Exercise sheet 2

Show that the scalar product of two time-like vectors can be expressed as $a \cdot b = ab \cosh \eta$, where $\eta$ is the rapidity connecting the two frames where $a^\mu \equiv (a, 0)$ and $b^\mu \equiv (b, 0)$ are valid.

Performing a Lorentz transformation from the frame $A$ where $a = (a, 0)$ to the frame $B$ gives
$$\tilde{a} = (\cosh \eta a, \sinh \eta a, 0, 0)$$
where we assumed that the boost is in $x^1$ direction. Taking then the scalar product results in
$$a \cdot b = \eta_{\mu\nu} \tilde{a}^\mu \tilde{b}^\nu = ab \cosh \eta.$$

Remarks: i) The result is in line with the usual $a \cdot b = ab \cos \alpha$ and our interpretation of $\eta$ as imaginary rotation angle. ii) Since $\cosh \eta \geq 1$, two time-like vectors cannot be orthogonal.

2. Relativity of simultaneity.
Draw a space-time diagram (in $d = 2$) for two inertial frames connected by a boost with velocity $\beta$: What are the angles between the axes $t$ and $t'$, $x$ and $x'$? Draw lines of constant $t$ and $t'$ and convince yourself that the time order of two space-like events is not invariant.

We parametrize a boost along the $x$ direction by
$$\tilde{t} = t \cosh \eta + x \sinh \eta, \quad (1)$$
$$\tilde{x} = t \sinh \eta + x \cosh \eta, \quad (2)$$
with $\tilde{y} = y$ and $\tilde{z} = z$.
We obtain the lines of constant $\tilde{t}$ (i.e. the $\tilde{x}$ axis and its parallels) in the $x - t$ plane by solving (1) for $t$,
$$t = -\beta x + \tilde{t}_0 \quad (3)$$
Now we recall that a straight-line with $y = mx$ has the angle $\tan \alpha = m$ to the $x$ axis. Thus the rotation angle in our case is given by $\eta$: Since $\beta \in [-1 : 1]$, $\tilde{x}$ axis has the angle between 0 and $45^\circ$ with the $x$ axis.
In the same way, the $\tilde{t}$ axis and its parallels follows from (2) as
$$x = -\beta^{-1} t + \tilde{x}_0. \quad (4)$$
Now $1/\beta$ is in the range $[-\infty : 1]$ and $[1 : \infty]$; the $\tilde{t}$ axis has thus the angle between 0 and $45^\circ$ to the $t$ axis.

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In the frame $(t', x')$ moving with $\beta = \tanh(\eta)$ relative to the frame $(t, x)$, the axes are rotated by $\eta$. Space-time events like $B$ that are inbetween $t = \text{const}$ and $t' = \text{const}$ have a different time order relative to $A$: $t'_B > t'_A$ and $t_A > t_B$. Since $|\eta| < 45^\circ$, time-like events are time-ordered.

Note that the $t', x'$ axes are orthogonal to each other (in space-time), although they are not plotted in the Euclidean geometry of $\mathbb{R}^2$.

### 3. Uniformly accelerated observer.

Consider a particle moving on the $x$ axis along a world-line parametrised by

\[ t(\sigma) = \frac{1}{a} \sinh \sigma, \quad x(\sigma) = \frac{1}{a} \cosh \sigma. \]

a.) Find the connection between $\sigma$ and proper-time $\tau$; express the world-line as function of $\tau$.

b.) Calculate the four-velocity $u^\alpha$ and the three-velocity $v^1$ of the particle. Check their normalisation.

c.) Calculate the four-acceleration $a^\alpha$ of the particle.

d.) Draw a space-time diagram.

a.)

\[
\begin{align*}
\text{d}r^2 &= \text{d}s^2 = \text{d}t^2 - \text{d}x^2 = \left(\frac{1}{a} \cosh \sigma \text{d}\sigma\right)^2 - \left(\frac{1}{a} \sinh \sigma \text{d}\sigma\right)^2 = \left(\cosh^2 \sigma - \sinh^2 \sigma\right) \left(\frac{\text{d}\sigma}{a}\right)^2 = \left(\frac{\text{d}\sigma}{a}\right)^2.
\end{align*}
\]

Setting $\tau = \sigma/a$, i.e. choosing the constant to zero, it follows

\[ t(\tau) = \frac{1}{a} \sinh(a\tau), \quad x(\tau) = \frac{1}{a} \cosh(a\tau). \]

The worldline of uniformly accelerated observers $O$; in the limit $a \rightarrow \infty$. Note the presence of 2 horizons: One half of Minkowski space cannot influence the future of $O$, while $O$ b.) We differentiate w.r.t. $\tau$,

\[ u^0 = \frac{\text{d}t}{\text{d}\tau} = \cosh(a\tau), \quad u^1 = \frac{\text{d}x}{\text{d}\tau} = \sinh(a\tau), \]

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Checking the normalisation gives $\mathbf{u} \cdot \mathbf{u} = \cosh^2(a\tau) - \sinh^2(a\tau) = 1$. The three-velocity is

$$v^1 = \frac{dx}{dt} = \frac{dx}{d\tau} \frac{d\tau}{dt} = \frac{u^1}{u^0} = \tan(a\tau),$$

and thus bounded by one, $|v| \leq 1$: The particle has the velocity $v = -c$ for $t \to -\infty$, decelerates to $v(t = 0) = 0$, followed by acceleration to $v = c$ for $t \to \infty$.

c.) We differentiate another time w.r.t. $\tau$,

$$a^0 = \frac{du^0}{d\tau} = a \sinh(a\tau), \quad a^1 = \frac{du^1}{d\tau} = a \cosh(a\tau),$$

The norm is $|\mathbf{a} \cdot \mathbf{a}|^{1/2} = a$, thus the acceleration is constant.

d.)

![Diagram](image.html)

Extra: If you need some extra training, you can solve the inverse problem: Determine the trajectory of a particle which is uniformly accelerated in its restframe.

In the rest-frame of an uniformly accelerated observer, the four-acceleration is given by $a^\alpha = \ddot{x}^\alpha = (0, a)$ with $|\mathbf{a}| = a = \text{const}$. We can convert this condition into a covariant form, writing

$$\eta_{\alpha\beta} \ddot{x}^\alpha \ddot{x}^\beta = -a^2. \quad (5)$$

In order to determine the trajectory $x^\alpha(\tau)$ of the accelerated observer, it is convenient to change to light-cone coordinates,

$$u = t - x \quad \text{and} \quad v = t + x.$$  

(We will suppress the transverse coordinates $y$ and $z$.) Forming the differentials $du$ and $dt$, we see that the line element in the new coordinates is $ds^2 = du dv$. The normalisation condition $\eta_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 1$ of the four-velocity becomes therefore $\dot{u} \dot{v} = 1$, while the acceleration equation (5) results in $\ddot{u} \ddot{v} = -a^2$. Differentiating then $\dot{u} = 1/\dot{v}$, we obtain $\ddot{u} = -\ddot{v}/\dot{v}^2$ or

$$\frac{\ddot{v}}{\dot{v}} = \pm a.$$

Integrating results in

$$v(\tau) = \frac{A}{a} \exp(a\tau) + C$$

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and, using \( \dot{u} = 1/\dot{v} \), in
\[
-u(\tau) = \frac{1}{Aa} \exp(-a\tau) + D.
\]

Going back to the original Cartesian coordinates, we obtain
\[
t(\tau) = \frac{1}{a} \sinh(a\tau) \quad \text{and} \quad x(\tau) = \frac{1}{a} \cosh(a\tau),
\]
where we set the integration constants \( A = 1 \) and \( C = D = 0 \), which selects the trajectory with \( t(0) = 0 \) and \( x(0) = 1/a \).

### 4. Infinitesimal Lorentz transformation.

Symmetry transformations form groups; continuous transformations in physics depend analytically on their parameters (e.g. as \( \cos \vartheta \) and \( \sin \vartheta \) on the rotation angle \( \vartheta \)). An element \( g \) of such a group (called "Lie group") can be expanded as a power series,
\[
g(\vartheta) = 1 + \sum_{a=1}^{n} i \vartheta^{a} T^{a} + \mathcal{O}(\vartheta^2) \equiv 1 + i \vartheta^{a} T^{a} + \mathcal{O}(\vartheta^2).
\]

The linear transformation in the arbitrary direction \( \vartheta^{a} \) is called an infinitesimal transformation, the \( T^{a} \) the (infinitesimal) generators of the transformation. The generators \( T^{a} \) can be obtained by differentiation, \( T^{a} = -i \partial g(\vartheta)/\partial \vartheta^{a} \mid_{\vartheta=0} \). Conversely, analyticity implies that the group element \( g(\vartheta) \) can be obtained by exponentiation,
\[
g(\vartheta) = \lim_{n \to \infty} [1 + i \vartheta^{a} T^{a}/n]^{n}
\]

a.) Calculate the generators of Lorentz transformations.
b.) Determine their “Lie algebra”, i.e. calculate the real numbers \( f^{abc} \) called structure constants in
\[
[T^{a}, T^{b}] = i f^{abc} T^{c}.
\]

Applied to the finite boost \( B_{x}(\eta) \) along the \( x \) direction given in (1) we find as generator \( K_{x} \)
\[
B_{x}(\eta) = \begin{pmatrix}
\cosh \eta & \sinh \eta & 0 & 0 \\
\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad K_{x} = \frac{1}{i} \frac{\partial B_{x}(\eta)}{\partial \eta} \mid_{\eta=0} = -i \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

and similarly for the other two boosts. The 4-dim. generators of rotations are obtained simply by adding \( (1,0,0,0) \) as zeroth column and row to the known 3-dim. rotations, e.g.
\[
R_{z}(\alpha) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha & 0 \\
0 & -\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad J_{z}(\alpha) = -i \frac{\partial R_{z}(\alpha)}{\partial \alpha} \mid_{\alpha=0} = -i \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

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Calculating then their commutation relations, one finds

\[
\begin{align*}
[J_i, J_j] &= i\varepsilon_{ijk} J_k, \\
[J_i, K_j] &= i\varepsilon_{ijk} K_k, \\
[K_i, K_j] &= -i\varepsilon_{ijk} J_k.
\end{align*}
\] (11a) (11b) (11c)

You should recognise in the first relation the anticommutation relation of the angular momentum operators; thus angular momentum are the generator of rotations. Moreover, their anticommutation relation are closed (i.e. contain no \(K_i\)). Thus rotations from a closed subgroup of the Lorentz group. This is not true for the boosts.

Remarks:

- The structure constant \(\varepsilon_{ijk}\) are those of SU(2). Introducing the linear combinations \(J^\pm = (J \pm iK)/2\), one finds that their mixed commutators are zero, while both \(J^\pm\) form a SU(2) group. Thus SO(1,3) ∼ SU(2) ⊗ SU(2).

- You may have encountered SU(2) when you discussed rotations of Pauli spinors. Thus in addition of Lorentz transformations acting on tensors, we could construct using SU(2) ⊗ SU(2) Lorentz transformations acting on spinors.

- Writing \(U = \exp(iT^a \varphi^a)\), we factored out an \(i\). Then generators like \(J\) are Hermetian (i.e. \((J_j)^\dagger = (J_j)^T, J_j\) and \(U\) is an unitary transformation. An exception are boosts \(K\). They cannot be implemented by unitary matrices; physically, this is clear since the number density \(j^0\) of particles is not Lorentz invariant.