Exercise sheet 2

1. Scalar product of time-like vectors.
Show that the scalar product of two time-like vectors can be expressed as $\boldsymbol{a}\cdot\boldsymbol{b}=ab\cosh\eta$, where η is the rapidity connecting the two frames where $a^\mu\equiv(a,\mathbf{0})$ and $\tilde{b}^\mu\equiv(b,\mathbf{0})$ are valid.

Performing a Lorentz transformation from the frame A where $\boldsymbol{a}=(a,\mathbf{0})$ to the frame B gives

$$\tilde{a}=(a\cosh\eta,a\sinh\eta,0,0)$$

where we assumed that the boost is in x^1 direction. Taking then the scalar product results in

$$\boldsymbol{a}\cdot\boldsymbol{b}=\eta_{\mu\nu}\tilde{a}^\mu\tilde{b}^\nu=ab\cosh\eta.$$

Remarks: i) The result is in line with the usual $\boldsymbol{a}\cdot\boldsymbol{b}=ab\cos\alpha$ and our interpretation of η as imaginary rotation angle. ii) Since $\cosh\eta\geq 1$, two time-like vectors cannot be orthogonal – as expected for two elements of an one-dimensional subspace.

2. Action of free relativistic particle.
Consider $S=\alpha\int d\tau$ as action for a free relativistic particle.
a.) Determine the constant α requiring the correct non-relativistic limit.
b.) Does a classically allowed path maximise or minimise the action?

a.) We ask that the action has the correct non-relativistic limit/ Then

$$S_0=\alpha\int_a^b ds=\alpha\int_a^b dt\,\sqrt{1-v^2}=\int_a^b dt\,\left(-m+\frac{1}{2}mv^2+\mathcal{O}(v^4)\right),\tag{1}$$

if we set $\alpha=-m$. The mass m corresponds to a potential energy in the non-relativistic limit and has therefore a negative sign in the Lagrangian. Moreover, a constant drops out of the equations of motion, and thus the term $-m$ can be omitted in the non-relativistic limit.

b.) Compare with Fig. 3.1 in the script and the corresponding discussion.

3. Uniformly accelerated observer.
Consider a particle moving on the x axis along a world-line parametrised by

$$t(\sigma)=\frac{1}{a}\sinh\sigma,\qquad x(\sigma)=\frac{1}{a}\cosh\sigma.$$

a.) Find the connection between σ and proper-time τ; express the world-line as function of τ.
b.) Calculate the four-velocity u^α and the three-velocity v^1 of the particle. Check their normalisation.

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c.) Calculate the four-acceleration a^α of the particle.
d.) Draw a spacetime diagram including $x^\mu(\sigma)$.

a.) We form the differential,

$$d\tau^2 = ds^2 = dt^2 - dx^2 = \left(\frac{1}{a}\cosh\sigma d\sigma\right)^2 - \left(\frac{1}{a}\sinh\sigma d\sigma\right)^2 = \left(\cosh^2\sigma - \sinh^2\sigma\right)\left(\frac{d\sigma}{a}\right)^2 = \left(\frac{d\sigma}{a}\right)^2.$$

Setting $\tau = \sigma/a$, i.e. choosing the constant to zero, it follows

$$t(\tau) = \frac{1}{a}\sinh(a\tau), \qquad x(\tau) = \frac{1}{a}\cosh(a\tau).$$

b.) We differentiate w.r.t. τ,

$$u^0 = \frac{dt}{d\tau} = \cosh(a\tau), \qquad u^1 = \frac{dx}{d\tau} = \sinh(a\tau),$$

Checking the normalisation gives $\boldsymbol{u}\cdot\boldsymbol{u} = \cosh^2(a\tau) - \sinh^2(a\tau) = 1$. The three-velocity is

$$v^1 = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{dt}{d\tau} = \frac{u^1}{u^0} = \tan(a\tau),$$

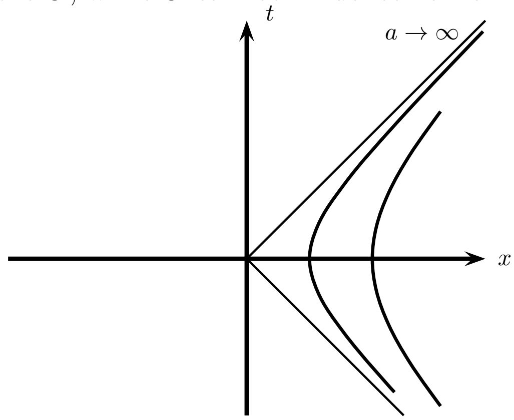
and thus bounded by one, $|v| \leq 1$: The particle has the velocity $v = -c$ for $\to -\infty$, decelerates to $v(t=0) = 0$, followed by acceleration to $v = c$ for $\to \infty$.

c.) We differentiate another time w.r.t. τ,

$$a^0 = \frac{du^0}{d\tau} = a\sinh(a\tau), \qquad a^1 = \frac{du^1}{d\tau} = a\cosh(a\tau),$$

The norm is $|\boldsymbol{a}\cdot\boldsymbol{a}|^{1/2} = -a$, thus the acceleration is constant.

d.) The worldline of an uniformly accelerated observers O; in the limit $a \to \infty$, the world-line become a triangle. Note the presence of 2 horizonts: One half of Minkowski space cannot influence the O, while O cannot influence half of Minkowski space.



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Extra: If you want some additional training, you can solve the inverse problem: Determine the trajectory of a particle which is uniformly accelerated in its rest-frame.
In the rest-frame of an uniformly accelerated observer, the four-acceleration is given by $a^\alpha = \ddot{x}^\alpha = (0, \boldsymbol{a})$ with $|\boldsymbol{a}| = a = \text{const}$. We can convert this condition into a covariant form, writing

$$\eta_{\alpha\beta}\ddot{x}^\alpha\ddot{x}^\beta = -a^2. \tag{2}$$

In order to determine the trajectory $x^\alpha(\tau)$ of the accelerated observer, it is convenient to change to light-cone coordinates,

$$u = t - x \quad \text{and} \quad v = t + x.$$

(We will suppress the transverse coordinates y and z.) Forming the differentials $\mathrm{d}u$ and $\mathrm{d}t$, we see that the line element in the new coordinates is $\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}v$. The normalisation condition $\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1$ of the four-velocity becomes therefore $\dot{u}\dot{v} = 1$, while the acceleration equation (2) results in $\ddot{u}\ddot{v} = -a^2$. Differentiating then $\dot{u} = 1/\dot{v}$, we obtain $\ddot{u} = -\ddot{v}/\dot{v}^2$ or

$$\frac{\ddot{v}}{\dot{v}} = \pm a.$$

Integrating results in

$$v(\tau) = \frac{A}{a}\exp(a\tau) + C$$

and, using $\dot{u} = 1/\dot{v}$, in

$$u(\tau) = -\frac{1}{Aa}\exp(-a\tau) + D.$$

Going back to the original Cartesian coordinates, we obtain

$$t(\tau) = \frac{1}{a}\sinh(a\tau) \quad \text{and} \quad x(\tau) = \frac{1}{a}\cosh(a\tau),$$

where we set the integration constants $A = 1$ and $C = D = 0$, which selects the trajectory with $t(0) = 0$ and $x(0) = 1/a$.

4. Infinitesimal Lorentz transformation.
Symmetry transformations form groups; continuous transformations in physics depend analytically on their parameters (e.g. as $\cos\vartheta$ and $\sin\vartheta$ on the rotation angle ϑ). An element g of such a group (called "Lie group") can be therefore expanded as a power series,

$$g(\vartheta) = 1 + \sum_{a=1}^{n} \mathrm{i}\vartheta^a T^a + \mathcal{O}(\vartheta^2) \equiv 1 + \mathrm{i}\vartheta^a T^a + \mathcal{O}(\vartheta^2). \tag{3}$$

The linear transformation in the arbitrary direction ϑ^a is called an infinitesimal transformation, the T^a the (infinitesimal) generators of the transformation. The generators T^a can be obtained by differentiation, $T^a = -\mathrm{i}\,\mathrm{d}g(\vartheta)/\mathrm{d}\vartheta^a|_{\vartheta=0}$. Conversely, analyticity implies that the group element $g(\vartheta)$ (more precisely, thus connected to the unity element) can be obtained by exponentiation,

$$g(\vartheta) = \lim_{n\to\infty}\left[1 + \mathrm{i}\vartheta^a T^a/n\right]^n \tag{4}$$

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a.) Calculate the generators of Lorentz transformations.
b.) Determine their "Lie algebra", i.e. calculate the real numbers f^{abc} called structure constants in

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c. \tag{5}$$

Applied to the finite boost $B_x(\eta)$ along the x direction given in (**??**) we find as generator K_x

$$B_x(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad K_x = \frac{1}{\mathrm{i}} \left.\frac{\partial B_x(\eta)}{\partial \eta}\right|_{\eta=0} = -\mathrm{i} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

and similarly for the other two boosts. The 4-dim. generators of rotations are obtained simply by adding $(1, 0, 0, 0)$ as zeroth colum and raw to the known 3-dim. rotations, e.g.

$$R_z(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad J_z(\alpha) = -\mathrm{i} \left.\frac{\partial R_z(\alpha)}{\partial \alpha}\right|_{\alpha=0} = -\mathrm{i} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

Calculating then their commutation relations, one finds

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk} J_k, \tag{8a}$$

$$[J_i, K_j] = \mathrm{i}\varepsilon_{ijk} K_k, \tag{8b}$$

$$[K_i, K_j] = -\mathrm{i}\varepsilon_{ijk} J_k. \tag{8c}$$

You should recognise in the first relation the anticommutation relation of the angular momentum operators; thus angular momentum are the generator of rotations. Moreover, their anticommutation relation are closed (i.e. contain no K_i). Thus rotations from a closed subgroup of the Lorentz group. This is not true for the boosts.

Remarks:

- The structure constant ε_{ijk} are those of SU(2). Introducing the linear combinations $\boldsymbol{J}^{\pm} = (\boldsymbol{J} \pm \mathrm{i}\boldsymbol{K})/2$, one finds that their mixed commutators are zero, while both $\boldsymbol{J}^{\pm}$ form a SU(2) group. Thus SO(1,3)$\sim$ SU(2)$\otimes$SU(2).

- You may have encountered SU(2) when you discussed rotations of Pauli spinors. Thus in addition of Lorentz transformations acting on tensors, we could construct Lorentz transformations acting on spinors using SU(2)$\otimes$SU(2).

- Writing $U = \exp(\mathrm{i}T^a\vartheta^a)$, we factored out an i. Then generators like J are Hermetian (i.e. $(J_z)^\dagger = (J_z)^{\mathrm{T},*} = J_z$) and U is an unitary transformation. An exception are boosts K: They cannot be implemented by unitary (finite-dimensional) matrices; physically, this is clear since the number density j^0 of particles is not Lorentz invariant.

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