Exercise sheet 10

1. Fierz identity for SU(n)

Derive the Fierz identity for SU(n),

\[ T_{ij}^a T_{kl}^a = \frac{1}{2} \left( \delta_{il} \delta_{jk} - \frac{1}{n} \delta_{ij} \delta_{kl} \right). \]

(Hint: Use that the generators \( T^a \) and the unit matrix 1 form as basis of the hermitian \( n \times n \) matrices.)

We can expand any hermitian matrix \( M \) as

\[ M = c_0 1 + c_a T^a. \]

Taking the trace using \( \text{tr}(T_{ij}^a) = 0 \) determines

\[ c_0 = \frac{1}{n} \text{tr}(M). \]

Next we take the trace of \( MT^b \) using \( \text{tr}(T^a T^b) = \delta^{ab}/2 \),

\[ \text{tr}(MT^b) = 0 + c_a \delta^{ab}/2 \]

or \( c_a = 2 \text{tr}(MT^a) \). Inserting \( c_i \) into the expansion of \( M \) gives

\[ M = c_0 1 + c_a T^a = \frac{1}{n} \text{tr}(M) 1 + 2 \text{tr}(MT^a) T^a \]

or adding indices

\[ 2M_{kl} T_{lk}^a T_{ij}^a = M_{ij} - \frac{1}{n} M_{kk} \delta_{ij}. \]

Finally, we divide by 2 and factor out \( M_{kl} \) on the RHS,

\[ M_{kl} T_{lk}^a T_{ij}^a = \frac{1}{2} \left[ M_{ij} - \frac{1}{n} M_{kk} \delta_{ij} \right] = \frac{1}{2} \left[ \delta_{ik} \delta_{jl} - \frac{1}{n} \delta_{ij} \delta_{kl} \right] M_{kl}. \]

Since this holds for any \( M \), the Fierz identity follows.

2. Three gauge boson vertex.

Derive the structure \( V^{rst}(k_1^a, k_2^a, k_3^a) \) of the three-gluon vertex. (Either by Fourier-transforming the part of \( \mathcal{L}_I \) containing three gluon fields to momentum space,

\[ F = \int d^4p_1 d^4p_2 d^4p_3 (2\pi)^4 \delta(p_1 + p_2 + p_3) f(A_{\mu}^a(p_1) A_{\nu}^b(p_2) A_{\rho}^c(p_3)) \]

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and then eliminating the fields by functional derivatives with respect to them

\[ V^{rst}(k_1^\rho, k_2^\sigma, k_3^\tau) = \frac{\delta^3 F}{\delta A_\rho^r(k_1) \delta A_\sigma^s(k_2) \delta A_\tau^t(k_3)} \]

Or by using symmetry arguments.

With \( \text{tr}(T^a[T^b, T^c]) = i f^{bcd} \text{tr}(T^a T^d) = i f^{abc} / 2 \) we obtain

\[ -\frac{1}{2} \text{tr}(F^2) = -2 \times \frac{1}{2}(\partial_\mu A_\rho^a - \partial_\nu A_\rho^a)(ig) \frac{i f^{abc}}{2} A^{b\mu} A^{c\nu} + \cdots = gf^{abc} \partial_\mu A_\rho^a A_{b\mu} A_{c\nu} \]

Then we go to momentum space plugging in Fourier transformed fields, differentiate and perform the space integration,

\[ F = gf^{abc} \int d^4 p_1 d^4 p_2 d^4 p_3 (2\pi)^4 \delta(p_1 + p_2 + p_3) \partial_\mu A_\rho^a(p_1) A_{b\mu}(p_2) A_{c\nu}(p_3) \]

\[ = -ig f^{abc} p_1 \eta_{\lambda\nu} A_\lambda^a(p_1) A_{b\mu}(p_2) A_{c\nu}(p_3). \]

(If the vertex contains derivatives, we have to fix arbitrarily the momentum flow.) Now we can extract the vertex by taking derivatives w.r.t. to three gauge fields,

\[ V^{rst}(k_1^\rho, k_2^\sigma, k_3^\tau) = \frac{\delta^3 F}{\delta A_\rho^r(k_1) \delta A_\sigma^s(k_2) \delta A_\tau^t(k_3)}. \]

Alternatively, we can argue as follows: The antisymmetry of \( f^{abc} \) in (1) implies that the index pairs \( (p_1, \lambda) \), \( (p_2, \mu) \) and \( (p_3, \nu) \) are antisymmetric; we make this antisymmetry explicit by the replacement

\[ gf^{abc} p_1 \eta_{\lambda\nu} A_\lambda^a(p_1) A_{b\mu}(p_2) A_{c\nu}(p_3) \rightarrow \frac{1}{3!} gf^{abc} [(p_3 - p_2) \lambda \eta_{\mu\nu} + (p_1 - p_3) \nu \eta_{\lambda\mu} + (p_2 - p_1) \mu \eta_{\lambda\nu}] \]

Finally, we note that the factor \( \frac{1}{3!} \) is cancelled by the 3! possible permutations of the LHS and add the factor \( i \) from \( e^{i S} \).

3. Gauge boson propagator in axial gauge.

The axial gauge condition is \( n^\mu A_\mu^a = 0 \) where \( n \) is a fixed vector.

a.) Show that the Fadeev-Popov term is independent of \( A_\mu^a \) and, thus, can be absorbed in the normalisation of the path integral.

b.) Derive the gauge boson propagator using

\[ \mathcal{L}_{gt} = \frac{1}{2\xi} (n^\mu A_\mu)^2 \]

as the corresponding gauge-fixing term.

0.) It may be useful to recall the case of the Maxwell equations. Choosing in \( \partial_\mu F^{\mu\nu} = \partial_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial_\mu \partial^\nu A^\mu = j^\nu \) a covariant gauge as e.g. the Lorenz gauge \( \partial_\mu A^\mu = 0 \) reduces the Solutions are discussed Thursday, 02.04.20
d.o.f. by one to three. We can still add to the potential \( A^\mu \) any \( \partial^\mu \chi \) satisfying \( \Box \chi = 0 \), setting e.g. \( A^0 = 0 \), giving \( \nabla \cdot A = 0 \). Now we have only 2 d.o.f., but explicit Lorentz invariance is lost. This choice is with \( n^\mu = (1,0,0,0) \) a special case of the axial gauges.

a.) We have to evaluate
\[
\det \left( \frac{\delta g^a}{\delta y^b} \right) \prod_{x,a} \delta (g^a)
\]
for the gauge condition \( g^a = n^\mu A^a_\mu = 0 \). An infinitesimal gauge transformation leads to
\[
\delta g^a = \delta(n^\mu A^a_\mu) = -n^\mu D^a_{\mu \nu} \partial^{\nu} = n^\mu (\delta^{ac} \partial_\mu - g f^{abc} A^b_\mu) \partial^c = n^\mu \partial_\mu \partial^a,
\]
where we used \( n^\mu A^a_\mu = 0 \) in the last step. Thus the Fadeev-Popov Lagrangian is independent of the gauge fields,
\[
\mathcal{L}_{FP} = -\bar{c} \delta g^a \delta \vartheta^b = -\bar{c} n^\mu \partial_\mu \vartheta,
\]
and changes only the normalisation of the generating functional.

b.) We have to consider only the quadratic terms, thus the non-abelian terms play no role, and we can suppress the group index \( a \).
\[
\mathcal{L}_{eff} = \mathcal{L}_{cl} + \mathcal{L}_{gf} = \frac{1}{2} A_\mu (\eta^{\mu \nu} \Box - \partial^{\mu} \partial^{\nu}) A_\nu + \frac{1}{2 \xi} A_\mu n^\mu n^\nu A_\nu = \frac{1}{2} A_\mu P^{\mu \nu} A_\nu.
\]
The propagator \( D^{\mu \nu} \) has to satisfy
\[
P_{\mu \nu} D^{\nu \lambda}(x-y) = \delta_\mu^\lambda \delta(x-y).
\]
FT gives \( P_{\mu \nu}(k)D^{\nu \lambda}(k) = \delta_\mu^\lambda \) with
\[
P^{\mu \nu}(k) = -\eta^{\mu \nu} k^2 + k^\mu k^\nu + \frac{1}{\xi} n^\mu n^\nu.
\]
Plugging the tensor decomposition
\[
D^{\mu \nu}(k) = A \eta^{\mu \nu} + B (n^\mu k^\nu + n^\nu k^\mu) + C k^\mu k^\nu + D n^\mu n^\nu
\]
into the ansatz and comparing the coefficients leads to
\[
A = -\frac{1}{k^2}, \quad B = -\frac{1}{nk} \frac{1}{k^2}, \quad C = -\frac{n^2 - \xi k^2}{(nk)^2 k^2}, \quad \text{and} \quad D = 0.
\]
Thus the gauge boson propagator in axial gauge is given by
\[
D^{\mu \nu}(k) = \frac{1}{k^2} \left[ -\eta^{\mu \nu} + \frac{1}{nk} (n^\mu k^\nu + n^\nu k^\mu) - \frac{n^2 - \xi k^2}{(nk)^2 k^2} k^\mu k^\nu \right].
\]

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