Exercise sheet 3

1. Transformation between inertial frames.

Consider two inertial frames K and K' with parallel axes at $t = t' = 0$ that are moving with the relative velocity v in the x direction.
a.) Show that the linear transformation between the coordinates in K and K' is given by

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} At + Bx \\ Dt + Ex \\ y \\ z \end{pmatrix} = \begin{pmatrix} At + Bx \\ A(x - vt) \\ y \\ z \end{pmatrix} \tag{1}$$

b.) Show that requiring the invariance of

$$\Delta s^2 \equiv c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \tag{2}$$

leads to Lorentz transformations.
c.) What is the condition leading to Galilean transformations?

a.) The origin $x' = 0$ of K' correspond in K to $x = vt$. Then

$$0 = Dt + Evt \quad \Rightarrow \quad D = -Ev$$

The origin $x = 0$ of K correspond in K' to $x' = -vt'$. Then

$$t' = At \tag{3}$$

$$-vt' = Dt \quad \Rightarrow \quad t' = -\frac{D}{v}t = At \tag{4}$$

and thus $A = -D/v$ and hence $A = E$.

Remark: The transformation has to be linear, if we assume that space and time are uniform.

b.) Square the terms on the RHS of

$$t^2 - x^2 = (At - Bx)^2 - A^2(x - vt)^2 \,,$$

order them as $t^2(\cdots) - x^2(\cdots) + 2tx(\cdots)$ and compare coefficients to the LHS. This gives $A = \gamma$ and $B = -\beta\gamma$.

c.) Absolute time $t = t'$ requires $A = 1$ and $B = 0$.

2. Index gymnastics.

Splitt the tensor $T^{\mu\nu}$ into its symmetric part $S^{\mu\nu} = S^{\nu\mu}$ and its anti-symmetric part $A^{\mu\nu} = -A^{\nu\mu}$.
a.) Show that this splitting is invariant under Lorentz transformations.
b.) Show that $S^{\mu\nu} A_{\mu\nu} = 0$.

Solutions are discussed Thursday, 11.09.19

a.) The antisymmetric tensor components are given by

$$A_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right) .$$

The definition is invariant under a Lorentz transformation $\Lambda^\mu{}_\nu$, since

$$2\tilde{A}_{\mu\nu} = \tilde{T}_{\mu\nu} - \tilde{T}_{\nu\mu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu T_{\rho\sigma} - \Lambda^\rho{}_\nu \Lambda^\sigma{}_\mu T_{\rho\sigma} \tag{5}$$

$$= \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu T_{\rho\sigma} - \Lambda^\sigma{}_\nu \Lambda^\rho{}_\mu T_{\sigma\rho} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu (T_{\rho\sigma} - T_{\sigma\rho}) = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu 2A_{\rho\sigma} \tag{6}$$

Here we used first the transformation law for a tensor of rank 2, and exchanged then dummy indices in the second term.
Note that this holds also for a general coordinate transformation, $\Lambda^\mu{}_\nu \to \frac{\partial \tilde{x}^\nu}{\partial x^\mu}$.

b.) To see that the contraction of a symmetric tensor $S_{\mu\nu}$ with an antisymmetric tensor $A_{\mu\nu}$ gives zero, consider

$$S_{\mu\nu}A^{\mu\nu} = -S_{\mu\nu}A^{\nu\mu} = -S_{\nu\mu}A^{\nu\mu} = -S_{\mu\nu}A^{\mu\nu} . \tag{7}$$

Here we used first the antisymmetry of $A^{\mu\nu}$, then the symmetry of $S_{\mu\nu}$, and finally exchanged the dummy summation indices. Note that this remains true, if the tensor expression contains additional indices.

3. Two particle decay.
A particle at rest decays into two particles, $A \to B + C$.
a.) Find the energy and momenta of B and C.
b.) What happens for $m_A < m_B + m_C$?

a.) Squaring $p_c^\mu = p_a^\mu - p_b^\mu$ and using $\boldsymbol{p}_a = 0$ gives

$$m_c^2 = m_a^2 - 2m_a E_b + m_b^2 \tag{8}$$

Solving for the energies of the decay products,

$$E_b = \frac{m_a^2 + m_b^2 - m_c^2}{2m_a} \quad \text{and} \quad E_c = \frac{m_a^2 + m_c^2 - m_b^2}{2m_a} . \tag{9}$$

Then the momenta follow as

$$\boldsymbol{p}_b^2 = E_b^2 - m_b^2 = \left(\frac{m_a^2 + m_b^2 - m_c^2}{2m_a}\right)^2 - \frac{4m_a^2 m_b^2}{4m_a^2} \tag{10}$$

or

$$|\boldsymbol{p}_b| = |\boldsymbol{p}_c| = \frac{1}{2m_a}\left[m_a^4 + m_b^4 + m_c^4 - 2m_a m_b - 2m_a m_c - 2m_b m_c\right]^{1/2} \tag{11}$$

$$\equiv \frac{1}{2m_a}\,\lambda^{1/2}(m_a^2, m_b^2, m_c^2) . \tag{12}$$

The Kibble or triangle function $\lambda(x, y, z)$ satisfies

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz \tag{13a}$$
$$= \left[x^2 - (y+z)^2\right]\left[x^2 - (y-z)^2\right] . \tag{13b}$$

b.) The decay is kinematically forbidden, since the momenta $|\boldsymbol{p}_b|$ and $|\boldsymbol{p}_c|$ become imaginary, cf. with Eq. (13b).

4. Threshold energy.
Particle A hits particle B at rest, producing the particles $C_1, \ldots, C_n$.
a.) Calculate the threshold (i.e. the minimal energy of A) for this reaction.
b.) What is the threshold energy for $pp \to X\pi^0$, where X is the minmal set of particles such that the reaction is allowed?
c.) What is the threshold energy for $pp \to X\bar{p}$?

a.) Denote the total momentum by P^μ. The initial state in the lab frame has $P^\mu = (E_a + m_b, \boldsymbol{p}_a)$ and thus

$$P^2 = 2E_a m_b + m_a^2 + m_b^2 .$$

After the collision, in the CMS frame at threshold

$$P^2 = (\sum_i m_i, \mathbf{0})^2 \equiv (M, \mathbf{0})^2 = M^2$$

Thus

$$E_a = \frac{M^2 - m_a^2 - m_b^2}{2m_b} .$$

b.) Since baryon number is conserved, $X = \{pp\}$, and thus $M = 2m_p + m_\pi \simeq 2011\,\text{MeV}$ or $E_{\text{th}} \simeq 1.2\,\text{GeV}$.

c.) Since baryon number is conserved, $X = \{ppp\bar{p}\}$, and thus $M^2 = 16m_p^2$ or $E_{\text{th}} = 7m_p \simeq 6.5\,\text{GeV}$.