Sketch of solutions for sheet 1

Hartle 5-6.

Consider a particle moving along the x-axis whose velocity as function of time is

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1+g^2t^2}}$$

where g is a constant.
a. Does the particle's speed ever exceeds the speed of light?
b. Calculate the four-velocity u.
c. Express x and t as function of the proper time τ along the trajectory.
d. Find the components of the four-force and the three-force acting on the particle.

a. Since $(gt)^2 < (1+g^2t^2)$ for all t, the velocity dx/dt is always smaller than the speed of light, $dx/dt < 1$.
b. The components of u are with $v_x = dx/dt$

$$u^t = \frac{dt}{d\tau} = \gamma = \sqrt{1+(gt)^2}$$

$$u^x = \frac{v}{\sqrt{1+v^2}} = gt$$

$$u^x = u^z = 0\,.$$

c. The proper time τ is the time measured by a clock travelling along the trajectory. Inverting

$$\tau = \int_0^t dt\sqrt{1-v^2} = \int_0^t \frac{dt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\mathrm{arcsinh} gt$$

we find $t(\tau) = \frac{1}{g}\sinh gt$. Next we integrate the velocity

$$x(t) - x_0 = \int_0^t dt \frac{gt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\sqrt{1+(gt)^2} =$$

d. From the definitions and u^α we find

$$f^\alpha = m\frac{d^2u^\alpha}{d\tau^2} = mg(\sinh(gt), \cosh(gt), 0, 0)$$

$$F^i = m\frac{du^i}{dt} = (mg, 0, 0)\,.$$

Charged pion decay.

A charged pion decays mainly via the reaction $\pi^\pm \to \mu^\pm + \nu_\mu$. Calculate the energy of the muon if the pion decays at rest. Calculate the maximal and minimal energy of the muon

Solutions were discussed Monday, 25.01.10

if the pion decays in flight with an energy of 1 TeV.
Use as masses $m_{\pi^\pm} = 139$ MeV, $m_{\mu^\pm} = 106$ MeV, and $m_\nu \approx 0$.

Combining energy conservation $m_\pi = E_\nu + E_\mu$, *smallness of neutrino masses* $E_\nu = p_\nu$ *and the cms condition* $\mathbf{p}_\nu = -\mathbf{p}_\mu$ *gives*

$$(E_\mu - m_\pi)^2 = |\mathbf{p}_\mu|^2$$

$$\underbrace{E_\mu^2 - |\mathbf{p}_\mu|^2}_{=m_\mu^2} - 2E_\mu m_\pi + m_\pi^2 = 0$$

$$m_\mu^2 + m_\pi^2 = 2E_\mu m_\pi$$

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = \frac{m_\pi}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right)$$

The energy in the lab system follows from the general LT $E' = \gamma(E + \beta p \cos\vartheta)$ *with* ϑ *as the angle between the velocity* β *of the pion and the emitted muon. The maximal/minimal values of* E' *follow for* $\cos\vartheta = \pm 1$*, i.e. if the muon emitted parallel and anti-parallel to the direction of flight of the pion.*

Inserting $E = (m_\pi/2)(1 + m_\mu^2/m_\pi^2)$ *and* $p = \sqrt{E'^2 - m_\mu^2} = (m_\pi/2)(1 - \underbrace{m_\mu^2/m_\pi^2}_{r})$ *gives*

$$E^{\max,\min} = \frac{\gamma m_\pi}{2}\left(1 + r \pm \beta(1-r)\right)$$

With $\gamma = 1\,TeV/m_\pi \approx 72{,}000$ *becomes* $\beta = \sqrt{1-\gamma^{-2}} \approx 1 - 1/2\gamma^2 \approx 1 - 10^{-10}$*. Hence the energy spectrum of muons is* $[0 : E_\pi]$*, with high-energy muons moving in forward and low-energy muons moving in backward direction.*

Planck units.
The four fundamental constants $\hbar$ (Planck's constant), c (velocity of light), G_N (gravitational constant) and k (Boltzmann constant) can be combined to obtain the dimension of a length, time, mass and temperature. Calculate their numerical values.

A formal way to derive e.g. the Planck time t_{Pl} *is to solve*

$$[c]^\alpha[\hbar]^\beta[G]^\gamma = [cm/s]^\alpha[g\,cm^2/s]^\beta[cm^3/(g\,s^2]^\gamma = s\,.$$

Simpler: i) note that k *contains as only one the temperature* T *and enters therefore only* T_{Pl}
ii) we need a combination of $\hbar G$ *to cancel the gram.*
iii) multiply with $1/c^5$ *to eliminate the centimeter,*

$$[\hbar G/c^5] = s^2$$

and thus

$$t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.4 \times 10^{-44} s\,.$$

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