## Exercise sheet 3

## 1. Measurements.

The universe is filled with cosmic microwave background (CMB) photons. Assume for simplicity that they have a single energy $\omega_{0}$. Which energy measures the uniformly accelerated observer from the last exercise sheet at time $\tau$ ?
a.) For an observer at rest, the time-like basis vector $\boldsymbol{e}_{0}(\tau)$ agrees with its four-velocity $\boldsymbol{u}_{\text {obs }}$ of the observer. A measurement of a particle with four-momentum $k^{\mu}=(\omega, \boldsymbol{k})$ performed by the observer at rest results in the energy $\omega$. We can rewrite this as tensor equations,

$$
\begin{equation*}
\omega=\boldsymbol{k} \cdot \boldsymbol{u}_{\mathrm{obs}} \tag{1}
\end{equation*}
$$

and thus the RHS is valid also for a moving observer.
b.) In exercise sheet 2, we found as the four-velocity of an observer accelerated along the $x$ axis

$$
u^{0}=\frac{\mathrm{d} t}{\mathrm{~d} \tau}=\cosh (a \tau), \quad u^{1}=\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\sinh (a \tau)
$$

For the photon, we have

$$
k^{\alpha}=\left(\omega_{0}, \omega_{0} \boldsymbol{n}\right)
$$

where $\boldsymbol{n}$ is a unit 3 -vector. Thus the observed frequency is

$$
\omega=\boldsymbol{k} \cdot \boldsymbol{u}_{\mathrm{obs}}=\omega_{0}[\cosh (a \tau)-\sinh (a \tau) \cos \vartheta]
$$

where $\vartheta$ is angle between the $x$ axis and the photon 3 -momentum. In particular, for $\vartheta=0$ and $180^{\circ}$, it follows

$$
\omega=\omega_{0} \exp (-a \tau) \quad \text { and } \quad \omega=\omega_{0} \exp (+a \tau)
$$

Thus for $\tau<0$ and $\vartheta=0$ (photons and observer approaching), the frequency is blue-shifted, while for $\tau>0$ and $\vartheta=0$ (photons and observer moving away), the frequency is redshifted. The minimal and maximal photon energies measured follow as

$$
\omega \in\left[\omega_{0} \mathrm{e}^{-a|\tau|}: \omega_{0} \mathrm{e}^{a|\tau|}\right] .
$$

## 2. Line-element.

Show that the line-element

$$
\mathrm{d} s^{2}=\mathrm{d} t^{2}-2 \mathrm{~d} x \mathrm{~d} t-\mathrm{d} y^{2}-\mathrm{d} z^{2}
$$

corresponds to a flat spacetime.
Option 1: Calculate Christoffel symbols. If they are zero, the spacetime is flat. If they are nonzero, you have to calculate additionally the curvature - something we still have to learn how to do.

Option 2: Find a coordinate transformation to flat space: We can eliminate the mixed terms setting

$$
t=t^{\prime}+x^{\prime}, \quad x=x^{\prime}, \quad y=y^{\prime}, \quad z=z^{\prime},
$$

what gives

$$
\begin{aligned}
\mathrm{d} s^{2} & =\left(\mathrm{d} t^{\prime}+\mathrm{d} x^{\prime}\right)^{2}-2 \mathrm{~d} x^{\prime}\left(\mathrm{d} t^{\prime}+\mathrm{d} x^{\prime}\right)-\mathrm{d} y^{\prime 2}-\mathrm{d} z^{\prime 2} \\
& =\mathrm{d} t^{\prime 2}-d x^{\prime 2}-\mathrm{d} y^{\prime 2}-\mathrm{d} z^{\prime 2} .
\end{aligned}
$$

## 3. Cylinder coordinates I.

Calculate for cylinder coordinates $x=(\rho, \phi, z)$ in $\mathbb{R}^{3}$

$$
\begin{aligned}
x_{1}^{\prime} & =\rho \cos \phi, \\
x_{2}^{\prime} & =\rho \sin \phi, \\
x_{3}^{\prime} & =z,
\end{aligned}
$$

the basis vectors $\boldsymbol{e}_{i}$, the components of $g_{i j}$ and $g^{i j}$, and $g \equiv \operatorname{det}\left(g_{i j}\right)$.
From $\boldsymbol{e}_{i}=\partial x^{\prime j} / \partial x^{i} \boldsymbol{e}_{j}^{\prime}$, it follows

$$
\begin{aligned}
& \boldsymbol{e}_{1}=\frac{\partial x_{j}}{\partial \rho} \boldsymbol{e}_{j}^{\prime}=\cos \phi \boldsymbol{e}_{1}^{\prime}+\sin \phi \boldsymbol{e}_{2}^{\prime}=\boldsymbol{e}_{\rho}, \\
& \boldsymbol{e}_{2}=\frac{\partial x_{j}}{\partial \phi} \boldsymbol{e}_{j}^{\prime}=-\rho \sin \phi \boldsymbol{e}_{1}^{\prime}+\rho \cos \phi \boldsymbol{e}_{2}^{\prime}=\rho \boldsymbol{e}_{\phi} . \\
& \boldsymbol{e}_{3}=\frac{\partial x_{j}}{\partial z} \boldsymbol{e}_{j}^{\prime}=\boldsymbol{e}_{3}^{\prime} .
\end{aligned}
$$

Since the $\boldsymbol{e}_{i}$ are orthogonal to each other, the matrices $g_{i j}$ and $g^{i j}$ are diagonal. From the definition $g_{i j}=\boldsymbol{e}_{i} \cdot \boldsymbol{e}_{j}$ one finds $g_{i j}=\operatorname{diag}\left(1, \rho^{2}, 1\right)$ Inverting $g_{i j}$ gives $g^{i j}=\operatorname{diag}\left(1, \rho^{-2}, 1\right)$. The determinant is $g=\operatorname{det}\left(g_{i j}\right)=\rho^{2}$. Note that the volume integral in cylindrical coordinates is given by

$$
\int d^{3} x^{\prime}=\int d^{3} x J=\int d^{3} x \sqrt{g}=\int \mathrm{d} \rho \mathrm{~d} \phi \mathrm{~d} z \rho,
$$

since $g_{i j}=\frac{\partial x^{k \prime}}{\partial \hat{x}^{\prime}} \frac{\partial x^{l \prime}}{\partial \tilde{x}^{\prime}} g_{k l}^{\prime}$ and thus $\operatorname{det}(g)=J^{2} \operatorname{det}\left(g^{\prime}\right)=J^{2}$ with $\operatorname{det}\left(g^{\prime}\right)=1$.

## 4. Hyperbolic plane $H^{2}$

The line-element of the hyperbolic plane $H^{2}$ is given by

$$
\mathrm{d} s^{2}=y^{-2}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}\right) \quad \text { and } \quad y \geq 0 .
$$

a.) Show that points on the $x$-axis are an infinite distance from any point $(x, y)$ in the upper plane. [The length $s$ of a line between $a$ and $b$ along $x$ is given by $s=\int_{a}^{b} \sqrt{g_{x x}}$.] b.) Deduce the Christoffel symbols $\Gamma^{a}{ }_{b c}$.
c.) Write out the geodesic equations and solve them to find $x$ and $y$ as function of the length $s$ of these curves.
a.) The length of a $x=$ const. line is

$$
s=\int \mathrm{d} y \sqrt{g_{y y}}=\int \frac{\mathrm{d} y}{y}
$$

what diverges if the lower integration limit approaches zero.
b.) Using as Lagrange function $L$ the kinetic energy $T$ instead of the line-element $\mathrm{d} s$ makes calculations a bit shorter. From $L=y^{-2}\left(\dot{x}^{2}+\dot{y}^{2}\right)$ we find

$$
\begin{array}{rll}
\frac{\partial L}{\partial x}=0 & , & \frac{\mathrm{~d}}{\mathrm{~d} t} \frac{\partial L}{\partial \dot{x}}=\frac{\mathrm{d}}{\mathrm{~d} t}\left(2 \dot{x} y^{-2}\right)=2 \ddot{x} y^{-2}-4 \dot{x} y^{-3} \dot{y} \\
\frac{\partial L}{\partial y}=-\frac{2}{y^{3}}\left(\dot{x}^{2}+\dot{y}^{2}\right) & , & \frac{\mathrm{d}}{\mathrm{~d} t} \frac{\partial L}{\partial \dot{y}}=\frac{\mathrm{d}}{\mathrm{~d} t}\left(2 \dot{y} y^{-2}\right)=2 \ddot{y} y^{-2}-4 \dot{y} y^{-3} \dot{y}
\end{array}
$$

and thus the solutions of the Lagrange equations are

$$
\ddot{x}-2 \frac{\dot{x} \dot{y}}{y}=0 \quad \text { and } \quad \ddot{y}-\frac{\dot{y}^{2}}{y}+\frac{\dot{x}^{2}}{y}=0 .
$$

Comparing with the given geodesic equation, we read off the non-vanishing Christoffel symbols as $-\Gamma^{x}{ }_{x y}=-\Gamma^{x}{ }_{y x}=\Gamma^{y}{ }_{x x}=-\Gamma^{y}{ }_{y y}=1 / y$. (Remember that $-2 y^{-1} \dot{x} \dot{y}=\Gamma^{x}{ }_{x y} \dot{x} \dot{y}+\Gamma^{x}{ }_{x y} \dot{x} \dot{y}$.)
c.) The geodesic equations written more explicitly are

$$
\frac{\mathrm{d}^{2} x}{\mathrm{~d} s^{2}}=\frac{2}{y} \frac{\mathrm{~d} x}{\mathrm{~d} s} \frac{\mathrm{~d} y}{d s}
$$

and

$$
\frac{\mathrm{d}^{2} y}{\mathrm{~d} s^{2}}=\frac{1}{y}\left(\frac{\mathrm{~d} x}{\mathrm{~d} s}\right)^{2}-\frac{1}{y}\left(\frac{\mathrm{~d} y}{d s}\right)^{2} .
$$

We can rewite the first equation as

$$
y^{2} \frac{1}{\mathrm{~d} s}\left(\frac{1}{y^{2}} \frac{\mathrm{~d} x}{\mathrm{~d} s}\right)=-\frac{2}{y} \frac{\mathrm{~d} y}{\mathrm{~d} s} \frac{\mathrm{~d} x}{\mathrm{~d} s}+\frac{\mathrm{d}^{2} y}{\mathrm{~d} s^{2}}=0 .
$$

Dividing by $y^{2}$, integrating and calling the integration constant $1 / r$ results in

$$
\frac{\mathrm{d} x}{\mathrm{~d} s}=\frac{y^{2}}{r}
$$

Next we use $\boldsymbol{u} \cdot \boldsymbol{u}=1$,

$$
\frac{1}{y^{2}}\left[\left(\frac{\mathrm{~d} x}{\mathrm{~d} s}\right)^{2}+\left(\frac{\mathrm{d} y}{\mathrm{~d} s}\right)^{2}\right]=1
$$

insert $\frac{\mathrm{d} x}{\mathrm{~d} s}=\frac{y^{2}}{r}$ and solve for

$$
\frac{\mathrm{d} y}{\mathrm{~d} s}= \pm \sqrt{y^{2}-y^{4} / r^{2}}
$$

Solutions are discussed Thursday, 01.02.24

Finally we obtain an equation for $x(y)$ writing

$$
\frac{\mathrm{d} x}{\mathrm{~d} y}=\frac{\mathrm{d} x / \mathrm{d} s}{\mathrm{~d} y / \mathrm{d} s}= \pm \frac{y}{\sqrt{r^{2}-y^{2}}}
$$

Integrating we find the equation of a circle centered at the integration constant $x_{0}$,

$$
\left(x-x_{0}\right)^{2}+y^{2}=r^{2} .
$$

Remark: This result was historically important as first counter-example to Euclid's 5.th postulate. It states that for a straight line L and a point P there is only one straight lines (i.e. a geodesics) through P that does not intersect L (namely the straight line parallel to L). The hyperbolic plane is an example where an infinite number of straight lines go through P that do not intersect L .

## 5. Killing vector fields of Minkowski space

Find the Killing vector fields of Minkowski space and specify the corresponding symmetries and conserved quantities. [Hint: Differentiate the Killing equation, permute the indices and find an equation for a single term which you can integrate.]

Extra for the dedicated student: Since the condition $\tilde{g}^{\mu \nu}(\tilde{x})=g^{\mu \nu}(\tilde{x})$ defines an isometry, that is, a distance conserving map between two spaces, $\mathrm{d} s^{2}=\mathrm{d} \tilde{s}^{2}$, the Killing vectors in Minkowski space are the generators of Poincaré transformations. The symmetry group of the Maxwell equations is, however, the larger conformal group. In this case only the light-cone structure is kept invariant: $\tilde{g}^{\mu \nu}(\tilde{x})=\Omega^{2}(\tilde{x}) g^{\mu \nu}(\tilde{x})$, or $\mathrm{d} s^{2}=0$. Repeating the steps in the lectues, one finds the conformal Killing equation which simplifies in Minkowski space to

$$
\begin{equation*}
\partial_{\mu} \xi_{\nu}+\partial_{\nu} \xi_{\mu}=\kappa \eta_{\mu \nu} \tag{2}
\end{equation*}
$$

where $\kappa$ is a function to be determined. Derive the additional Killing vector fields in $d$ dimensions.
The Killing equation $\nabla_{\mu} \xi_{\nu}+\nabla_{\nu} \xi_{\mu}=0$ simplifies in Minkowski space to

$$
\begin{equation*}
\partial_{\mu} \xi_{\nu}+\partial_{\nu} \xi_{\mu}=0 \tag{3}
\end{equation*}
$$

Following the hint, we take one more derivative,

$$
\begin{equation*}
\partial_{\rho} \partial_{\mu} \xi_{\nu}+\partial_{\rho} \partial_{\nu} \xi_{\mu}=0 \tag{4}
\end{equation*}
$$

Cycling the indices to generate three equations, then adding two and subtracting one gives

$$
\begin{equation*}
2 \partial_{\mu} \partial_{\rho} \xi_{\nu}=0 . \tag{5}
\end{equation*}
$$

Thus the solution $\xi_{\nu}$ is linear in the coordinates. Integrating twice, we find

$$
\begin{equation*}
\xi^{\mu}=\omega_{\nu}^{\mu} x^{\nu}+a^{\mu} . \tag{6}
\end{equation*}
$$

Because of

$$
\partial^{\nu} \xi^{\mu}=\omega_{\rho}^{\mu} \partial^{\nu} x^{\rho}=\omega_{\rho}^{\mu} g^{\nu \rho}=\omega^{\mu \nu}
$$

the matrix $\omega^{\mu \nu}$ has to be antisymmetric in order to satisfy Eq. (3). Thus the Killing vector fields are determined by ten integration constants. They agree with the infinitesimal generators of Poincaré transformations:
The four parameters $a^{\mu}$ generate translations, $x^{\mu} \rightarrow x^{\mu}+a^{\mu}$, described by four Killing vector fields which can be chosen as the Cartesian basis vectors of Minkowski space,

$$
\boldsymbol{T}_{t}=\boldsymbol{e}_{t}, \quad \boldsymbol{T}_{x}=\boldsymbol{e}_{x}, \quad \boldsymbol{T}_{y}=\boldsymbol{e}_{y}, \quad \boldsymbol{T}_{z}=\boldsymbol{e}_{z}
$$

For a particle with momentum $p^{\mu}=m u^{\mu}$ moving along a geodesics $x^{\mu}(\lambda)$, the existence of a Killing vector $\boldsymbol{T}_{\mu}$ implies ( $\mu$ is just labelling the four vector fields)

$$
\frac{\mathrm{d}}{\mathrm{~d} \lambda}\left(\boldsymbol{T}_{(\mu)} \cdot \boldsymbol{u}\right)=\frac{\mathrm{d}}{m \mathrm{~d} \lambda}\left(\boldsymbol{T}_{(\mu)} \cdot \boldsymbol{p}\right)=0
$$

i.e. the conservation of the four-momentum component $p_{\mu}$.

Consider next the $i j$ (=spatial) components of the Killing equation, which satisfy $\omega^{i}{ }_{j}=-\omega_{i}^{j}$. Setting $\omega_{2}^{1}=-\omega_{1}^{2}=1$ and all other ones to zero,

$$
\xi^{0}=0, \quad \xi^{1}=\omega_{2}^{1} x^{2}, \quad \xi^{2}=\omega_{1}^{2} x^{1}, \quad \xi^{3}=0
$$

or

$$
\boldsymbol{\xi}=y \boldsymbol{e}_{x}-x \boldsymbol{e}_{y} \propto \boldsymbol{J}_{z}, \quad \text { and cyclic permutations. }
$$

The existence of Killing vectors $\boldsymbol{J}_{i}$ implies that $\boldsymbol{J}_{i} \cdot \boldsymbol{p}$ is conserved along a geodesics of particle. But

$$
\boldsymbol{J}_{x} \cdot \boldsymbol{p}=-\left(y p_{z}-z p_{y}\right)=-J_{x}
$$

and thus the angular momentum around the origin of the coordinate system is conserved. The other three components satisfy the $0 \alpha$ component of the Killing equations $\left(\omega_{1}^{0}=\omega_{0}^{1}\right)$,

$$
\begin{equation*}
\boldsymbol{K}_{1}=t \partial_{z}+z \partial_{t}, \quad \text { and cyclic permutations } \tag{7}
\end{equation*}
$$

The conserved quantity $t p_{z}-z E=$ const. now depends on time and is therefore not as popular as the previous ones. Its conservation implies that the centre-of-mass of a system of particles moves with constant velocity $v_{\alpha}=p_{\alpha} / E$.

Extra: Taking the trace $\kappa=2 \partial_{\mu} \xi^{\mu} / d$ follows. Inserted into the Killing equation, we obtain

$$
\begin{equation*}
\partial_{\mu} \xi_{\nu}+\partial_{\nu} \xi_{\mu}=\frac{2}{d} \partial_{\rho} \xi^{\rho} \eta_{\mu \nu}=\kappa \eta_{\mu \nu} \tag{8}
\end{equation*}
$$

Next we take one more derivative $\partial_{\rho}$, exchange then indices $\rho \leftrightarrow \mu$ and subtract the two equations, arriving at

$$
\begin{equation*}
\partial_{\rho}\left(\partial_{\mu} \xi_{\nu}-\partial_{\nu} \xi_{\mu}\right)=\partial_{\mu} \kappa \eta_{\nu \rho}-\partial_{\nu} \kappa \eta_{\mu \rho} . \tag{9}
\end{equation*}
$$

Integrating this equation, we find

$$
\begin{equation*}
\partial_{\mu} \xi_{\nu}-\partial_{\nu} \xi_{\mu}=\int\left(\partial_{\mu} \kappa \mathrm{d} x_{\nu}-\partial_{\nu} \kappa \mathrm{d} x_{\mu}\right)+2 \omega_{\mu \nu} \tag{10}
\end{equation*}
$$

where $\omega_{\mu \nu}$ is an antiymmetric tensor containing the integration constant. Now we add to this to the conformal Killing eq. (2). Integrating then again, we obtain

$$
\begin{equation*}
\xi^{\mu}=a^{\mu}+\omega_{\nu}{ }^{\mu} x^{\nu}+\alpha x^{\mu} \kappa+\frac{1}{2} \int \mathrm{~d} x^{\nu} \int\left(\partial_{\nu} \kappa \mathrm{d} x^{\mu}-\partial_{\mu} \kappa \mathrm{d} x^{\nu}\right) . \tag{11}
\end{equation*}
$$

Finally, we have to determine the function $\kappa$. Acting with $\partial^{\mu}$ on Eq. (8) we find

$$
\begin{equation*}
d \square \xi_{\nu}=(2-d) \partial_{\nu} \partial_{\mu} \xi^{\mu} . \tag{12}
\end{equation*}
$$

Hence in $d=2$ any harmonic function (i.e. a function satisfying $\square \xi_{\nu}=0$ ) determines a conformal Killing vector field and the conformal group $C(1,1)$ is thus infinite dimensional. In contrast, for $d>2$ the condition $\partial_{\nu} \partial_{\mu} \xi^{\mu}=0$ follows. Hence the function $\kappa$ can be at most linear in the coordinates, and we choose it as

$$
\begin{equation*}
\kappa=-2 \alpha+4 \beta_{\mu} x^{\mu} . \tag{13}
\end{equation*}
$$

Inserting $\kappa$ into Eq. (10), we find the conformal Killing vector fields as

$$
\begin{equation*}
\xi^{\mu}=a^{\mu}+\omega_{\nu}{ }^{\mu} x^{\nu}+\alpha x^{\mu}+\beta_{\nu}\left(2 x^{\mu} x^{\nu}-\eta^{\mu \nu} x^{2}\right) . \tag{14}
\end{equation*}
$$

They depend on $(d+1)(d+2) / 2$ parameters: $d$ translations, $d(d-1) / 2$ Lorentz transformations, one dilatation and $d$ special conformal transformations.

