Exercise sheet 2

1. Scalar product of time-like vectors.
Show that the scalar product of two time-like vectors can be expressed as $\boldsymbol{a} \cdot \boldsymbol{b} = ab \cosh \eta$, where η is the rapidity connecting the two frames where $a^\mu \equiv (a, \mathbf{0})$ and $\tilde{b}^\mu \equiv (b, \mathbf{0})$ are valid.

Performing a Lorentz transformation from the frame A where $\boldsymbol{a} = (a, \mathbf{0})$ to the frame B gives

$$\tilde{a} = (a \cosh \eta, a \sinh \eta, 0, 0)$$

where we assumed that the boost is in x^1 direction. Taking then the scalar product results in

$$\boldsymbol{a} \cdot \boldsymbol{b} = \eta_{\mu\nu} \tilde{a}^\mu \tilde{b}^\nu = ab \cosh \eta.$$

Remarks: i) The result is in line with the usual $\boldsymbol{a} \cdot \boldsymbol{b} = ab \cos \alpha$ and our interpretation of η as imaginary rotation angle. ii) Since $\cosh \eta \geq 1$, two time-like vectors cannot be orthogonal – as expected for two elements of an one-dimensional subspace.

2. Action of free relativistic particle.
Consider $S = \alpha \int d\tau$ as action for a free relativistic particle.
a.) Determine the constant α requiring the correct non-relativistic limit.
b.) Does a classically allowed path maximise or minimise the proper-time?

a.) We ask that the action has the correct non-relativistic limit/ Then

$$S_0 = \alpha \int_a^b ds = \alpha \int_a^b dt \sqrt{1 - v^2} = \int_a^b dt \left(-m + \frac{1}{2} m v^2 + \mathcal{O}(v^4) \right), \tag{1}$$

if we set $\alpha = -m$. The mass m corresponds to a potential energy in the non-relativistic limit and has therefore a negative sign in the Lagrangian. Moreover, a constant drops out of the equations of motion, and thus the term $-m$ can be omitted in the non-relativistic limit.

b.) Compare with Fig. 3.1 in the script and the corresponding discussion. Or, for Minkowski space, recall the twin paradox: Thus space-like geodesics maximise the proper-time

3. Uniformly accelerated observer.
Consider a particle moving on the x axis along a world-line parametrised by

$$t(\sigma) = \frac{1}{a} \sinh \sigma, \qquad x(\sigma) = \frac{1}{a} \cosh \sigma.$$

a.) Find the connection between σ and proper-time τ; express the world-line as function of τ.
b.) Calculate the four-velocity u^α and the three-velocity v^1 of the particle. Check their normalisation.

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c.) Calculate the four-acceleration a^α of the particle.

d.) Draw a spacetime diagram including $x^\mu(\sigma)$.

a.) We form the differential,

$$d\tau^2 = ds^2 = dt^2 - dx^2 = \left(\frac{1}{a}\cosh\sigma d\sigma\right)^2 - \left(\frac{1}{a}\sinh\sigma d\sigma\right)^2 = \left(\cosh^2\sigma - \sinh^2\sigma\right)\left(\frac{d\sigma}{a}\right)^2 = \left(\frac{d\sigma}{a}\right)^2.$$

Setting $\tau = \sigma/a$, i.e. choosing the integration constant to zero, it follows

$$t(\tau) = \frac{1}{a}\sinh(a\tau), \qquad x(\tau) = \frac{1}{a}\cosh(a\tau).$$

b.) We differentiate w.r.t. τ,

$$u^0 = \frac{dt}{d\tau} = \cosh(a\tau), \qquad u^1 = \frac{dx}{d\tau} = \sinh(a\tau),$$

Checking the normalisation gives $\boldsymbol{u} \cdot \boldsymbol{u} = \cosh^2(a\tau) - \sinh^2(a\tau) = 1$. The three-velocity is

$$v^1 = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{dt}{d\tau} = \frac{u^1}{u^0} = \tan(a\tau),$$

and thus bounded by one, $|v| \leq 1$: The particle has the velocity $v = -c$ for $\to -\infty$, decelerates to $v(t = 0) = 0$, followed by acceleration to $v = c$ for $\to \infty$.

c.) We differentiate another time w.r.t. τ,

$$a^0 = \frac{du^0}{d\tau} = a\sinh(a\tau), \qquad a^1 = \frac{du^1}{d\tau} = a\cosh(a\tau),$$

The norm is $|\boldsymbol{a} \cdot \boldsymbol{a}|^{1/2} = -a$, thus the acceleration is constant.

d.) The worldline of an uniformly accelerated observers O; in the limit $a \to \infty$, the world-line become a triangle. Note the presence of 2 horizonts: One half of Minkowski space cannot influence the O, while O cannot influence half of Minkowski space.

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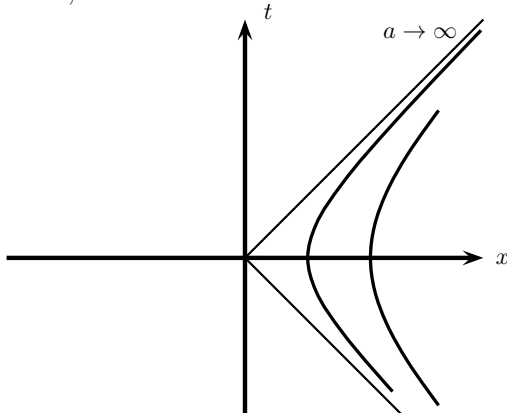
Extra: If you want some additional training, you can solve the inverse problem: Determine the trajectory of a particle which is uniformly accelerated in its rest-frame.

In the rest-frame of an uniformly accelerated observer, the four-acceleration is given by $a^\alpha = \ddot{x}^\alpha = (0, \boldsymbol{a})$ with $|\boldsymbol{a}| = a = \text{const}$. We can convert this condition into a covariant form, writing

$$\eta_{\alpha\beta}\ddot{x}^\alpha\ddot{x}^\beta = -a^2. \tag{2}$$

In order to determine the trajectory $x^\alpha(\tau)$ of the accelerated observer, it is convenient to change to light-cone coordinates,

$$u = t - x \quad \text{and} \quad v = t + x.$$

(We will suppress again the transverse coordinates y and z.) Forming the differentials du and dt, we see that the line element in the new coordinates is $ds^2 = dudv$. The normalisation condition $\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1$ of the four-velocity becomes therefore $\dot{u}\dot{v} = 1$, while the acceleration equation (2) results in $\ddot{u}\ddot{v} = -a^2$. Differentiating then $\dot{u} = 1/\dot{v}$, we obtain $\ddot{u} = -\ddot{v}/\dot{v}^2$ or

$$\frac{\ddot{v}}{\dot{v}} = \pm a.$$

Integrating results in

$$v(\tau) = \frac{A}{a}\exp(a\tau) + C$$

and, using $\dot{u} = 1/\dot{v}$, in

$$u(\tau) = -\frac{1}{Aa}\exp(-a\tau) + D.$$

Going back to the original Cartesian coordinates, we obtain

$$t(\tau) = \frac{1}{a}\sinh(a\tau) \quad \text{and} \quad x(\tau) = \frac{1}{a}\cosh(a\tau),$$

where we set the integration constants $A = 1$ and $C = D = 0$, which selects the trajectory with $t(0) = 0$ and $x(0) = 1/a$.

4. Lorentz group.

Lorentz transformations $\Lambda \in O(1,3)$ are all those coordinate transformations $x^\mu \to \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ that keep the norm $\eta_{\mu\nu}x^\mu x^\nu$ of space-time points x^μ invariant. Apart from rotations and boosts which are continuously connected to the unit element, there are other, unconnected pieces of the Lorentz group. Determine these unconnected pieces by considering the relation $\eta = \Lambda^T\eta\Lambda$. What is their meaning? [Hint: A relation like $[f(\Lambda)]^2 = 1$ shows that the Lorentz group consists of at least two disconnected pieces, one with $f = 1$, another one with $f = -1$.]

We rewrite first the fact that the metric tensor $\eta_{\mu\nu}$ is invariant under Lorentz transformations, $\tilde{\eta}_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$, in matrix form, $\eta = \Lambda^T\eta\Lambda$. Taking then the determinant, we obtain

$$(\det\Lambda)^2 = 1 \quad \text{or} \quad \det\Lambda = \pm 1. \tag{3}$$

Thus O(1,3) consists of at least two disconnected pieces. The part with $\det\Lambda = 1$ contains the proper Lorentz transformations and is called SO(1,3). The second part contains the improper

Lorentz transformations that can be written as the product of a proper transformation and a discrete transformation changing the sign of an odd number of coordinates.

Next we consider the 00 component of the equation $\eta = \Lambda^T \eta \Lambda$. With $\eta_{00} = \eta_{\mu\nu}\Lambda^\mu{}_0\Lambda^\nu{}_0$, it is

$$1 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3}(\Lambda^i{}_0)^2. \tag{4}$$

Thus $(\Lambda^0{}_0)^2 \geq 1$ and both the proper and the improper Lorentz transformations consist of two disconnected pieces. The transformations with $\Lambda^0{}_0 \geq 1$ are called orthochronous since they do not change the direction of time. In contrast, transformations with $\Lambda^0{}_0 \leq -1$ transform a future-directed time-like vector into a past-directed one and vice versa, and are therefore called antichronous. These properties are Lorentz invariant, and thus we can split Lorentz transformations into four categories:

- proper, orthochronous (= restricted) transformations $L \in \mathfrak{L}^\uparrow_+$ with $\det\Lambda = 1$ and $\Lambda^0{}_0 \geq 1$,

- proper, antichronous transformations $TPL \in \mathfrak{L}^\downarrow_+$ with $\det\Lambda = 1$ and $\Lambda^0{}_0 \leq -1$,

- improper, orthochronous transformation $PL \in \mathfrak{L}^\uparrow_-$ with $\det\Lambda = -1$ and $\Lambda^0{}_0 \geq 1$,

- improper, antichronous transformations $TL \in \mathfrak{L}^\downarrow_-$ with $\det\Lambda = -1$ and $\Lambda^0{}_0 \leq -1$.

Only those Lorentz transformations that are elements of the restricted Lorentz group, $L \in \mathfrak{L}^\uparrow_+$, are connected smoothly to the identity and can thus be built up from infinitesimal transformations. The other three disconnected pieces of the Lorentz group can be obtained as the product of an element $L \in \mathfrak{L}^\uparrow_+$ and additional discrete P and T transformations with $P(x^0, \boldsymbol{x}) = (x^0, -\boldsymbol{x})$ and $T(x^0, \boldsymbol{x}) = (-x^0, \boldsymbol{x})$.

Remark: No need to memorize the names. The important point to note is that an arbitrary Lorentz transformation can be obtained by combining those connected to the unity, plus a discrete parity and/or time-inversion transformation.

Solutions are discussed Thursday, 23.01.25