interpretation, if we interpret z as the coordinate of a one-dimensional dynamical system and q as time: acceleration $\ddot{z}$, friction $\dot{z}$ and potential energy $V = \mathrm{e}^z - 2z$. Hence the system (alas the density distribution $\rho(r)$) will move towards the minimum of the potential V, slowed by the friction term. But the minimum of V at $z = \ln 2$ or $y = 2q - z = 2\ln x + \text{const.}$ corresponds to the isothermal sphere.

Thus we found the important result that a self-gravitating system will evolve towards the density distribution (3.15) of an isothermal system, if degeneracy pressure or energy production play no important role. Such systems are for instance objects dominated by dark matter (galaxies, dark matter clumps inside galaxies) and stellar clusters.