Exercise sheet 10

1. Christoffel symbols and the Ricci tensor

Find the Christoffel symbols and the Ricci tensor for the metric $\mathrm{d}l^2 = S(t)\,[B(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega]$.

We solve the Lagrange equations for a test particle,

$$L = B(r)\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\phi}^2)\,,$$

where we neglected the overall factor $S(t)$. Comparing the Lagrange equations

$$\ddot{r} - \frac{B'}{2B}\dot{r}^2 - (\dot{\vartheta}^2 + \sin^2\vartheta\dot{\phi}^2)\frac{r}{B} = 0\,, \quad \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\vartheta\dot{\vartheta}\dot{\phi} = 0\,, \quad \ddot{\vartheta} + \frac{2\dot{r}\dot{\vartheta}}{r} - \cos\vartheta\sin\vartheta\dot{\phi}^2 = 0$$

with the geodesic equation $\ddot{x}^\kappa + \Gamma^\kappa{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, we can read off the non-vanishing Christoffel symbols as

$$\Gamma^r{}_{rr} = -B'/(2B)\,, \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\vartheta/B \quad \text{and} \quad \Gamma^r{}_{\vartheta\vartheta} = -r/B$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\vartheta{}_{r\vartheta} = 1/r, \qquad \Gamma^\phi{}_{\phi\vartheta} = \cot\vartheta \quad \text{and} \quad \Gamma^\vartheta{}_{\phi\phi} = -\cos\vartheta\sin\vartheta.$$

Since the metric is diagonal, the non-diagonal elements of the Ricci tensor are zero too. We calculate with

$$R_{ab} = R^c{}_{acb} = \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ac} + \Gamma^c{}_{ab}\Gamma^d{}_{cd} - \Gamma^d{}_{bc}\Gamma^c{}_{ad}$$

for instance the rr component as

$$R_{rr} = 0 + \Gamma^c{}_{rr}\Gamma^d{}_{cd} - \Gamma^d{}_{rc}\Gamma^c{}_{rd} = \Gamma^r{}_{rr}(\Gamma^\phi{}_{r\phi} + \Gamma^\vartheta{}_{r\vartheta}) = -\frac{B'}{rB}. \tag{1}$$

Similarly, we find $R_{\vartheta\vartheta} = 1 + \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{1}{B}$ and $R_{\phi\phi} = \sin^2\vartheta R_{\vartheta\vartheta}$.

2. Redshift

Derive the redshift of a photon in the FLRW metric analogous to the redshift of a photon in the Schwarzschild metric, using the fact that homogenity leads to the existence of three space-like Killing vector fields. [For simplicity, restrict yourself to the flat case $k = 0$.]

The metric of the flat FLRW spacetime depends only on time, and admits therefore three space-like Killing vector fields, $\xi_x = (0,1,0,0)$, $\xi_x = (0,0,1,0)$, and $\xi_z = (0,0,0,1)$. With p as four-momentum of the photon, it follows $\xi_i \cdot p = \text{const}$. Consider e.g. the x component,

$$\xi_x \cdot p = g_{\mu\nu}\xi_x^\mu p^\mu = -a^2 p^x = \text{const}.$$

and thus $p^x \propto 1/a^2$. Next we use that p is a null vector,

$$(p^t)^2 - a^2(p^x)^2 = 0$$

or $p^t \propto ap^x \propto 1/a$. An observer at rest, $u = (1,0,0,0)$ measures thus as frequency

$$\omega = p \cdot u = p^t \propto 1/a$$

3. New galaxies.
As the universe expands, the horizon grows. *Estimate* the time it takes for one galaxy entering the horizon assuming the universe to be matter-dominated.

We need to estimate the amount of mass crossing the horizon per time, $\dot{M}_h(t)$. The time δt between new galaxies with mass $M_h \sim 10^{12} M_\odot$ enter our horizon can then be estimated as $\delta t \simeq M_{\rm gal}/\dot{M}_h$.
As first step, we estimate the mass inside the present horizon as

$$M_h(t_0) = \frac{4\pi}{3}\rho_m(t_0) d_h^3(t_0)$$

with ($k = 0$)

$$d_h(t) = a(t)\int_0^t \frac{\mathrm{d}t'}{a(t')}$$

Inserting and differentiating,

$$\frac{\mathrm{d}M_h(t)}{\mathrm{d}t} = \underbrace{\frac{4\pi}{3}\rho_m(t)a^3(t)}_{\text{const.}}\, 3\left(\int_0^t \frac{\mathrm{d}t'}{a(t')}\right)^2 \frac{1}{a(t)}$$

or

$$\frac{\mathrm{d}M_h(t)}{\mathrm{d}t} = 4\pi\rho_m(t) d_h^2(t)$$

For a matter-dominated universe, $d_h(t_0) = 2/H_0$ (and $d_h(t_0) \simeq 3.3/H_0$ for ΛCDM). With $\rho_m = \Omega_m \rho_{\rm cr}$ it follows

$$\dot{M}_h \simeq 4\pi\, \Omega_m \frac{3H_0^2}{8\pi G}\, \frac{10}{H_0^2}$$

i.e. the result is independendt of the expansion rate H_0. Finally,

$$\Delta t \simeq \frac{M_{\rm gal}}{\dot{M}_h} \simeq 0.05\text{yr}.$$

Thus ~ 20 galaxies enter each year the visible volume of the universe.

Solutions are discussed Thursday, 18.04.24