Exercise sheet 6

1. Scalar electrodynamics.
a.) Write down the Lagrangian of scalar QED, i.e. a complex scalar field coupled to the photon via $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$. Derive the Noether current(s) and the current to which the photon couples (defined by $\partial_\mu F^{\mu\nu} = j^\nu$).
b.) Show that $\mathscr{L} = -F^2/4$ corresponds to a canonically normalised field, i.e. that

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} A^\mu D_{\mu\nu} A^\nu,$$

where $D_{\mu\nu}$ is a differential operator.

Excluding a scalar self-interaction, the Lagrangian is

$$\mathscr{L} = (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi - \frac{1}{4} F^2 .$$

or expanded with $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$,

$$\mathscr{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \underbrace{- \mathrm{i}qA_\mu \phi^\dagger \partial^\mu \phi + \mathrm{i}qA^\mu (\partial_\mu \phi^\dagger)\phi + q^2 A_\mu A^\mu \phi^\dagger \phi}_{\mathscr{L}_I} - \frac{1}{4} F^2 .$$

We obtain the current on the RHS of the wave-equation for the photon from

$$j^\mu = -\frac{\partial \mathscr{L}_I}{\partial A_\mu} = \mathrm{i}\left[\phi^\dagger D^\mu \phi - (D^\mu \phi)^\dagger \phi\right] = \mathrm{i}\left[\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi\right] - 2q^2 A^\mu \phi^\dagger \phi . \tag{1}$$

Note that j^μ contains A^μ, because $\mathscr{L}_I$ is quadratic in A^μ. As a result, the splitting (we are used from fermions) into external currents and the resulting electromagnetic field becomes dubious.
The Noether current follows with $\delta\phi = \mathrm{i}q\phi$, $\delta\phi^\dagger = -\mathrm{i}q\phi^\dagger$, and $\delta A_\mu = 0$,

$$j^\mu = \frac{\delta \mathscr{L}}{\delta \partial_\mu \phi} \delta\phi + \frac{\delta \mathscr{L}}{\delta \partial_\mu \phi^\dagger} \delta\phi^\dagger = \mathrm{i}q \left[\phi^\dagger D^\mu \phi - (D^\mu \phi^\dagger)\phi\right] . \tag{2}$$

where we used that A^μ is invariant under global gauge transformations. Thus the two currents agree as expected. They are conserved and gauge invariant.
Note that the ususal $j_\mu A^\mu$ coupling rule applies only for the linear terms. For quadratic terms in A, $\partial\mathscr{L}_I/\partial A_\mu$ implies an additional factor 2.

b.) We write

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) \\ &= \frac{1}{2} (A_\nu \partial_\mu \partial^\mu A^\nu - A_\nu \partial_\mu \partial^\nu A^\mu) = \frac{1}{2} A_\mu \left[\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu\right] A_\nu = \frac{1}{2} A^\mu D^{-1}_{\mu\nu} A^\nu,\end{aligned} \tag{3}$$

where we made a partial integration, dropping as usual the surface term. Note that $D_{\mu\nu}A^\nu = 0$ corresponds to the free Maxwell equation (without imposing a gauge constraint).

2. Dimension of a scalar field ϕ.
We set $\hbar = c = 1$ and choose as the unique dimension the mass unit m.
a.) Find the dimension (i.e. the power m^α) of a scalar field ϕ in d spacetime dimensions.
b.) For which d has $\mathscr{L}_I = \lambda\phi^3$ a dimensionless coupling constant?
c.) For which d has $\mathscr{L}_I = \lambda\phi^4$ a dimensionless coupling constant?

a.) The Lagrange density $\mathscr{L}$ has the dimension m^d in d dimensions, because the action $S = \int d^d x \mathscr{L}$ has to be dimensionless. We denote this briefly as $[\mathscr{L}] = d$. From $\mathscr{L}_0 \propto (\partial_\mu\phi)^2$ we conclude that $[\partial_\mu\phi] = d/2$ and thus $[\phi] = (d-2)/2$. For instance, ϕ has the dimension 1 in $d = 4$ and 2 in $d = 6$ dimensions.
(The field dimension obtained from dimensional analysis as above is often called the "canonical dimension" of the field. If you take a QFT course, you will see that renormalised fields and thus Green functions acquire an "anomalous dimension".)

b.) Plotting $d = n(d-2)/2$ for $n = 3$ ($n = 4$) gives $d = 6$ ($d = 4$).

3. Stress tensor for the electromagnetic field.
a.) Determine the stress tensor $T^{\mu\nu}$ of the electromagnetic field using either i) $f^\mu = -\frac{\partial T^{\mu\nu}}{\partial x^\nu}$, where $T^{\mu\nu}$ is the stress tensor of the field acting with the force density f_μ on external currents, or ii) converting $T^{00} = (\boldsymbol{E}^2 + \boldsymbol{B}^2)/2$ into an valid tensor expression.
b.) Confirm that T^{00} corresponds to the energy-density ρ. Find the trace of $T^{\mu\nu}$ and the Equation of State (EoS) defined by $w = P/\rho$.

a.) We use the force law connecting f^μ and $T^{\mu\nu}$,

$$f^\mu = -\frac{\partial T^{\mu\nu}}{\partial x^\nu},$$

where $T^{\mu\nu}$ is the stress tensor of the field acting with the force density f_μ on external currents. Inserting the Lorentz force $f_\mu = F_{\mu\nu}j^\nu$ and Maxwell's equation $j^\nu = -\partial_\lambda F^{\nu\lambda}$ gives

$$f_\mu = F_{\mu\nu}j^\nu = -F_{\mu\nu}\,\frac{\partial F^{\nu\lambda}}{\partial x^\lambda}. \tag{4}$$

We use now the product rule to rewrite this as

$$-f_\mu = \frac{\partial}{\partial x^\lambda}\left(F_{\mu\nu}F^{\nu\lambda}\right) - F^{\nu\lambda}\frac{\partial F_{\mu\nu}}{\partial x^\lambda}. \tag{5}$$

We should rewrite the second term as a symmetric divergence. Starting from

$$F^{\nu\lambda}\frac{\partial F_{\mu\nu}}{\partial x^\lambda} = \frac{1}{2}\left(F_{\nu\lambda}\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + F_{\lambda\nu}\frac{\partial F_{\mu\lambda}}{\partial x^\nu}\right) \tag{6}$$

we have exchanged the indices λ and ν in the second term. Then we use first in both factors of the second term the antisymmetry of F,

$$= \frac{1}{2}\, F_{\nu\lambda} \left(\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} \right) \tag{7}$$

then $\partial_\mu \tilde{F}^{\mu\nu} = 0$ and finally the product rule,

$$= \quad -\frac{1}{2}\, F_{\nu\lambda}\, \frac{\partial F_{\nu\lambda}}{\partial x^\mu} \tag{8}$$

$$= \quad -\frac{1}{4}\, \frac{\partial}{\partial x^\mu} \left(F_{\nu\lambda} F^{\nu\lambda} \right) = -\frac{1}{4}\, \delta^\lambda_\mu\, \frac{\partial}{\partial x^\lambda} \left(F_{\sigma\tau} F^{\sigma\tau} \right) \tag{9}$$

Combining, we get

$$-f_\mu = \frac{\partial}{\partial x^\lambda} \left(F_{\mu\nu} F^{\nu\lambda} + \frac{1}{4}\, \delta^\lambda_\mu\, F_{\sigma\tau} F^{\sigma\tau} \right) = \frac{\partial T^\lambda_\mu}{\partial x^\lambda}\,. \tag{10}$$

or

$$T_{\mu\nu} = -F_{\mu\lambda} F_\nu{}^\lambda + \frac{1}{4}\, \eta_{\mu\nu}\, F_{\sigma\tau} F^{\sigma\tau} \tag{11}$$

We convert $\rho = T_{00} = (\boldsymbol{E}^2 + \boldsymbol{B}^2)/2$ into a tensor equation noting that this agrees with

$$\rho = T_{\alpha\beta} u^\alpha u^\beta$$

for an observer at rest, $u^\alpha = (1, 0)$. Our task is to massage the $\boldsymbol{E}^2 + \boldsymbol{B}^2$ term into the same form: First we use

$$\mathscr{L} = -\frac{1}{4} F^2 = \frac{1}{2} (\boldsymbol{E}^2 - \boldsymbol{B}^2)\,,$$

obtaining with $u_\alpha u^\alpha = 1$

$$\boldsymbol{B}^2 = \boldsymbol{E}^2 + \frac{1}{2} F^2 = \boldsymbol{E}^2 + \frac{1}{2} F^2 \eta_{\alpha\beta} u^\alpha u^\beta\,.$$

The energy density becomes

$$\rho = \boldsymbol{E}^2 + \frac{1}{4} F^2 \eta_{\alpha\beta} u^\alpha u^\beta\,.$$

Next we work on the $\boldsymbol{E}^2$: We use $F_{0k} = E_k$ or $u^\alpha F_{\gamma\alpha} = E_\gamma$, obtaining

$$\boldsymbol{E}^2 = -E_\gamma E^\gamma = -u^\alpha F_{\gamma\alpha} F^\gamma{}_\beta u^\beta\,.$$

Combining the terms gives

$$\rho = (-F_{\gamma\alpha} F^\gamma{}_\beta + \frac{1}{4} \eta_{\alpha\beta} F^2) u^\alpha u^\beta$$

or

$$T_{\alpha\beta} = -F_{\alpha\gamma} F_\beta{}^\gamma + \frac{1}{4} \eta_{\alpha\beta} F^2\,.$$

b.) Its trace is zero, $T^\mu_\mu = 0$ – a general result for theories without dimensionfull parameter. The 00 component is

$$T^{00} = -F^{0\sigma}\, F^0{}_\sigma + \frac{1}{4}\,\eta^{00} F_{\sigma\tau} F^{\sigma\tau}\,. \tag{12}$$

with $F^{0k}\, F^0{}_k = -\boldsymbol{E}^2$ and $F_{\sigma\tau} F^{\sigma\tau} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2)$,

$$T^{00} = -F^{0k}\, F^0{}_k + \frac{1}{2}\,(\boldsymbol{B}^2 - \boldsymbol{E}^2) = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) \geq 0\,. \tag{13}$$

We identify $\rho = T^{00}$ and $P\delta^{ij} = T^{ij}$ comparing to the ideal fluid or a scalar field. Using then $T^\mu_\mu = \rho - 3P = 0$, the EoS $w = 1/3$ follows.