Exercise sheet 3

Scalar fields.
The most general expression for the Lagrange density $\mathcal{L}$ of $N$ scalar fields $\phi_i$ which is Lorentz invariant and at most quadratic in the fields is

$$\mathcal{L} = \frac{1}{2} A_{ij} \partial^\mu \phi_i \partial_\mu \phi_j - \frac{1}{2} B_{ij} \phi_i \phi_j - C.$$  

What constraints have the coefficients $A_{ij}, B_{ij}, C$ to satisfy? Show that $\mathcal{L}$ can be recast into “canonical form”

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi_i \partial_\mu \phi_i - \frac{1}{2} b_i \phi_i \phi_i - C$$

by linear field redefinitions.

$\phi_i \phi_j$ and $\partial^\mu \phi_i \partial_\mu \phi_j$ are symmetric, and thus a possible antisymmetric part of $A$ and $B$ drops out. Conservation of probability requires that the Hamiltonian density $\mathcal{H}$ is Hermitian. Thus $A, B$ and $C$ should be real. The energy should bounded from below, and thus $A$ and $B$ are (semi-) positive definite matrices: Their eigenvalues are $\geq 0$ and thus we can form $A^{1/2}$. Then we redefine the fields as

$$\tilde{\phi}_i = (A^{1/2} \phi)_i$$

which makes the kinetic term diagonal, while the mass term becomes

$$\tilde{B}_{ij} = (A^{-1/2} BA^{-1/2})_{ij}.$$  

Next we diagonalise $\tilde{B}$ by an orthogonal transformation, $\phi' = O \tilde{\phi}$, and $O^\top \tilde{B} O = \text{diag}(b_1, \ldots, b_n)$ with $b_i \equiv m_i^2 \geq 0$.

Note that it is sufficient to consider only linear transformations which maintain second-order equations of motion.

Maxwell Lagrangian.
a.) Derive the Lagrangian for the photon field $A_\mu$ from the source-free Maxwell equation $\partial_\mu F^{\mu\nu} = 0$. (Compare to the procedure for a scalar field in the notes.) b.) What is the meaning of the unused set of Maxwell equations?

a.) We multiply the free field equation $\partial_\mu F^{\mu\nu} = 0$ by a variation $\delta A$ that vanishes on the boundary $\partial \Omega$ of $\Omega = V \times [t_a : t_b]$. Then we integrate over $\Omega$, and perform a partial integration,

$$\int_\Omega d^4x \, \partial_\mu F^{\mu\nu} \delta A_\nu = - \int_\Omega d^4x \, F^{\mu\nu} \delta (\partial_\mu A_\nu) = 0. \quad (1)$$

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Next we note that
\[ (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = 2(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\partial^\alpha A^\beta \]  
(2)
and thus
\[ F^{\mu\nu} \delta(\partial_\mu A_\nu) = \frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu}. \]  
(3)
Applying the product rule, we obtain as final result
\[ -\frac{1}{4} \delta \int d^4x \ F_{\mu\nu} F^{\mu\nu} = \delta S[A_\mu] = 0 \]  
(4)
and thus
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \]  
(5)
Note that we expressed \( \mathcal{L} \) through \( F \), but \( \mathcal{L} \) should be viewed nevertheless as function of \( A \). In order to convince us that the sign of the Lagrangian is correct, we would have to calculate the Hamiltonian density \( \mathcal{H} \) and to show that it is bounded from below.

b.) The other (homogeneous) set of Maxwell equations corresponds to
\[ \partial_\alpha F^{\beta\gamma} + \partial_\beta F^{\gamma\alpha} + \partial_\gamma F^{\alpha\beta} = 0. \]  
(6)
They are constraints: if e.g. \( \nabla \cdot \textbf{B} = 0 \) is satisfied at one time, it holds for all times. Introducing the dual field-strength tensor \( F^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \) with \( \varepsilon^{\alpha\beta\gamma\delta} \) as the completely anti-symmetric tensor, we can rewrite this (noting that all three terms are even permutations of the indices) as
\[ \partial_\alpha \tilde{F}^{\alpha\beta} = 0. \]  
(7)
We can view Eqs. (6-7) as the condition which allows us the express the field-strength tensor by the potential, \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \). Having done this, any \( A_\mu \) satisfies Eq. (6),
\[ \partial_\alpha \tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha F_{\gamma\delta} = \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\gamma A_\delta = 0, \]  
(8)
since we contract a symmetric tensor \( (\partial_\alpha \partial_\gamma) \) with an anti-symmetric one \( (\varepsilon^{\alpha\beta\gamma\delta}) \).
This is analogue to the familiar case of a “conservative” vector field \( \textbf{V} \): If \( \textbf{V} \) is rotation-free, we can express it as a gradient of a scalar \( \phi \). On the other hand, any vector field \( \partial_\mu \phi \) is rotation-free.

Green functions.
Show that the connected and the unconnected \( n \)-point Green functions are identical for \( n = 2 \),
\[ G(x_1, x_2) = \mathcal{G}(x_1, x_2), \]
while they differ in general for \( n \geq 4 \). [Recall that \( \langle 0 | 0 \rangle = 1 \) and \( \langle 0 | \phi(x) | 0 \rangle = 0 \].

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We start from the definition of the connected 2-point Green function and perform the differentiations,

\[
G(x_1, x_2) = \left. \frac{1}{\hbar^2} \frac{\delta^2 \ln Z[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0} = \left. \frac{1}{\hbar^2} \frac{\delta}{\delta J(x_1)} \left[ 1 - \frac{\delta Z[J]}{Z[J]} \right] \right|_{J=0} = \left. \frac{1}{\hbar^2} \left[ -\frac{\delta Z[J]}{Z[J]^2} \delta J(x_1) \delta J(x_2) + \frac{1}{Z[J]} \delta Z[J] \right] \right|_{J=0}.
\] (9)

The Green functions agree, \( G(x_1, x_2) = G(x_1, x_2) \), if \( \delta Z[J] / \delta J(x) \big|_{J=0} = 0 \) vanishes and \( Z[0] = 1 \). The latter holds, since we require \( \langle 0 | 0 \rangle = 1 \). The former holds, because the vacuum should be empty, \( \langle 0 | \phi(x) | 0 \rangle = 0 \). Taking further derivatives, the extra terms will not vanish anymore for \( J = 0 \), and thus in general \( G(x_1, \ldots, x_n) \neq G(x_1, \ldots, x_n) \) for \( n > 3 \).

**Volume of a sphere in arbitrary dimensions.**

a.) Calculate the volume of the unit sphere \( S^{n-1} \) defined by \( x_1^2 + \ldots + x_n^2 = 1 \) in \( \mathbb{R}^n \).

b.) Generalize the result to arbitrary (not necessarily integer) dimensions and show that it agrees with the familiar results for \( n = 1, 2 \) and 3. (Recall that mathematicians distinguish between a sphere \( S^n \) and a ball \( B^n \).)

In usual language, the “volume of the unit sphere \( S^{n-1} \)” means the volume enclosed by the unit sphere \( S^{n-1} \), or more precisely the volume of the ball \( B^n \) defined by \( x_1^2 + \ldots + x_n^2 \leq 1 \).

Two standard methods are the following ones:

1. The volume \( \text{vol}(B^n(R)) \equiv V_{B^n}(R) \) of a ball \( B^n(R) \) with radius \( R \) is connected by \( V_{B^n}(R) = R^n V_{B^n}(1) \) to the volume of the unit ball \( B^n \). Thus \( dV_{B^n}(R) = n R^{n-1} V_{B^n}(1) dR \). With

\[
I_n = \int \prod_{i=1}^{n} dy_i = \prod_{i=1}^{n} \exp\left( -\sum_i y_i^2 \right) = \pi^{n/2},
\] (10)

\[
= n V_{B^n}(1) \int_0^\infty dR \ R^{n-1} \exp(-R^2)
\] (11)

and the substitution \( t = R^2 \) we can reduce the integral to a Gamma function\(^1\), \( \Gamma(n/2) / 2 \). Solving then for the volume of a ball \( B^n \) gives

\[
V_{B^n}(1) = \frac{2 \pi^{n/2}}{n \Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}.
\]

2. Starting from \( V_{B^1}(1) = 2 \) one sees that

\[
V_{B^2}(1) = V_{B^1}(1) \int_{-1}^{1} dt \ (1 - t^2)^{1/2}
\]

\(^1\Gamma(x + 1) = \int_0^\infty dt \ e^{-t^2}

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and generally
\[ V_{B^n}(1) = V_{B^{n-1}}(1) \int_{-1}^{1} \frac{dt}{1 - t^2} \frac{(n-1)!}{2} \cdot \left( n \right) . \]

Solving the integrals by partial integrations leads to recurrence relations, which in turn leads to the desired formula for the volume. Expressing then the (double) factorials by Gamma functions, the formula become valid for arbitrary \( n \)

We proceed to the for us more interesting volume of a sphere \( S^{n-1} \). The volume of a shell between \( r \) and \( r + \, d\, r \) equals \( V_{S^{n-1}}(R) \, dR \). Moreover, the volume \( \text{vol}(S^{n-1}(R)) \equiv V_{S^{n-1}}(R) \) of a sphere \( S^{n-1}(R) \) with radius \( R \) is connected by \( V_{S^{n-1}}(R) = R^{n-1} V_{S^{n-1}}(1) \) to the volume of the unit sphere \( S^{n-1} \). Then
\[ I_n = \int_0^\infty \frac{dR}{R} V_{S^{n-1}}(R) \exp(-R^2) = V_{S^{n-1}}(1) \int_0^\infty \frac{dR}{R} \, R^{n-1} \exp(-R^2) \quad (12) \]
and a comparison gives now
\[ V_{S^{n-1}}(1) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad (13) \]

In order to compute the volume of the unit spheres \( S^n \) with \( n = 1, \ldots, 3 \), we have to determine the required values of the Gamma function. Starting from \( \Gamma(1) = 1 \) and \( \Gamma(1/2) = \sqrt{\pi} \), we find the other values from the recurrence relation
\[ \Gamma(x + 1) = x\Gamma(x) \quad (14) \]

Then it is \( \Gamma(n) = (n - 1)! \), \( \Gamma(3/2) = \sqrt{\pi}/2 \), and \( \Gamma(5/2) = 3\sqrt{\pi}/4 \).

Thus \( V(S^1) = 2\pi/\Gamma(1) = 2\pi \), \( V(S^2) = 2\pi^{3/2}/\Gamma(3/2) = 4\pi \), and \( V(S^3) = 2\pi^2/\Gamma(2) = 2\pi^2 \), while \( V(B^2) = \pi/\Gamma(2) = \pi \), \( V(B^3) = \pi^{3/2}/\Gamma(5/2) = 4\pi/3 \).

The value \( V(S^3) = 2\pi^2 \) corresponds to the value of the angular integral in a four-dimensional integral.

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