Exercise sheet 6

1. Stress tensor for the electromagnetic field.
Determine the stress tensor $T^{\mu\nu}$ of the free Maxwell field. Symmetrize $T^{\mu\nu}$, if necessary. Confirm that $T^{\mu\mu}$ corresponds to the energy-density $\rho$. Find the trace of $T^{\mu\nu}$ and the Equation of State (EoS) defined by $w = P/\rho$.

[Possible ways to find $T^{\mu\nu}$: i) use Noether’s theorem, ii) use Newton’s law, iii) convert $\rho = (E^2 + B^2)/2$ into a tensor law, iv) use that the stress tensor is the source of the gravitational field, i.e. $T^{\mu\nu}$ is the lineare response of a variation of the matter action w.r.t. to the metric.]

i.) We start with the standard method, Noether’s theorem, using the definition

$$T^\nu_\mu = \frac{\partial A_\sigma}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial (\partial A_\sigma / \partial x^\nu)} - \delta^\nu_\mu \mathcal{L}.$$  \hspace{1cm} (1)

Since $\mathcal{L}$ depends only on the derivatives $A^\mu_\nu$, we can use the following short-cut: We know already that

$$\delta \mathcal{L} = -\frac{1}{4} \delta(F_{\mu\nu}F^{\mu\nu}) = F^{\mu\nu} \delta(\partial_\nu A_\mu).$$  \hspace{1cm} (2)

Thus

$$\frac{\partial \mathcal{L}}{\partial (\partial A_\sigma / \partial x^\nu)} = F^{\sigma\nu} = -F^{\mu\sigma}$$  \hspace{1cm} (3)

and

$$T^\nu_\mu = -\frac{\partial A_\sigma}{\partial x^\mu} F^{\mu\sigma} + \frac{1}{4} \delta^{\nu}_\mu F_\sigma F^{\sigma\tau}.$$  \hspace{1cm} (4)

Rearranging the indices, we have

$$T^{\mu\nu} = -\frac{\partial A_\sigma}{\partial x^\mu} F^{\nu}_\sigma + \frac{1}{4} \eta^{\mu\nu} F_\sigma F^{\sigma\tau}.$$  \hspace{1cm} (5)

This result in neither gauge invariant (it contains $A^\mu$) nor symmetric. To symmetrize it, we should add

$$\frac{\partial A^\mu_\sigma}{\partial x^\sigma} F^{\nu}_\sigma = \frac{\partial}{\partial x^\sigma}(A^\mu F^{\nu}_\sigma).$$  \hspace{1cm} (6)

The last step is possible for a free electromagnetic field, $\partial_\sigma F^{\mu\sigma} = 0$. And since the term is a divergence and antisymmetric in $\nu\sigma$, it is $\partial_\nu \partial_\sigma F^{\mu\nu} = 0$, and we can add it without changing the conservation law for $T^{\mu\nu}$. The two terms combine to $F$, and we get

$$T^{\mu\nu} = -F^{\mu\sigma} F^{\nu}_\sigma + \frac{1}{4} \eta^{\mu\nu} F_\sigma F^{\sigma\tau}.$$  \hspace{1cm} (7)

It is gauge invariant and symmetric. Its trace is zero, $T^{\mu}_\mu = 0$ – a general result for theories without dimensionfull parameter. The 00 component is

$$T^{00} = -F^{0\sigma} F^0_\sigma + \frac{1}{4} \eta^{00} F_\sigma F^{\sigma\tau}.$$  \hspace{1cm} (8)

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with \( F^{\alpha k} F_k^0 = -E^2 \) and \( F_{\sigma \tau} F^{\sigma \tau} = 2(B^2 - E^2) \),

\[
T^{00} = -F^{\alpha k} F_k^0 + \frac{1}{2} (B^2 - E^2) = \frac{1}{2} (E^2 + B^2) \geq 0.
\]

(9)

We identify \( \rho = T^{00} \) and \( P \delta^{ij} = T^{ij} \) comparing to the ideal fluid or a scalar field. Using then \( T_{\mu} = \rho - 3P = 0 \), the EoS \( w = 1/3 \) follows.

ii.) We use Newton’s law

\[
f_\mu = -\frac{\partial T^{\mu \nu}}{\partial x^\nu},
\]

where \( T^{\mu \nu} \) is the stress tensor of the field acting with the force density \( f_\mu \) on external currents. Inserting the Lorentz force \( f_\mu = F_{\mu \nu} j^\nu \) and Maxwell’s equation \( j^\nu = -\partial_\lambda F^{\nu \lambda} \) gives

\[
f_\mu = F_{\mu \nu} j^\nu = -F_{\mu \nu} \frac{\partial F^{\nu \lambda}}{\partial x^\lambda}.
\]

(10)

We use now the product rule to rewrite this as

\[
-f_\mu = \frac{\partial}{\partial x^\lambda} \left( F_{\mu \nu} F^{\nu \lambda} \right) - F^{\nu \lambda} \frac{\partial F_{\mu \nu}}{\partial x^\lambda}.
\]

(11)

We should rewrite the second term as a symmetric divergence. Starting from

\[
F^{\nu \lambda} \frac{\partial F_{\mu \nu}}{\partial x^\lambda} = \frac{1}{2} \left( F^{\nu \lambda} \frac{\partial F_{\mu \nu}}{\partial x^\lambda} + F^{\nu \lambda} \frac{\partial F_{\mu \nu}}{\partial x^{\nu'}} \right),
\]

(12)

we have exchanged the indices \( \lambda \) and \( \nu \) in the second term. Then we use first in both factors of the second term the antisymmetry of \( F \),

\[
= \frac{1}{2} F^{\nu \lambda} \left( \frac{\partial F_{\mu \nu}}{\partial x^\lambda} + \frac{\partial F_{\mu \nu}}{\partial x^{\nu'}} \right)
\]

(13)

then \( \partial_\mu \tilde{F}^{\mu \nu} = 0 \) and finally \( \partial_\mu F^{\mu \nu} = 0 \),

\[
= -\frac{1}{2} F^{\nu \lambda} \frac{\partial F_{\mu \nu}}{\partial x^\mu}
\]

(14)

\[
= -\frac{1}{4} \frac{\partial}{\partial x^\mu} (F_{\nu \lambda} F^{\nu \lambda}) = -\frac{1}{4} \delta^\lambda_\mu \frac{\partial}{\partial x^\lambda} (F_{\sigma \tau} F^{\sigma \tau})
\]

(15)

Combining, we get

\[
-f_\mu = \frac{\partial}{\partial x^\lambda} \left( F_{\mu \nu} F^{\nu \lambda} + \frac{1}{4} \delta^\lambda_\mu F_{\sigma \tau} F^{\sigma \tau} \right) = \frac{\partial T^{\lambda \mu}}{\partial x^\lambda}.
\]

(16)

or

\[
T_{\mu \nu} = -F_{\mu \lambda} F^{\lambda \nu} + \frac{1}{4} \eta_{\mu \nu} F_{\sigma \tau} F^{\sigma \tau}
\]

(17)

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iii.) We convert \( \rho = T_{00} = (E^2 + B^2)/2 \) into a tensor equation noting that this agrees with

\[
\rho = T_{\alpha\beta} u^\alpha u^\beta
\]

for an observer at rest, \( u^\alpha = (1, 0) \). Our task is to massage the \( E^2 + B^2 \) term into the same form: First we use

\[
\mathcal{L} = -\frac{1}{4} F^2 = \frac{1}{2}(E^2 - B^2),
\]

obtaining

\[
B^2 = E^2 + \frac{1}{2} F^2 = E^2 + \frac{1}{2} F^2 \eta_{\alpha\beta} u^\alpha u^\beta.
\]

The energy density becomes

\[
\rho = E^2 + \frac{1}{4} F^2 \eta_{\alpha\beta} u^\alpha u^\beta.
\]

Next we work on the \( E^2 \): We use \( F_{0k} = E_k \), or [with \( E_\gamma = (0, -E_k) \)] \( u^\alpha F_{\gamma\alpha} = E_\gamma \), obtaining

\[
E^2 = -E_\gamma E^\gamma = -u^\alpha F_{\gamma\alpha} F^\gamma_{\beta} u^\beta.
\]

Combining the terms gives

\[
\rho = (-F_{\gamma\alpha} F^\gamma_{\beta} + \frac{1}{4} \eta_{\alpha\beta} F^2) u^\alpha u^\beta
\]

or

\[
T_{\alpha\beta} = -F_{\alpha\gamma} F_{\beta}^\gamma + \frac{1}{4} \eta_{\alpha\beta} F^2.
\]

iv.) The dynamical energy-momentum tensor is defined as the response of the action \( S_m \) of matter with respect to a variation of the metric tensor,

\[
\frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu}.
\]

You have to rewrite \( S_m \) such that the action is valid for arbitrary coordinates. In general, this implies the replacements \( \{\partial_\mu, \eta_{\mu\nu}\} \rightarrow \{\nabla_\mu, g_{\mu\nu}\} \) in the Langragian. Additionally, the Jacobi determinant \( \sqrt{|g|} \) has to be included into the action.

For the Maxwell field, the step \( \partial_\mu \rightarrow \nabla_\mu \) is not needed, because \( F_{\mu\nu} \) does not depend on the metric tensor. Use \( \delta \sqrt{|g|} = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu} \).

2. Scalar QED.

a.) Write down the Lagrangian of scalar QED, i.e. a complex scalar field coupled to the photon via \( D_\mu = \partial_\mu + ig A_\mu \). Derive the Noether current and the current to which the photon couples (defined by \( \square A^\mu = j^\mu \)).

b.) Find the vertices of this theory. [Pay attention to the sign of the momentum of scalar

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c.) Write down the matrix element for “scalar Compton scattering” $\phi_\gamma \to \phi_\gamma$ and show that it is gauge invariant: i.e. that after replacing $\varepsilon \to k$, the matrix element vanishes.

a.) Excluding a scalar self-interaction, the Lagrangian is

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi - \frac{1}{4} F^2.$$  

or expanded with $D_\mu = \partial_\mu + iqA_\mu$,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - iqA_\mu \phi^\dagger \partial^\mu \phi + iqA^\mu (\partial_\mu \phi^\dagger)\phi + q^2 A^\mu A^\nu \phi^\dagger \phi - \frac{1}{4} F^2.$$  

The wave equation for the photon is $\Box A_\mu = j_\mu$. The Maxwell Lagrange density depends only on gradients $\partial_\mu A_\nu$. Therefore we obtain the current on the RHS of the wave-equation for the photon from

$$j^\mu = \frac{\partial \mathcal{L}_1}{\partial A^\mu} = i \left[ \phi^\dagger D^\mu \phi - (D^\mu \phi)^\dagger \right] = i \left[ \phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi \right] - 2q^2 A^\mu \phi^\dagger \phi. \quad (18)$$  

Note that $j^\mu$ contains $A_\mu$, because $\mathcal{L}_1$ is quadratic in $A^\mu$. As a result, the splitting (we are used from fermions into external currents and the resulting electromagnetic field becomes dubious. The Noether current follows with $\delta \phi = iq\phi$, $\delta \phi^\dagger = -iq\phi^\dagger$, and $\delta A_\mu = 0$,

$$j^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^\dagger} \delta \phi^\dagger = iq \left[ \phi^\dagger D^\mu \phi - (D^\mu \phi^\dagger)\phi \right]. \quad (19)$$  

where we used that $A^\mu$ is invariant under global gauge transformations. Thus the two currents agree as expected. They are conserved and gauge invariant.

Note that the usual $j_\mu A^\nu$ coupling rule applies only for the linear terms. For quadratic terms in $A$, $\partial \mathcal{L}_1/\partial A_\mu$ implies an additional factor 2.

b.) There exists a $\phi^\dagger A$ vertex $(-iqA^\mu [\phi^\dagger \partial^\mu \phi - (\partial_\mu \phi^\dagger)\phi]$ and a $\phi^\dagger A^2$ vertex given by $q^2 \eta_{\mu\nu} A^\mu A^\nu \phi^\dagger \phi$. In the latter case we have to think about the symmetry factor (2, because the two photons can be permuted): Thus the vertex is $2\eta_{\mu\nu}$. In the former case about the sign of the momenta. We check it using the language of canonical quantisation: Starting from the operators for the two real fields, we obtain

$$a_{\pm}(k) = [a_1(k) \pm ia_2(k)]/\sqrt{2} \quad \text{and} \quad a_{\pm}^\dagger(k) = [a_1^\dagger(k) \pm ia_2^\dagger(k)]/\sqrt{2}$$  

with $a_{\pm}(k) |0\rangle = 0$ and $a_{\pm}^\dagger(k) |0\rangle = |k_\pm\rangle$. The field operators $\phi$ and $\phi^\dagger$ are

$$\phi(x) \sim [a_{+}(k)e^{-ikx} + a_{-}^\dagger(k)e^{ikx}] \quad \text{and} \quad \phi^\dagger(x) \sim [a_{+}^\dagger(k)e^{ikx} + a_{-}(k)e^{-ikx}]$$  

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Consider now e.g. an incoming particle with momentum $k$ and an outgoing particle with momentum $k'$. Then $\langle 0 | \phi(x) | k, + \rangle = N_k e^{-i k x}$ and thus

$$\langle k', + | \phi \partial_\mu \phi \rangle | k, + \rangle \propto -i k e^{i (k' - k) x}$$

and

$$\langle k', + | (\partial_\mu \phi^\dagger) \phi \rangle | k, + \rangle \propto i k' e^{i (k' - k) x}$$

Thus the vertex in this case $i (\partial_\mu \phi^\dagger) \phi \partial_\mu \phi = -i q (k + k')^\mu$. (A $k$ comes with an i, another one from $\exp(i S_{\text{int}})$, another one from the prefactor $-i q$.) This can be repeated for the other 3 types of vertices, with the result that the momentum of anti-particles enters with a minus sign – consistent with our interpretation of antiparticles as particles moving backward in time and crossing symmetry.

c.) At tree-level, three diagrams contribute: Two are analogous to the ones of fermionic QED, the third one is the so-called “seagull diagram”,

$$i \mathcal{A} = (-i q)^2 \left[ \epsilon' \cdot (2 p' + k') \frac{1}{(p + k)^2 - m^2} \frac{1}{(p - k')^2 - m^2} \epsilon \cdot (2 p + k) + \epsilon' \cdot (2 p - k') \epsilon \cdot (2 p' + k) - 2 \epsilon' \cdot \epsilon \right]. \quad (20)$$

Performing the gauge transformation $\Lambda(x) = -i \lambda \exp(-i k x)$, the polarisation vector of a photon changes as $\epsilon_\mu \rightarrow \epsilon'_\mu = \epsilon_\mu + \lambda k_\mu$. Since $\lambda$ is arbitrary, replacing $\epsilon$ by $k$ (or $\epsilon' \rightarrow k'$) in $\mathcal{A}$ has to give zero.

Inserting $\epsilon \rightarrow k$ and using $k^2 = 0$, $(p + k)^2 - m^2 = 2 p k$ and $(p - k')^2 - m^2 = -2 p k'$ gives

$$i \mathcal{A} \propto \epsilon' (2 p' + k') \frac{1}{2 p k} 2 p k + \epsilon'(2 p - k') \epsilon \cdot (2 p + k) - 2 \epsilon' \cdot \epsilon k \quad (21)$$

$$= \epsilon'(2 p' + k' - 2 p + k' - 2 k) = 2 \epsilon' (p' + k' - p - k) = 0. \quad (22)$$

3. Local U(1) transformation.

Show that the transformation law for the classical Lagrangian $\mathcal{L}$ of a complex scalar field under a local U(1) transformation $\phi(x) \rightarrow \tilde{\phi}(x) = e^{i \alpha(x)} \phi(x)$ can be expressed as

$$\delta \mathcal{L} = (\partial_\mu \alpha) j^\mu,$$

where $j^\mu$ is the Noether current.

Only problematic term is the kinetic energy $\mathcal{L} = -\phi^\dagger \Box \phi$. For an infinitesimal transformation, $\phi \rightarrow \tilde{\phi} = (1 + i \alpha) \phi$ we find

$$\Box \tilde{\phi} = \partial^\mu \partial_\mu \tilde{\phi} = \partial^\mu (\partial_\mu \phi + i \alpha \partial_\mu \phi + i \partial_\mu \alpha \phi) =$$

$$= \Box \phi + i \Box \alpha \phi + 2 i \partial^\mu \alpha \partial_\mu \phi + i \alpha \Box \phi.\quad (24)$$

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Multiplying with $\tilde{\phi}^\ast = (1 - i\alpha)\phi^\ast$, the $\alpha\phi^\ast \Box \phi$ terms cancel and we are left in linear order with

$$\tilde{\phi}^\ast \Box \tilde{\phi} = \phi^\ast \Box \phi + i \Box \alpha \phi^\ast \phi + 2i\partial^\mu \alpha \partial_\mu \phi + \mathcal{O}(\alpha^2).$$  \hspace{1cm} (25)

Thus the Lagrangian changes as

$$\delta \mathcal{L} = i [2\partial^\mu \alpha \phi^\ast \partial_\mu \phi + \Box \alpha \phi^\ast \phi]$$

$$= i \partial^\mu \alpha [\phi^\ast \partial_\mu \phi - \partial_\mu \phi^\ast \phi] = \partial^\mu \alpha j_\mu,$$  \hspace{1cm} (26)

If one prefers the form $\mathcal{L} = \partial_\mu \phi^\ast \partial^\mu \phi$ for the kinetic energy, then

$$\partial_\mu \phi^\ast \partial^\mu \phi = (\partial_\mu \phi^\ast - i\alpha \partial_\mu \phi^\ast - i\partial_\mu \alpha \phi^\ast)(\partial^\mu \phi + i\alpha \partial^\mu \phi + i\partial^\mu \alpha \phi) =$$

$$= \partial_\mu \phi^\ast \partial^\mu \phi + i\partial_\mu \phi^\ast (\alpha \partial^\mu \phi + \partial^\mu \alpha \phi) - i(\alpha \partial_\mu \phi^\ast + \partial_\mu \alpha \phi^\ast) \partial^\mu \phi =$$

$$= \partial_\mu \phi^\ast \partial^\mu \phi + i\partial_\mu \phi^\ast \partial^\mu \alpha \phi - i\partial_\mu \alpha \phi^\ast \partial^\mu \phi =$$

$$= \partial_\mu \phi^\ast \partial^\mu \phi - \partial^\mu \alpha j_\mu$$  \hspace{1cm} (27)

leads to the same $\delta \mathcal{L}$.