## Exercise sheet 1

## 1. Units.

a.) The four fundamental constants $\hbar$ (Planck's constant), $c$ (velocity of light), $G_{N}$ (gravitational constant) and $k_{B}$ (Boltzmann constant) can be combined to obtain the dimension of a length, time, mass, energy and temperature. Find the four relations and calculate the numerical values of two of these so-called "Planck units". Do you have a guess of their physical meaning?
b.) Natural units: We will set in the these lectures $c=\hbar=1$. Then the famous Einstein relation becomes $E=m$. Show how one can restore usual units.
a.) A formal way to derive e.g. the Planck time $t_{\mathrm{Pl}}$ is to solve

$$
[c]^{\alpha}[\hbar]^{\beta}[G]^{\gamma}\left[k_{B}\right]^{\delta}=[\mathrm{cm} / \mathrm{s}]^{\alpha}\left[\mathrm{g} \mathrm{~cm}^{2} / \mathrm{s}\right]^{\beta}\left[\mathrm{cm}^{3} /\left(\mathrm{g} \mathrm{~s}^{2}\right)\right]^{\gamma}\left[\mathrm{g} \mathrm{~cm}^{2} / \mathrm{K}\right]^{\delta}=\mathrm{s}
$$

Simpler: i) note that $k_{B}$ contains as only one the temperature $T$ and enters therefore only $T_{\mathrm{Pl}}$ ii) we need a combination of $\hbar G$ to cancel the gram.
iii) multiply with $1 / c^{5}$ to eliminate the centimeter,

$$
\left[\hbar G / c^{5}\right]=s^{2}
$$

and thus

$$
t_{\mathrm{Pl}}=\sqrt{\frac{\hbar G}{c^{5}}} \approx 5.4 \times 10^{-44} \mathrm{~s}
$$

Then $l_{\mathrm{Pl}}=c t_{\mathrm{Pl}}=1.6 \times 10^{-33} \mathrm{~cm}$, etc.
These values shound indicate the range of validity of classical relativity (containing only $c$ and $G$ ): At lengths and times smaller than the Planck length and time, "quantum gravity" is expected to lead to large corrections to classical gravity. In particular, our picture of a classical spacetime might become invalid. Similar, effects of "quantum gravity" should become important at energies above the Planck energy.
b.) We can restore usual units by comparing the units of $E=c^{\alpha} \hbar^{\beta} G^{\gamma} m$, or

$$
[E]=\frac{\mathrm{g} \cdot \mathrm{~cm}^{2}}{\mathrm{~s}^{2}}=[c]^{\alpha}[\hbar]^{\beta}[G]^{\gamma} \mathrm{g}
$$

leading to $\alpha=2$ and $\beta=\gamma=0$.

## 2. Transformation between inertial frames.

Consider two inertial frames $K$ and $K^{\prime}$ with parallel axes at $t=t^{\prime}=0$ that are moving with the relative velocity $v$ in the $x$ direction.
a.) Show that the linear transformation between the coordinates in $K$ and $K^{\prime}$ is given by

$$
\left(\begin{array}{c}
t^{\prime}  \tag{1}\\
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right)=\left(\begin{array}{c}
A t+B x \\
D t+E x \\
y \\
z
\end{array}\right)=\left(\begin{array}{c}
A t+B x \\
A(x-v t) \\
y \\
z
\end{array}\right)
$$

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b.) Show that requiring the invariance of

$$
\begin{equation*}
\Delta s^{2} \equiv c^{2} t^{2}-x^{2}-y^{2}-z^{2}=c^{2} t^{\prime 2}-x^{\prime 2}-y^{\prime 2}-z^{\prime 2} \tag{2}
\end{equation*}
$$

leads to Lorentz transformations.
a.) The origin $x^{\prime}=0$ of $K^{\prime}$ correspond in $K$ to $x=v t$. Then

$$
0=D t+E v t \Rightarrow D=-E v
$$

The origin $x=0$ of $K$ correspond in $K^{\prime}$ to $x^{\prime}=-v t^{\prime}$. Then

$$
\begin{align*}
t^{\prime} & =A t  \tag{3}\\
-v t^{\prime} & =D t \quad \Rightarrow \quad t^{\prime}=-\frac{D}{v} t=A t \tag{4}
\end{align*}
$$

and thus $A=-D / v$ and hence $A=E$.
Remark: The transformation has to be linear, if we assume that space and time are uniform (translation invariant).
b.) Square the terms on the RHS of

$$
t^{2}-x^{2}=(A t-B x)^{2}-A^{2}(x-v t)^{2},
$$

order them as $t^{2}(\cdots)-x^{2}(\cdots)+2 t x(\cdots)$ and compare coefficients to the LHS. This gives $A=\gamma$ and $B=-\beta \gamma$.

## 3. Doppler effect.

The photon is a massless particle. Consider a Lorentz transformation of its wave four-vector $k^{\mu}=p^{\mu} / \hbar=(\omega, \boldsymbol{k})$ to obtain the relativistic Doppler formula.

Denote by $k_{0}^{\alpha}=\left(\omega_{0}, \boldsymbol{k}_{0}\right)$ and $k^{\alpha}=(\omega, \boldsymbol{k})$ the wave-vectors of the photon in the rest-system $K_{0}$ of the source and $K$ of the observer, respectively. If the source moves with the velocity $v$, then $K$ moves relative to $K_{0}$ with $-v$. Applying then a Lorentz transformation and assuming that $v$ is parallel to the $x$-axis gives

$$
\omega_{0}=\gamma\left(\omega-v k_{x}\right)
$$

or using $k_{x}=\omega \cos \alpha$

$$
\frac{\omega}{\omega_{0}}=\frac{\sqrt{1-v^{2}}}{1-v \cos \alpha}
$$

## 4. Index gymnastics.

a.) Splitt the arbitrary tensor $T^{\mu \nu}$ into its symmetric part $S^{\mu \nu}=S^{\nu \mu}$ and its anti-symmetric part $A^{\mu \nu}=-A^{\nu \mu}$.
b.) Show that this splitting is invariant under Lorentz transformations.
c.) Show that $S^{\mu \nu} A_{\mu \nu}=0$.

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a.) We can split any tensor $T^{\mu \nu}$ into a symmetric and antisymmetric piece, $T^{\mu \nu}=S^{\mu \nu}+A^{\mu \nu}$ with $S^{\mu \nu}=S^{\nu \mu}$ and $A^{\mu \nu}=-A^{\nu \mu}$, writing

$$
T_{\mu \nu}=\frac{1}{2}\left(T_{\mu \nu}+T_{\nu \mu}\right)+\frac{1}{2}\left(T_{\mu \nu}-T_{\nu \mu}\right) \equiv S_{\mu \nu}+A_{\mu \nu} .
$$

b.) Let us consider e.g. the antisymmetric tensor components given by

$$
A_{\mu \nu}=\frac{1}{2}\left(T_{\mu \nu}-T_{\nu \mu}\right) .
$$

The definition is invariant under a Lorentz transformation $\Lambda^{\mu}{ }_{\nu}$, since

$$
\begin{align*}
2 \tilde{A}_{\mu \nu} & =\tilde{T}_{\mu \nu}-\tilde{T}_{\nu \mu}=\Lambda^{\rho}{ }_{\mu} \Lambda^{\sigma}{ }_{\nu} T_{\rho \sigma}-\Lambda^{\rho}{ }_{\nu} \Lambda^{\sigma}{ }_{\mu} T_{\rho \sigma}  \tag{5}\\
& =\Lambda^{\rho}{ }_{\mu} \Lambda^{\sigma}{ }_{\nu} T_{\rho \sigma}-\Lambda^{\sigma}{ }_{\nu} \Lambda^{\rho}{ }_{\mu} T_{\sigma \rho}=\Lambda^{\rho}{ }_{\mu} \Lambda^{\sigma}{ }_{\nu}\left(T_{\rho \sigma}-T_{\sigma \rho}\right)=\Lambda^{\rho}{ }_{\mu} \Lambda^{\sigma}{ }_{\nu} 2 A_{\rho \sigma} \tag{6}
\end{align*}
$$

Here we used first the transformation law for a tensor of rank 2, and exchanged then dummy indices in the second term.
Note that this holds also for a general coordinate transformation, $\Lambda^{\mu}{ }_{\nu} \rightarrow \frac{\partial \tilde{x}^{\nu}}{\partial x^{\mu}}$.
c.) To see that the contraction of a symmetric tensor $S_{\mu \nu}$ with an antisymmetric tensor $A_{\mu \nu}$ gives zero, consider

$$
\begin{equation*}
S_{\mu \nu} A^{\mu \nu}=-S_{\mu \nu} A^{\nu \mu}=-S_{\nu \mu} A^{\nu \mu}=-S_{\mu \nu} A^{\mu \nu} . \tag{7}
\end{equation*}
$$

Here we used first the antisymmetry of $A^{\mu \nu}$, then the symmetry of $S_{\mu \nu}$, and finally exchanged the dummy summation indices. Note that this remains true, if the tensor expression contains additional indices.

## 5. Lorentz group.

Lorentz transformations $\Lambda \in O(1,3)$ are all those coordinate transformations $x^{\mu} \rightarrow \tilde{x}^{\mu}=$ $\Lambda^{\mu}{ }_{\nu} x^{\nu}$ that keep the norm $\eta_{\mu \nu} x^{\mu} x^{\nu}$ of space-time points $x^{\mu}$ invariant. Apart from rotations and boosts which are continuously connected to the unit element, there are other, unconnected pieces of the Lorentz group. Determine these unconnected pieces by considering the relation $\eta=\Lambda^{T} \eta \Lambda$. [Hint: A relation like $[f(\Lambda)]^{2}=1$ shows that the Lorentz group consists of at least two disconnected pieces, one with $f=1$, another one with $f=-1$.]

We rewrite first the fact that the metric tensor $\eta_{\mu \nu}$ is invariant under Lorentz transformations, $\tilde{\eta}_{\mu \nu} \Lambda^{\mu}{ }_{\rho} \Lambda^{\nu}{ }_{\sigma}=\eta_{\rho \sigma}$, in matrix form, $\eta=\Lambda^{T} \eta \Lambda$. Taking then the determinant, we obtain

$$
\begin{equation*}
(\operatorname{det} \Lambda)^{2}=1 \quad \text { or } \quad \operatorname{det} \Lambda= \pm 1 \tag{8}
\end{equation*}
$$

Thus $\mathrm{O}(1,3)$ consists of at least two disconnected pieces. The part with $\operatorname{det} \Lambda=1$ contains the proper Lorentz transformations and is called $\mathrm{SO}(1,3)$. The second part contains the improper Lorentz transformations that can be written as the product of a proper transformation and a

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discrete transformation changing the sign of an odd number of coordinates.
Next we consider the 00 component of the equation $\eta=\Lambda^{T} \eta \Lambda$. With $\eta_{00}=\eta_{\mu \nu} \Lambda^{\mu}{ }_{0} \Lambda^{\nu}{ }_{0}$, it is

$$
\begin{equation*}
1=\left(\Lambda_{0}^{0}\right)^{2}-\sum_{i=1}^{3}\left(\Lambda_{0}^{i}\right)^{2} . \tag{9}
\end{equation*}
$$

Thus $\left(\Lambda_{0}^{0}\right)^{2} \geq 1$ and both the proper and the improper Lorentz transformations consist of two disconnected pieces. The transformations with $\Lambda_{0}^{0} \geq 1$ are called orthochronous since they do not change the direction of time. In contrast, transformations with $\Lambda_{0}^{0} \leq-1$ transform a future-directed time-like vector into a past-directed one and vice versa, and are therefore called antichronous. These properties are Lorentz invariant, and thus we can split Lorentz transformations into four categories:

- proper, orthochronous (= restricted) transformations $L \in \mathfrak{L}_{+}^{\uparrow}$ with $\operatorname{det} \Lambda=1$ and $\Lambda_{0}^{0} \geq 1$,
- proper, antichronous transformations $T P L \in \mathfrak{L}_{+}^{\downarrow}$ with $\operatorname{det} \Lambda=1$ and $\Lambda_{0}^{0} \leq-1$,
- improper, orthochronous transformation $P L \in \mathfrak{L}_{-}^{\uparrow}$ with $\operatorname{det} \Lambda=-1$ and $\Lambda_{0}^{0} \geq 1$,
- improper, antichronous transformations $T L \in \mathfrak{L}_{-}^{\downarrow}$ with $\operatorname{det} \Lambda=-1$ and $\Lambda_{0}^{0} \leq-1$.

Only those Lorentz transformations that are elements of the restricted Lorentz group, $L \in \mathfrak{L}_{+}^{\uparrow}$, are connected smoothly to the identity and can thus be built up from infinitesimal transformations. The other three disconnected pieces of the Lorentz group can be obtained as the product of an element $L \in \mathfrak{L}_{+}^{\uparrow}$ and additional discrete $P$ and $T$ transformations with $P\left(x^{0}, \boldsymbol{x}\right)=\left(x^{0},-\boldsymbol{x}\right)$ and $T\left(x^{0}, \boldsymbol{x}\right)=\left(-x^{0}, \boldsymbol{x}\right)$.

Remark: No need to memorize the names. The important point to note is that an arbitrary Lorentz transformation can be obtained by combining those connected to the unity, plus a discrete parity and/or time-inversion transformation.

