1. The $\lambda \phi^3$ theory.
Consider the theory of a massive real scalar field $\phi$ and a $\lambda \phi^3$ self-interaction in $d = 6$ dimensions.

a.) Write down the Lagrange density $\mathcal{L}$ and explain your choice of signs and pre-factors.

b.) Write down the corresponding generating functional for disconnected and connected Green functions. How does one obtain connected Green functions?

c.) Determine the dimension of the field $\phi$ and of the coupling $\lambda$.

d.) Draw the Feynman diagram(s) and write down the analytical expression for the self-energy $i\Sigma$ (i.e. the loop correction for the free propagator) at order $\mathcal{O}(\lambda^2)$ in momentum space.

e.) Determine the symmetry factor of $i\Sigma$.

f.) Calculate the self-energy $i\Sigma$ using dimensional regularisation, split the result into a divergent pole term and a finite remainder.

a.) The free Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2$$

the relative sign is fixed by the relativistic energy-momentum relation, the overall sign by the requirement that the Hamiltonian is bounded from below. As the self-interaction is odd, adding $+\frac{\lambda}{3!} \phi^3$ or $-\frac{\lambda}{3!} \phi^3$ is equivalent: both choices will lead to an unstable vacuum.

The prefactor 1/2 of the kinetic term corresponds to “canonically normalised field”, leading to the correct size of vacuum fluctuations.

The prefactor of the $\lambda \phi^3$ term can be chosen arbitrary, if the Feynman rule is adjusted accordingly: For $-i\lambda$, we should choose $\mathcal{L}_1 = -\frac{\lambda}{3!} \phi^3$.

b.) The generating functional $Z[J]$ of disconnected Green functions is obtained from the path integral by i) adding a linear coupling to an external source $J$, ii) taking the limit $t, -t' \to \infty$ with $m^2 - i\varepsilon$,

$$Z[J] = \left< 0 | 0 \right> = \mathcal{N} \int \mathcal{D} \phi \exp \left( \int_{\Omega} d^4x \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{3!} \phi^3 + J \phi \right) \right) = \exp i W[J].$$

The functional $W[J]$ generates connected Green functions,

$$G(x_1, \ldots, x_n) = \left. \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} i W[J] \right|_{J=0}. \quad (1)$$

page 1 of 3 pages
c.) The action $S = \int d^6x \mathcal{L}$ has to be dimensionless. Thus $[\mathcal{L}] = m^6$, $[\phi] = m^2$, and thus the coupling is dimensionless, $[\lambda] = m^0$. [That's the reason why we do this exercise in $d = 6$.]

Using the Feynman rules gives for

\[ k \quad \square \quad k \]

in momentum space

\[ i\Sigma(k^2) = S (-i\lambda)^2 \int \frac{d^6p}{(2\pi)^6} \frac{i}{(p + k)^2 - m^2 + i\varepsilon} \frac{i}{p^2 - m^2 + i\varepsilon} \]

where the symmetry factor $S$ is determined in e.) and the vertex $-i\lambda$ was used.

e.) The self-energy is a second order diagram, corresponding to the term

\[ \frac{1}{2!} \left( \frac{-i\lambda}{3!} \right)^2 \int d^4y_1 d^4y_2 \langle 0 | \phi(x_1) \phi(x_2) \phi^3(y_1) \phi^3(y_2) + | y_1 \leftrightarrow y_2 \rangle \]

in the perturbative expansion in coordinate space. The exchange graph $y_1 \leftrightarrow y_2$ is identical to the original one, canceling the factor $1/2!$ from the Taylor expansion. We count the number of possible ways to combine the fields in the time-ordered product into four propagators. We have three possibilities to contract $\phi(x_1)$ with a $\phi(y_1)$. Similarly, there are three possibilities for $\phi(x_2)\phi(y_2)$. The remaining pairs of $\phi(y_1)$ and $\phi(y_2)$ can be contracted in $2!$ ways. Thus the symmetry factor is

\[ S = \left( \frac{1}{2!} \times 2 \right) \left( \frac{1}{3!} \right)^2 (3 \times 3 \times 2!) = \frac{1}{2} \]

[The symmetry factor is given for the vertex $-i\lambda$.]

f.) We combine the two propagators (suppressing the $i\varepsilon$) using (9),

\[ \frac{1}{(p + k)^2 - m^2} \frac{1}{p^2 - m^2} = \int_0^1 dx \frac{1}{D^2} \]

with

\[ D = x[(p + k)^2 - m^2] + (1 - x)(p^2 - m^2) \]
\[ = (p + xk)^2 + x(1 - x)k^2 - m^2 = q^2 + f, \]

where we introduced $q = p + xk$ as new integration variable and set $f = x(1 - x)k^2 - m^2$. We go now to $d = 2\omega = 6 - \varepsilon$ dimensions,

\[ i\Sigma(k^2) = \frac{1}{2} \lambda^2 \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q + f)^2}. \]

Evaluating the integral with (10), using $\Gamma(2) = 1$ and $\omega = 3 - \varepsilon/2$ gives

\[ \Sigma(k^2) = -\frac{\lambda^2}{2} \frac{\Gamma(-1 + \varepsilon/2)}{(4\pi)^3} \int_0^1 dx f^{\varepsilon/2} \left( \frac{4\pi \mu^2}{f} \right)^{\varepsilon/2}. \]
Here, we added a mass scale $\mu$ in order to make the $\varepsilon$ dependent term dimensionless such that we can expand it using (11),
\[
\left(\frac{4\pi\mu^2}{f}\right)^{\varepsilon/2} = 1 + \frac{\varepsilon}{2} \ln \left(\frac{4\pi\mu^2}{f}\right) + \mathcal{O}(\varepsilon^2).
\]
Expanding also
\[
\Gamma(-1 + \varepsilon/2) = -\left[\frac{2}{\varepsilon} + 1 - \gamma + \mathcal{O}(\varepsilon)\right]
\]
we arrive at
\[
\Sigma(k^2) = \frac{\alpha}{2} \left[\left(\frac{2}{\varepsilon} + 1 - \gamma\right) \left(\frac{k^2}{6} - m^2\right) + \int_0^1 dx f \ln \left(\frac{4\pi\mu^2}{f}\right)\right]
\]
where we used $\int_0^1 dx f = k^2/6 - m^2$ and set $\alpha = \lambda^2/(4\pi)^3$. The obtained expression for the self-energy has the UV divergence isolated into an $1/\varepsilon$ pole which is ready for subtraction.

2. Fermions.
   a.) Define left- and right-chiral fields $\psi_L$ and $\psi_R$ as eigenfunctions of $\gamma^5$. Express
   \[
   \mathcal{L} = \bar{\psi}i\gamma^\mu\psi - m\bar{\psi}\gamma^0\psi
   \]
   in terms of $\psi_L$ and $\psi_R$. (7 pts)
   b.) Give an operator which commutes with the (free Dirac) Hamiltonian and can be used to classify the spin states of a fermion. Explain its meaning. (You don’t have to calculate the commutator.) (3 pts)

   a.) We can split any solution $\psi$ of the Dirac equation into
   \[
   \psi_L = \frac{1}{2}(1 - \gamma^5)\psi \equiv P_L\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi \equiv P_R\psi.
   \]
   Since $\gamma^5\psi_L = -\psi_L$ and $\gamma^5\psi_R = \psi_R$, $\psi_L, \psi_R$ are eigenfunctions of $\gamma^5$ with eigenvalue $\pm 1$. Expressing the mass term through these fields as
   \[
   \bar{\psi}\psi = \bar{\psi} \left(P_L^2 + P_R^2\right)\psi = \psi^\dagger \left(P_R \gamma^0 P_L + P_L \gamma^0 P_R\right)\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R
   \]
   and similarly for the kinetic term,
   \[
   \bar{\psi}\partial^\mu\psi = \bar{\psi} \left(P_L^2 + P_R^2\right)\partial^\mu\psi = \psi^\dagger \left(P_R \gamma^0 \gamma^\mu P_L + P_L \gamma^0 \gamma^\mu P_R\right)\partial^\mu\psi = \bar{\psi}_L\partial^\mu\psi_L + \bar{\psi}_R\partial^\mu\psi_R.
   \]
   the Dirac Lagrange density becomes
   \[
   \mathcal{L} = i\bar{\psi}_L\partial^\mu\psi_L + i\bar{\psi}_R\partial^\mu\psi_R - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L).
   \]
   b.) One possibility is the helicity operator $h = \mathbf{s} \cdot \mathbf{p}/|\mathbf{p}|$, or more generally, $\gamma^5\mathbf{\hat{s}}$. 

page 3 of 3 pages
3. Scattering.
Derive the optical theorem
\[ 2 \Im T_{ii} = \sum_n T_{in}^* T_{ni}. \]

Give a physical interpretation of this relation (less than 100 words).

The unitarity of the scattering operator, \( S^\dagger S = SS^\dagger = 1 \), expresses the fact that we (should) use a complete set of states for the initial and final states in a scattering process,
\[ 1 = \sum_n |n, +\infty \rangle \langle n, +\infty | = \sum_n |n, -\infty \rangle \langle n, -\infty | S^\dagger SS^\dagger. \]  

We split the scattering operator \( S \) into a diagonal part and the transition operator \( T \), \( S = 1 + iT \), and thus
\[ 1 = (1 + iT)(1 - iT^\dagger) = 1 + i(T - T^\dagger) + TT^\dagger \]  
or
\[ iTT^\dagger = T - T^\dagger. \]

We now consider matrix elements between the initial and final state,
\[ \langle f | T - T^\dagger | i \rangle = T_{fi} - T_{fi}^\dagger = i \langle f | TT^\dagger | i \rangle = i \sum_n T_{fn} T_{ni}^*. \]

If we set \( |i \rangle = |f \rangle \), we obtain optical theorem as a connection between the forward scattering amplitude \( T_{ii} \) and the scattering into all possible states \( n \),
\[ 2 \Im T_{ii} = \sum_n |T_{in}|^2. \]

It relates the attenuation of a beam of particles in the state \( i \), \( dN_i \propto -|\Im T_{ii}|^2 N_i \), to the probability that they scatter into all possible states \( n \): what is lost, should show up somewhere.

4. Gauge invariance.
Consider a local gauge transformation
\[ U(x) = \exp[ig \sum_{a=1}^m \partial^a(x) T^a] \]

which changes a vector of fermion fields \( \psi \) with components \( \{\psi_1, \ldots, \psi_k\} \) as
\[ \psi(x) \to \psi'(x) = U(x) \psi(x). \]

a.) Assume that \( U \) are elements of the non-abelian gauge group \( SU(n) \) and that \( \{\psi_1, \ldots, \psi_k\} \) transform with the fundamental representation. What are then the values of \( n \) and \( m \)? What is the physical interpretation of \( m \)?
b.) Derive the transformation law of $A_\mu = A_\mu^a T^a$ under a gauge transformation. One way to do this is to require that i) the covariant derivatives transform in the same way as $\psi$,

$$D_\mu \psi(x) \rightarrow [D_\mu \psi(x)]' = U(x)[D_\mu \psi(x)].$$

and ii) that the gauge field should compensate the difference between the normal and the covariant derivative,

$$D_\mu \psi(x) = [\partial_\mu + ig A_\mu(x)]\psi(x).$$

(8 pts)

c.) The non-abelian field-strength $F_{\mu\nu} = F_{\mu\nu}^a T^a$ transforms under a local gauge transformation $U(x)$ as

- $F_{\mu\nu} \rightarrow F_{\mu\nu}' = F_{\mu\nu}$
- $F_{\mu\nu} \rightarrow F_{\mu\nu}' = U(x) F_{\mu\nu} U^\dagger(x)$
- $F_{\mu\nu} \rightarrow F_{\mu\nu}' = U(x) F_{\mu\nu} U^\dagger(x) + \frac{i}{g}(\partial_\mu U(x))\partial_\nu U^\dagger(x)$
- $F_{\mu\nu} \rightarrow F_{\mu\nu}' = F_{\mu\nu} + [D_\mu, A_\nu]$

(2 pts)

a.) The fundamental representation of SU(n) is n-dimensional. Since $\{\psi_1, \ldots, \psi_5\}$ transforms with the fundamental representation, it is $n = 5$. Then $m = 5^2 - 1 = 24$ is the number of generators of SU(5), or more physically speaking, the number of gauge bosons.

b.) Combining both requirements gives

$$D_\mu \psi(x) \rightarrow [D_\mu \psi]' = UD_\mu \psi = UD_\mu U^{-1} U \psi = UD_\mu U^{-1} \psi',$$

and thus the covariant derivative transforms as $D'_\mu = UD_\mu U^{-1}$. Using its definition, we find

$$[D'_\mu \psi]' = [\partial_\mu + ig A'_\mu]U \psi = UD_\mu \psi = U[\partial_\mu + ig A_\mu] \psi.$$

(12)

We compare now the second and the fourth term, after having performed the differentiation $\partial_\mu (U \psi)$. The result

$$[(\partial_\mu U) + ig A'_\mu U] \psi = ig U A_\mu \psi$$

should be valid for arbitrary $\psi$ and hence after multiplying from the right with $U^{-1}$ we arrive at

$$A_\mu \rightarrow A'_\mu = U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U) U^{-1} = U A_\mu U^{-1} - \frac{i}{g} U \partial_\mu U^{-1}.$$  \hspace{1cm} (14)

Here we also used $\partial_\mu (UU^{-1}) = 0$. For SU(n), the gauge transformation $U$ is an unitary transformation and one sets $U^{-1} = U^\dagger$.

c.) Option two
Some formulas

\[ \{ \gamma^\mu, \gamma^\nu \} = 2 \eta^{\mu \nu}. \] (15)

\[ \gamma^5 \equiv \gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3. \] (16)

\[ \sigma^{\mu \nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \] (17)

\[ \Gamma = \gamma^0 \Gamma^0 \] (18)

\[ \frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1 - z)]^2}. \] (19)

\[ I(\omega, \alpha) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{[k^2 + 2p k + M^2 + i\varepsilon]^\alpha} \]  
\[ = i \frac{(-\pi)^\omega}{(2\pi)^2} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} \frac{1}{[M^2 - p^2 + i\varepsilon]^\alpha - \omega}. \] (20)

\[ f^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2} \ln f + O(\varepsilon^2). \] (21)

\[ \Gamma(n + 1) = n! \] (22)

\[ \Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left[ \frac{1}{\varepsilon} + \psi(n + 1) + O(\varepsilon) \right], \] (23)

\[ \psi(n + 1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma, \] (24)

\[ \phi \]

\[ \phi \]

\[ \phi \]

\[ -i\lambda \]

\[ p \]

\[ \frac{i}{p^2 - m^2 + i\varepsilon} \]