Exercise sheet 9

1. Energy losses.
a.) Find the time evolution of the energy, $E(t)$, of a particle suffering quadratic energy losses, $-\mathrm{d}E/\mathrm{d}t = bE^2$.
b.) Compare the energy density of a magnetic field with $B = 3\mu$G and of the CMB. Determine b.
c.) Make a $E(t)$ plot for a 0.1, 1, 10, 100 TeV electron. Assuming a diffusion coefficient $D(E) = D_0(E/E_0)^{1/3}$ with $D_0 = 5 \times 10^{26}\mathrm{cm}^2/\mathrm{s}$ and $E_0 = 10\,\mathrm{GeV}$, what do you conclude if 10 TeV electrons are observed at Earth?

a.) Integrating $-\mathrm{d}E/\mathrm{d}t = bE^2$ gives

$$E(t) = \frac{E_0}{1 + bE_0 t} = \frac{E_0}{1 + t/\tau_{1/2}}$$

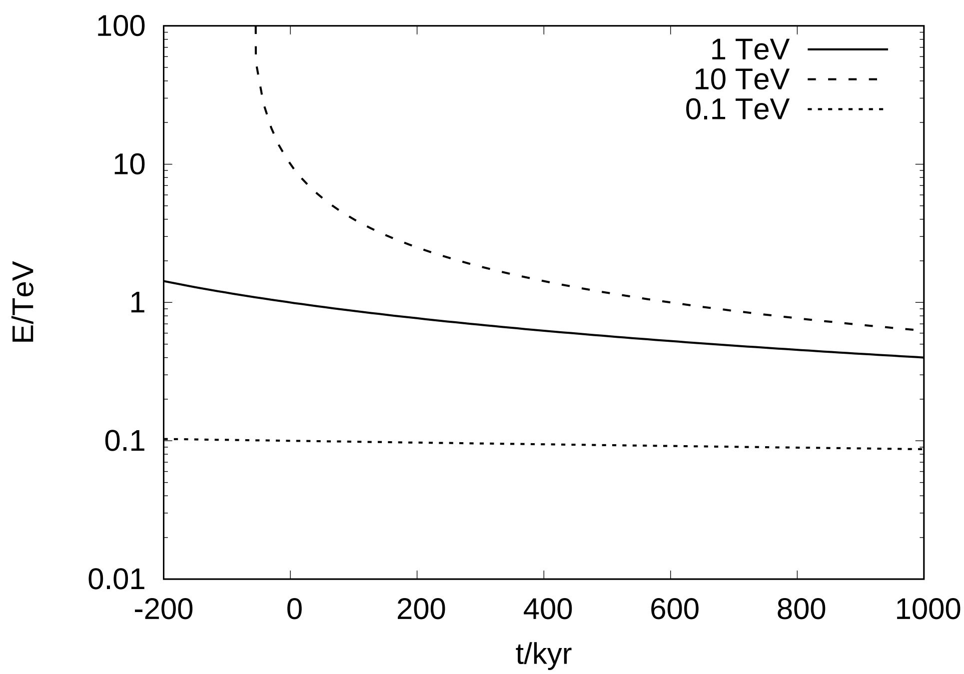
with $\tau_{1/2} = 1/(bE_0)$. Looking forward in time, $\tau_{1/2}$ is the time after which the energy of the particle is halved, $E(\tau_{1/2}) = E_0/2$.

b.) The energy density u of a uniform magnetic field B is

$$u = \frac{B^2}{8\pi} \simeq 0.45\,\frac{\mathrm{eV}}{\mathrm{cm}^3}\left(\frac{B}{3\mu\mathrm{G}}\right)^2 \tag{1}$$

The energy density $u = aT^4$ of CMB photons with temperature $T = 2.7\,\mathrm{K}$ is $u \simeq 0.45\,\mathrm{eV/cm}^3$. Thus

$$b = \frac{3}{4}\sigma_{\mathrm{Th}}\frac{cu}{(m_e c^2)^2} \simeq 5 \times 10^{-17}\frac{1}{\mathrm{GeV}\cdot\mathrm{s}} = 1.5 \times 10^{-3}\frac{1}{\mathrm{TeV}\cdot\mathrm{kyr}}.$$



c.) The intersting point to note is that $E(t)$ diverges for $t \to -\tau_{1/2}$. Thus we can interprete $\tau_{1/2}$ also as the earliest creation time of an electron observed with energy E_0 today ($t = 0$). For $E_0 = 10\,\mathrm{TeV}$, it is $\tau_{1/2} \simeq 70\,\mathrm{kyr}$.
If the particles diffuses (random walk in a turbulent magnetic field), then the distance travelled is $r = \langle x^2 \rangle^{1/2} = \sqrt{2D\tau_{1/2}}$. For simplicity, we ignore that $E = E(t)$ and evaluate $D(10\,\mathrm{TeV}) \simeq 5 \times 10^{27}\mathrm{cm}^2/\mathrm{s}$ and thus $r \simeq 50\,\mathrm{pc}$. Hence 10 TeV electrons should come from sources in our very local neighbourhood. [The diffusion equation including energy losses can be solved analytically. The result differs not much: the source should be within 100 pc.]

2. Charged pion decay.
A charged pion decays mainly via the reaction $\pi^\pm \to \mu^\pm + \nu_\mu$. Discuss this decay analogous to the decay of a neutral pions.

a.) Combining energy conservation $m_\pi = E_\nu + E_\mu$, smallness of neutrino masses $E_\nu = p_\nu$ and the cms condition $\boldsymbol{p}_\nu = -\boldsymbol{p}_\mu$ gives

$$(E_\mu - m_\pi)^2 = |\boldsymbol{p}_\mu|^2$$

$$\underbrace{E_\mu^2 - |\boldsymbol{p}_\mu|^2}_{=m_\mu^2} - 2E_\mu m_\pi + m_\pi^2 = 0$$

$$m_\mu^2 + m_\pi^2 = 2E_\mu m_\pi$$

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = \frac{m_\pi}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right) = \frac{m_\pi}{2}(1+r)$$

with $r \equiv m_\mu^2/m_\pi^2 \simeq 0.58$. The momentum follows as

$$|\boldsymbol{p}_\mu|^2 = E_\mu^2 - m_\mu^2 = \frac{m_\pi^2}{4}(1 + 2r + r^2) - \frac{4m_\pi^2 m_\mu^2}{4m_\pi^2} = \frac{m_\pi^2}{4}(1 - 2r + r^2)$$

or $|\boldsymbol{p}_\mu| = (m_\pi/2)(1-r)$.

b.) The energy in the lab system follows from the general Lorentz transformation $E' = \gamma(E + \beta p \cos\vartheta)$ with ϑ as the angle between the velocity β of the pion and the emitted muon. The maximal/minimal values of E' follow for $\cos\vartheta = \pm 1$, i.e. if the muon is emitted parallel and anti-parallel to the direction of flight of the pion.
Inserting $E = (m_\pi/2)(1+r)$ and $p = (m_\pi/2)(1-r)$ gives

$$E_{\min}^{\max} = \frac{\gamma m_\pi}{2}(1 + r \pm \beta(1-r)).$$

In the ultra-relativistic limit, $\beta \to 1$, and thus $E_{\max} \simeq \gamma m_\pi = E_\pi$ and $E_{\min} \simeq rE_\pi \simeq 0.58E_\pi$.

c.) In the rest-frame of the pion, the muon is emitted isotropically, $\mathrm{d}N/\mathrm{d}\Omega = 1/(4\pi)$, since there is no preferred direction. In a frame where the pion is moving, we differentiate $E' = \gamma(E + \beta p \cos\vartheta)$, obtaining

$$\mathrm{d}E' = \gamma\beta p \mathrm{d}(\cos\vartheta).$$

Thus

$$dN = \frac{1}{2}d(\cos\vartheta) = \frac{dE'}{2\gamma\beta p}$$

or

$$\frac{dN}{dE'} = \frac{1}{2\gamma\beta p}.$$

The energy distribution is again a flat box, with the boundaries found in b.)

3. Muon decay.
Consider the decay of the muon $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ (at rest): find the condition that the electron energy is maximal.

Denoting the four-momenta as $\mu^-(p_1) \to e^- + (p_2)\bar{\nu}_e(p_3) + \nu_\mu(p_4)$, it is

$$(p_1 - p_2)^2 = (p_3 + p_4)^2$$

Squaring and solving for E_e, it follows

$$E_e = \frac{m_\mu^2 + m_e^2 - (p_3 + p_4)^2}{2m_\mu}$$

Thus E_e is maximal, if

$$(p_3 + p_4)^2 = 2E_3E_4(1 - \cos\vartheta)$$

is minimal, or $\cos\vartheta = 1$. Hence the neutrino 3-momenta are parallel, opposite to the electron momentum.