Exercise sheet 5

1. Perturbation of unstable circular orbit

A small perturbation of an unstable circular orbit around a Schwarzschild BH will grow (initially) exponentially with time. Show that a displacement δr will grow as $\delta r \propto \exp(\tau/\tau_*)$, where τ is the proper-time along the trajectory. Evaluate τ_* and explain its behavior for $r_{\max} \to 6M$.

Expand V_{eff} around the unstable maximum,

$$V_{\text{eff}}(r) = V_{\text{eff}}(r_{\max}) + \underbrace{\frac{1}{2}\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}r^2}\bigg|_{r_{\max}}}_{\equiv -K^2 > 0} (\delta r)^2 + \dots$$

with $\delta r = r - r_{\max}$. The radial equation becomes

$$\frac{1}{2}\left(\frac{\mathrm{d}(\delta r)}{\mathrm{d}\tau}\right)^2 - \frac{1}{2}K^2(\delta r)^2 = 0$$

with solution $\delta r \propto \exp(\pm K\tau)$. Thus $\tau_* = 1/K = [2V''_{\text{eff}}(r_{\max})]^{-1/2}$.
To obtain an explicit expression, calculate

$$V''_{\text{eff}}(r) = \frac{-2Mr^2 + 3l^2 r - 12Ml^2}{r^5},$$

what should be evaluated at

$$r_{\max} = \frac{l^2}{2M}\left[1 + \sqrt{1 - 12M^2/l^2}\right].$$

Square the expression and combine terms,

$$-2Mr_{\max}^2 = -2l^2 r_{\max} + 6Ml^2.$$

Inserting this gives

$$V''_{\text{eff}}(r_{\max}) = \frac{l^2 r_{\max} - 12Ml^2}{r_{\max}^5},$$

Thus

$$\tau_* = \left[\frac{1}{2}\frac{r_{\max}^5}{l^2(r_{\max} - 6M)}\right]^{1/2}.$$

The instability disappears, $\tau_* \to \infty$, for $r_{\max} \to 6M$ when stable and unstable orbit coalesce.

2. Survival time.

What is the maximal proper-time an observer can enjoy inside a Schwarzschild black-hole after crossing the horizon?

Solutions are discussed Thursday, 22.02.24

A time-like geodesic maximizes the proper time between two points P_1 and P_2 relative to other time-like paths. Thus using fuel in a rocket trying to escape is counter-prodoctive! Using the radial energy equation, it is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = e^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{l^2}{r^2}\right). \tag{1}$$

Thus

$$\tau = -\int_{2M}^{0} \mathrm{d}r\, \frac{\mathrm{d}\tau}{\mathrm{d}r} = \int_0^{2M} \mathrm{d}r \left[e^2 + \underbrace{\left(\frac{2M}{r}-1\right)}_{>0}\left(1+\frac{l^2}{r^2}\right)\right]^{-1/2}. \tag{2}$$

Thus τ is maximised for $e = l = 0$. Setting $\xi = r/(2M)$, it follows

$$\tau = \int_0^{2M} \mathrm{d}r \left(1-\frac{2M}{r}\right)^{-1/2} = 2M\int_0^1 \mathrm{d}\xi\, \frac{\sqrt{\xi}}{\sqrt{1-\xi}} = \pi M. \tag{3}$$

3. Entropy and evaporation time of BHs

a.) Calculate the (photon) luminosity and the life-time of a Schwarzschild BH, if its temperature is $T = 1/(8\pi M)$. Assume that the area from which the photons are emitted is the horizon area. What is the mass of a primordial (i.e. created at time $t \simeq 0$) BH that evaporates today?

b.) What are potential detection methods?

c.) Estimate the entropy of a $10M_\odot$ star before collapse and compare it to the one of the resulting BH.

d.) Try to estimate the total entropy of the observable universe.

a) By assumption the Stefan-Boltzmann law is valid,

$$L = \frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{c^2\mathrm{d}M}{\mathrm{d}t} = 4\pi R_S^2\sigma T^4 = 4\pi\left(\frac{2GM}{c^2}\right)^2\left(\frac{k^4\pi^5}{60c^2\hbar^3}\right)\left(\frac{\hbar c^3}{8\pi kGM}\right)^4 = \frac{1}{15360\pi}\,\frac{\hbar c^6}{(GM)^2}.$$

Its evaporation time is

$$t = \int_0^M \frac{c^2 dM}{L} = 2560\pi^2\left(\frac{GM}{c^2}\right)^2\left(\frac{M}{h}\right) = 2\times 10^{67} yr \left(\frac{M}{M_\odot}\right)^3.$$

Thus a BH with mass $M \simeq 10^{14}$g has a life-time comparable to the age of the universe. Such ("primordial") BHs might be produced in the early Universe.

b.) The emission of particles with $m \lesssim T$ is exponentially surpressed. As M increases, $T \propto 1/M$ increases and thus heavier particles are also emitted. Thus primordial BHs can be searched for by looking for flashes of high-energy particles. Or by looking at cumulative effect on, e.g. Big Bang Nucleosynthesis.

c.) We use $R = 4R_\odot$ and (somewhat arbitrary) as representative temperature $T = 2 \times 10^6\,\mathrm{K}$ of a B2 star for the estimate. Then (with $1\mathrm{K} \simeq 4.34/\mathrm{cm}$) it is

$$S = \frac{4\pi^2}{45} V T^3 \sim 10^{56}$$

This 15 orders lower than the entropy of a $10M_\odot$ BH.

d.) We first estimate the entropy of a single Milky Way-like galaxy. The entropy in the central SMBH with mass $10^6 M_\odot$ is $S \sim 10^{82}$. This is much larger than the entropy of $N_\odot = 10^{11}$ stars. We neglect stellar and intermediate BHs.
With $n \sim 0.1/\mathrm{Mpc}^3$ as density of normal galaxies, and $4000\,\mathrm{Mpc}$ as radius of the visible universe, there are $N \sim 3 \times 10^{10}$ SMBH, so the total entropy in BHs is at least 10^{92}.
The entropy of CMB photons (with $T = 2.7\,\mathrm{K}$) is

$$S = \frac{4\pi^2}{45} V T^3 \sim 10^{88}$$

thus smaller, but not much smaller.