Exercise sheet 3b: Additional exercises

1. Proper distance, area and volumes.

Consider the metric

$$ds^2 = -(1-Ar^2)^2 dt^2 + (1-Ar^2)^2 dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2)$$

i) Calculate the proper distance from $r=0$ to $r=R$.
ii) Calculate the area of a sphere with coordinate radius R.
iii) Calculate the three-volume of a sphere with coordinate radius R.
iv) Calculate the four-volume of a sphere with coordinate radius R bounded by two t=const planes separated by time difference T.

The distance is given by

$$s = \int_0^R \mathrm{d}r\,(1-Ar^2) = R\left(1 - \frac{1}{3}AR^2\right),$$

the area by

$$A = \int_{S^2} Rd\vartheta\, R\sin\vartheta \mathrm{d}\phi = R^2 \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\phi\, \sin\vartheta = 4\pi R^2,$$

the 3-volume by

$$V = \int_{B^2} \mathrm{d}r(1-Ar^2)\, rd\vartheta\, r\sin\vartheta \mathrm{d}\phi = \int_0^R \mathrm{d}r(1-Ar^2)r^2 \int_0^\pi \mathrm{d}\vartheta\, \sin\vartheta \int_0^{2\pi} \mathrm{d}\phi = \frac{4\pi}{3}R^3\left(1 - \frac{3AR^2}{5}\right),$$

and the 4-volume by

$$V_4 = \int_{T\times B^2} \mathrm{d}t(1-Ar^2)\,\mathrm{d}r(1-Ar^2)\, rd\vartheta\, r\sin\vartheta \mathrm{d}\phi = \tag{1}$$

$$= \int_0^T \mathrm{d}t \int_0^R \mathrm{d}r(1-Ar^2)^2 r^2 \int_0^\pi \mathrm{d}\vartheta\, \sin\vartheta \int_0^{2\pi} \mathrm{d}\phi \tag{2}$$

$$= \frac{4\pi}{3}R^3 T\left(1 - \frac{6AR^2}{5} + \frac{3(AR^2)^2}{7}\right). \tag{3}$$

2. Newtonian gravity as a spacetime phenomenon

Since Newtonian gravity is a special case of Einstein gravity, it should be possible to replace the Newtonian gravitational force by a deformation of Minkowski space. Confirm that the metric describing gravitational effects in the Newtonian limit can be chosen as

$$\mathrm{d}s^2 = \left(1 + 2\Phi/c^2\right) c^2 \mathrm{d}t^2 - \left(1 - 2\Phi/c^2\right) \mathrm{d}l^2 \tag{4}$$

by showing that the resulting action of a point particle is equivalent to the the one using the standard Lagrangian

$$L = \frac{1}{2}mv^2 - m\Phi = T + V \tag{5}$$

in the limit $v/c \to 0$. Here, Φ is the Newtonian gravitational potential and $\mathrm{d}l^2 = dx^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ is the Euclidean line-element.

With this metric, the action of a point particle becomes

$$S = \int_1^2 \mathrm{d}\tau = \int_1^2 \left[\left(1 + 2\Phi/c^2\right) \mathrm{d}t^2 - \left(1 - 2\Phi/c^2\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\right]^{1/2} \tag{6a}$$

$$= \int_1^2 \mathrm{d}t \left[\left(1 + 2\Phi/c^2\right) - \frac{1}{c^2}\left(1 - 2\Phi/c^2\right) v^2\right]^{1/2}. \tag{6b}$$

Keeping only terms of order $1/c^2$ and expanding the square root, the action becomes

$$S \simeq \int_1^2 \mathrm{d}t \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 - \Phi\right)\right]. \tag{7}$$

The first, constant term does not contribute to δS, so that this action is equivalent to the one using the standard Lagrangian

$$L = \frac{1}{2}mv^2 - m\Phi = T + V \tag{8}$$

for a non-relativistic particle with mass m and the gravitational potential energy $V = m\Phi$. Thus the geodesics of the metric (1) agree with the classical trajectories in the gravitational potential Φ. Note also that the coefficient of $\mathrm{d}l$ dropped out of Eq. (2): Thus an infinite number of spacetimes leads at lowest order in v/c to the same trajectories. Such metrics may however imply different trajectories of relativistic particles like photons.

3. Maximal velocity of a comet

A comet starts at $r \to \infty$, approaches a star of mass M and disappears again to $r \to \infty$. What is the maximal velocity v of the comet measured by a stationary observer at the radius R of nearest approach?
(2 ways of solution: 1. find v as function of R, l and b, connect then l to b and R. 2. Use the orthonormal basis associated with the observer.)

Solution way 1: By the same argument as in 2., the velocity v measured by a stationary observer is

$$E = \frac{m}{\sqrt{1-v^2}} = m\left(1 - \frac{2M}{r}\right)^{1/2} u^t,$$

if m is the mass and u^t the time component of the four-velocity of the comet. At the radius R of closest approach, it is

$$u^r = 0 \quad \text{and} \quad u^\phi = \frac{l}{R^2}.$$

Evaluating the normalisation condition $\boldsymbol{u}_{\text{obs}} \cdot \boldsymbol{u}_{\text{obs}} = 1$ gives

$$u^t = \left(1 - \frac{2M}{r}\right)^{-1/2} \left(\frac{l}{R^2} + 1\right)^{1/2}$$

and thus

$$v = \frac{l}{R\sqrt{1+(l/R)^2}}$$

As last step, we have to replace l by the impact parameter b. At large r, it is $\phi \simeq b/r$ and thus

$$l = r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -b\frac{\mathrm{d}r}{\mathrm{d}\tau} = +b\sqrt{e^2-1}$$

At the turning point, it is

$$e^2 = \left(1-\frac{2M}{r}\right)\left(1+\frac{l}{R^2}\right)^{1/2}$$

Combining everything, it follows

$$v = \left(1-\frac{2M}{r}\right)^{1/2}\frac{b}{R}.$$

Solution way 2: Choose the basis such that the ϕ direction of the observer frame agrees with *phi* coordinate. Then the normalised basis vectors are

$$\hat{\boldsymbol{e}}_t = \left(1-\frac{2M}{r}\right)^{1/2}\boldsymbol{e}_t$$

$$\hat{\boldsymbol{e}}_\phi = \frac{1}{r}\boldsymbol{e}_\phi.$$

The impact parameter is

$$b = \frac{l}{e} = \frac{r^2}{1-2M/r}\frac{u^\phi}{u^t}.$$

At R, the three-velocituy of the comet has only a ϕ component, hence $v = \hat{\boldsymbol{u}}_\phi/\hat{\boldsymbol{u}}_t$. Now we have only to connect the components in the different bases, using that $\langle v\rangle = v^\alpha\boldsymbol{e}_\alpha = \hat{v}^\alpha\hat{\boldsymbol{e}}_\alpha$. Thus

$$u^t = (1-2M/r)^{-1/2}\hat{u}^t$$

and

$$u^\phi = (1/r)\hat{u}^\phi$$

Combining everything, we obtain the same result.