Exercise sheet 4

1. Charged pion decay.
A charged pion decays mainly via the reaction $\pi^\pm \rightarrow \mu^\pm + \nu_\mu$.

a.) Calculate the energy and the momentum of the muon, if the pion decays at rest.

b.) Calculate the maximal and minimal energy of the muon, if the pion decays in flight with an energy of 1 TeV.

c.) Bonus: what is the energy distribution $dN/dE$ of the muons?

Use as masses $m_{\pi^\pm} = 139$ MeV, $m_{\mu^\pm} = 106$ MeV, and $m_\nu = 0$.

a.) Combining energy conservation $m_{\pi} = E_\nu + E_\mu$, smallness of neutrino masses $E_\nu = p_\nu$ and the cms condition $p_\nu = -p_\mu$ gives

$$(E_\mu - m_{\pi})^2 = |p_\mu|^2$$

$$E_\mu^2 - |p_\mu|^2 - 2E_\nu m_{\pi} + m_{\pi}^2 = 0$$

$$m_{\mu}^2 + m_{\pi}^2 = 2E_\mu m_{\pi}$$

$$E_\mu = \frac{m_{\mu}^2 + m_{\pi}^2}{2m_{\pi}} = \frac{m_{\pi}}{2} \left( 1 + \frac{m_{\mu}^2}{m_{\pi}^2} \right) = \frac{m_{\pi}}{2} \left( 1 + \frac{m_{\mu}^2}{m_{\pi}^2} \right)$$

with $r \equiv m_{\mu}^2/m_{\pi}^2 \simeq 0.58$. The momentum follows as

$$|p_\mu|^2 = E_\mu^2 - m_{\mu}^2 = \frac{m_{\pi}^2}{4} \left( 1 + 2r + r^2 \right) - \frac{4m_{\mu}^2m_{\pi}^2}{4m_{\pi}^2} = \frac{m_{\pi}^2}{4} \left( 1 - 2r + r^2 \right)$$

or $|p_\mu| = (m_{\pi}/2)(1 - r)$.

b.) The energy in the lab system follows from the general Lorentz transformation $E' = \gamma(E + \beta p \cos \theta)$ with $\theta$ as the angle between the velocity $\beta$ of the pion and the emitted muon. The maximal/minimal values of $E'$ follow for $\cos \theta = \pm 1$, i.e. if the muon is emitted parallel and anti-parallel to the direction of flight of the pion.

Inserting $E = (m_{\pi}/2)(1 + r)$ and $p = (m_{\pi}/2)(1 - r)$ gives

$$E_{\min}^\text{max} = \frac{\gamma m_{\pi}}{2} (1 + r \pm \beta(1 - r))$$

With $\gamma = 1$ TeV$/m_{\pi} \simeq 7200$ becomes $\beta = \sqrt{1 - \gamma^{-2}} \simeq 1$. Then

$$E_{\min}^\text{max} \simeq \frac{\gamma m_{\pi}}{2} (1 + r \pm (1 - r))$$

and thus $E_{\max} \simeq \gamma m_{\pi} = E_{\pi}$ and $E_{\min} \simeq r E_{\pi} \simeq 0.58E_{\pi}$.

Solutions are discussed Thursday, 20.09.18
c.) In the rest-frame of the pion, the muon is emitted isotropically, $dN/d\Omega = 1/(4\pi)$, since there is no preferred direction. In a frame where the pion is moving, we differentiate $E' = \gamma(E + \beta p \cos \vartheta)$, obtaining

$$dE' = \gamma \beta p d(\cos \vartheta).$$

Thus

$$dN = \frac{1}{2} d(\cos \vartheta) = \frac{dE'}{2\gamma \beta p}$$

or

$$\frac{dN}{dE'} = \frac{1}{2\gamma \beta p}$$

The energy distribution is a flat box, with boundaries given by Eq. (1).

2. Delta decay.
What are the possible values of the angular momentum $L$ of the final state in the decay $\Delta^{++} \rightarrow p + \pi^+$? (The delta has spin $s = 3/2$, the pion spin $s = 0$.)

The total angular momentum is conserved, which in the initial state equals $J = s = 3/2$. In the final state, we have to combine the proton spin and the angular momentum $L$ so that we get $3/2$.

option 1: $(s = 1/2) + (L = 1) \Rightarrow 3/2$ or $1/2$;

option 2: $(s = 1/2) + (L = 2) \Rightarrow 5/2$ or $3/2$.

Thus the angular momentum of the final state can be $L = 1$ or $2$.

3. SU(3).
The eight generators of SU(3) can be chosen as the Gell-Mann matrices $\lambda_i$,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(2)

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

(3)

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

(4)

a.) Convince yourself that the relation $\text{tr}\{\lambda_i \lambda_j\} = 2\delta_{ij}$ holds by calculating two examples.

b.) We set now $F_i \equiv \lambda_i/2$ and introduced the spherical representation,

$$T_+ = F_1 + iF_2, \quad T_3 = F_3$$

(5)

$$V_+ = F_4 + iF_5, \quad Y = \frac{2}{\sqrt{3}} F_8$$

(6)

$$U_+ = F_6 + iF_7.$$
Show that the following comutation relations are correct:

\[ [T_3, U_{\pm}] = \mp \frac{1}{2} U_{\pm}, \quad [U_+, U_-] = \frac{3}{2} Y - T_3 = 2U_3 \]

a.) Calculate for instance

\[ \lambda_3 \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

and thus \( \text{tr}(\lambda_3 \lambda_8) = 0 \), or

\[ \lambda_8 \lambda_8 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \]

or \( \text{tr}(\lambda_8 \lambda_8) = 6/3 = 2 \).

4. (Iso-) spin matrices for \( S = 1 \) and \( S = 3/2 \).

In the lectures, we used only an explicit representation of the spin matrices for \( S = 1/2 \) (or isospin \( T = 1/2 \)). A possible way to derive them for \( S = 1 \) and \( S = 3/2 \) is the following:

a.) Derive first \( S_z \) from the known eigenvalues.

b.) Construct the ladder operators \( S_{\pm} \) which have the property

\[ S_{\pm} |sm\rangle = \sqrt{(s(s+1) - m(m \pm 1)} |sm \pm 1\rangle \]

c.) Find \( S_x \) and \( S_y \).

a. For \( S = 1 \), choose three orthonormal unit vectors for \( \chi_i \) with \( i = \{+, 0, -\} \), \( \chi_+ = (1, 0, 0)^T \) etc. From \( S_2 \chi_+ = +\chi_+ \), \( S_2 \chi_0 = 0 \), and \( S_2 \chi_- = -\chi_- \), we find

\[ S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

b.) From \( S_+ |1 1\rangle = 0 \), \( S_+ |1 0\rangle = \sqrt{2} - 0 |1 1\rangle \) and \( S_+ |1 -1\rangle = \sqrt{2} - 0 |1 0\rangle \) we know that

\[ S_+ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0, \quad S_+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad S_+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \]

and thus

\[ S_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

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Next we construct $S_-$. From $S_- |1 1\rangle = \sqrt{2 + 0} |1 0\rangle$, $S_- |1 0\rangle = \sqrt{2 + 0} |1 -1\rangle$ and $S_- |1 -1\rangle = 0$ we obtain

$$S_- = \sqrt{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$$

c.) Adding and subtracting them results in

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad S_+ = \frac{1}{2i}(S_+ - S_-) = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -i & 0 \\
i & 0 & -i \\
0 & i & 0
\end{pmatrix}.$$

Calculation for $S = 3/2$ proceeds completely analogous. Starting from four unit vectors, we have

$$S_z = \frac{1}{2} \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3
\end{pmatrix},$$

From $S_+ |3/2 3/2\rangle = 0$, $S_+ |3/2 1/2\rangle = \sqrt{15/4 - 3/4} |3/2 3/2\rangle$ $S_+ |3/2 -1/2\rangle = \sqrt{15/4 + 3/4} |3/2 3/2\rangle$, and $S_+ |3/2 -3/2\rangle = \sqrt{15/4 - 3/4} |3/2 1/2\rangle$ we know that

$$S_+ = \begin{pmatrix}
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & \sqrt{3} \\
0 & 0 & 0 & 0
\end{pmatrix},$$

In the same way,

$$S_- = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \sqrt{3} & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0
\end{pmatrix},$$

and finally

$$S_x = \frac{1}{2} \begin{pmatrix}
0 & \sqrt{3} & 0 & 0 \\
\sqrt{3} & 0 & 2 & 0 \\
0 & 2 & 0 & \sqrt{3} \\
0 & 0 & \sqrt{3} & 0
\end{pmatrix}, \quad S_- = \frac{1}{2} \begin{pmatrix}
0 & -\sqrt{3}i & 0 & 0 \\
\sqrt{3}i & 0 & -2i & 0 \\
0 & 2i & 0 & -\sqrt{3}i \\
0 & 0 & \sqrt{3}i & 0
\end{pmatrix}.$$

Remark: in the case $S = 1$, we can use 2 alternative methods: First, we know the finite transformations (i.e. the rotation matrices) by heart. Then we can obtain the generators $T^a$ by differentiation, $T^a = -i \frac{d g(\theta)}{d \theta} |_{\theta = 0}$. Second, for any Lie group we can form the so-called adjoint representation defined by

$$(T_{adj})_{bc} = -i f^{abc} \quad (8)$$

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Since for SU(2), \( a = \{1, 2, 3\} \), we obtain in this way three \( 3 \times 3 \) matrices, as required. Both alternatives give

\[
J_1 = i \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} \quad J_2 = i \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix} \quad J_3 = i \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

This representation is not identical to the previous one (it doesn’t know that we wish \( S_z \) diagonal), but equivalent and thus connected by a similarity transformation, \( S_i = V J_i V^\dagger \). Requiring \( S_3 = V J_3 V^\dagger \) fixes \( V \) as

\[
V = \frac{1}{2} \begin{pmatrix}
-1 & 0 & 1 \\
-i & 0 & -i \\
0 & 1 & 0
\end{pmatrix}
\]

from which \( S_x \) and \( S_y \) can be calculated.

Alternatively, we could realise that the 2 sets of matrices corresponds to the same operator, one (\( J \)) in a Cartesian and one (\( S \)) in a spherical basis. Or for the basis vectors \( \chi_+ = 1/\sqrt{2}(e_x + ie_y) \), \( \chi_- = -1/\sqrt{2}(e_x - ie_y) \) and \( \chi_0 = e_z \). Then the \( V \) above corresponds to the (complex conjugated) transformation matrix \( \chi = V^*e \).

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