Exercise sheet 4

1. Light deflection?
In another theory of gravity, the metric outside a star is

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left[dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2)\right] .$$

a.) Calculate the deflection of photons in this theory.
b.) Use a general argument why you should get this result.

a.) The condition $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ gives

$$\left(1 - \frac{2M}{r}\right)\left[\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\vartheta}{d\lambda}\right)^2\right] = 0$$

for $\vartheta = \pi/2$. The conserved energy and angular momentum are

$$e = \boldsymbol{\xi} \cdot \boldsymbol{u} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}$$

and

$$l = -\boldsymbol{\eta} \cdot \boldsymbol{u} = \left(1 - \frac{2M}{r}\right) r^2 \frac{d\phi}{d\lambda}$$

Following the same steps as in the Schwarzschild case, one finds

$$\frac{d\phi}{dr} = \frac{1}{r^2}\left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{-1/2}$$

which is independent of M. Thus light propagates as in Minkowski space, i.e. is not deflected.

b.) The metric is conformally flat.

2. Circular orbits
A spaceship is moving without power in a circular orbit about a star with mass M. The radius in Schwarzschild coordinates is $r = 7M$.
a.) Show that $\Omega = d\phi/dt = (M/r^3)^{1/2}$.
b.) What is the period measured by an observer at infinity?
c.) What is the period measured by a clock onboard the spaceship?

a.) The coordinate time t agrees for $r \to \infty$ with proper-time τ; thus Ω is the angular velocity as measured by an observer at infinity. Using the chain rule and introducing the conserved quantities e and l, we find

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\frac{l}{e}.$$

For a circular orbit, the ratio l/e cannot be arbitrary. We fix it by using $u'' = 0$ in the orbital and $\mathrm{d}r/\mathrm{d}\tau = 0$ in the energy equation: Solving

$$u = \frac{M}{l^2} + 3Mu^2$$

for l gives

$$l^2 = \frac{Mr}{1 - 3M/r}.$$

(This shows also again that no circular orbits with $r < 3M$ exist.) Next we evaluate the radial equation for a circular orbit, $\mathcal{E} = V_{\text{eff}}$

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right]$$

or

$$e^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right).$$

Thus the ratio is

$$\frac{l}{e} = (Mr)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1}.$$

Inserting gives $\Omega = (M/r^3)^{1/2}$, what agrees with the Newtonian result.

b.) The period measured by an observer at infinity is $P_\infty = 2\pi/\Omega$. With $r = 7M$, it follows

$$P_\infty = \frac{2\pi}{\Omega} = 2\pi\left(\frac{r^3}{M}\right)^{1/2} = 14\pi\sqrt{7}M.$$

c.) In the spaceship, one measures $P = 2\pi/\omega$ with $\omega = \mathrm{d}\phi/\mathrm{d}\tau$. Thus we use

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \Omega\frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

For a circular orbit in the equatorial plane, the normalisation condition $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ becomes

$$1 = \left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \tag{1}$$

$$= \left(1 - \frac{2M}{r} - r^2\Omega^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \tag{2}$$

With $r^2\Omega^2 = M/r$ and $r = 7M$, it follows $\mathrm{d}t/\mathrm{d}\tau = \sqrt{7/4}$. Hence the periode measured onboard is $P = 28\pi M$, i.e. 32% smaller.

3. Falling into a BH

An observer falls radially into a Schwarzschild BH of mass M. The observer starts from rest relative to a stationary observer at $r = 10M$. How much time elapses on the clock of the falling observer before hitting the singularity at $r = 0$?

Solutions are discussed Thursday, 15.02.24

The energy equation with $l = 0$ is

$$(1 - 2M/r) = e^2 - (\mathrm{d}r/\mathrm{d}\tau)^2$$

For $r = 10M$ and $\mathrm{d}r/\mathrm{d}\tau = 0$ it follows $e = 2/\sqrt{5}$. For $r \neq 10M$, it is

$$\mathrm{d}r/\mathrm{d}\tau = \pm(2M/r - 1/5)^{1/2}.$$

Integrating gives

$$\tau = \int_0^{10M} \mathrm{d}r[2M/r - 1/5]^{-1/2} = 5\sqrt{5}\pi M.$$

4. Force of a hovering rocket.
A stationary observer hovers on a radial orbit around a Schwarzschild black-hole.
a.) Argue that

$$f^\alpha = m\left(\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma\right)$$

is the correct generalisation of Newton's second law to curved spacetime.
b.) Using $\Gamma^r{}_{tt} = (1 - 2M/r)(M/r^2)$, find the radial force required to stay on a stationary orbit.
c.) The result from b.) is the radial force in the coordinate basis. Relate it to the force measured by the observer in its Cartesian inertial frame.

a.) First, we note that we obtain for $f^\alpha = 0$ the geodesic equation. Second, choosing flat space with Cartesian coordinates, $\Gamma^\alpha{}_{\beta\gamma} = 0$, and we are back to the standard version of Newton's second law. Thus the force law has the correct limiting cases.
More formally, we should rewrite this equation as a tensor equation. In order to do so, we introduce the concept of parallel transport: We say that a tensor $\boldsymbol{T}$ is parallel transported along the curve $x(\sigma)$, if its components $T^{\mu\cdots}_{\nu\cdots}$ stay constant. In flat space, this means simply

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}T^{\mu\dots}_{\nu\dots} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\partial_\alpha T^{\mu\dots}_{\nu\dots} = 0. \tag{3}$$

In curved space, we have to replace the normal derivative by a covariant one. We define the directional covariant derivative along $x(\sigma)$ as

$$\frac{D}{\mathrm{d}\sigma} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\nabla_\alpha. \tag{4}$$

Then the requirement of parallel transport for u^μ becomes

$$\frac{D}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0. \tag{5}$$

Introducing $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, we obtain the geodesic equation in its standard form. Thus this equation can also seen as the requirement that the four-velocity is paraller transported along $x^\mu(\sigma)$.

b.) For a stationary observer (with $u^\alpha = [(1-2M/r)^{-1/2}, 0, 0, 0])$, the $\mathrm{d}^2x^\alpha/\mathrm{d}\tau^2$ term vanishes by definition. Then

$$f^r = m\Gamma^r{}_{\beta\gamma}u^\beta u^\gamma = m\Gamma^r{}_{tt}u^t u^t = m(1-2M/r)(M/r^2)(1-2M/r)^{-1} = mM/r^2.$$

c.) A normalised basis in the observer system is $\boldsymbol{e}'_\mu = \boldsymbol{e}_\mu/\sqrt{g_{\mu\mu}}$ (no summation): this guarantees for $g_{\mu\nu} = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu$ and $g_{\mu\nu}$ diagonal that $\boldsymbol{e}'_\mu \cdot \boldsymbol{e}'_\nu = \eta_{\mu\nu}$.
With $\boldsymbol{e}'^\alpha_r = (0, (1-2M/r)^{1/2}, 0, 0)$, it follows for the radial force measured by the observer

$$f^r_{\text{obs}} = \boldsymbol{e}'^r \cdot \boldsymbol{f} = m(1-2M/r)^{-1/2}M/r^2.$$