Exercise sheet 7

Linear Sigma model I.
Consider the Lagrangian
\[ \mathcal{L} = \frac{1}{2} \left[ (\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2 \right] - \frac{1}{2} m^2 (\pi^2 + \sigma^2) - \frac{\lambda}{4} (\pi^2 + \sigma^2)^2. \]

[You may interpret the fields \( \pi = (\pi_1, \pi_2, \pi_3) \) as pions with isospin 1, and \( \sigma \) as a sigma meson with isospin 0; the corresponding SU(2) symmetry has then the physical interpretation of isospin.]

a) Show that the Lagrangian \( \mathcal{L} \) is invariant under the symmetry transformation
\[ \Sigma \to \Sigma' = U \Sigma U^\dagger, \tag{1} \]
where
\[ \Sigma \equiv \sigma + i \tau \cdot \pi, \]
\( U = \exp(i\alpha \cdot \tau/2) \) and \( \tau = (\tau^1, \tau^2, \tau^3) \) are the Pauli matrices. Find the corresponding conserved Noether currents. [Hint: Calculate first \( \Sigma \Sigma^\dagger \).]

b) Give the Feynman rules (i.e. specify propagators and vertices in momentum space) for this Lagrangian.

Remark: This Lagrangian called the “linear sigma model” is both useful to understand the effect of spontaneous symmetry breaking and, adding terms describing nucleons, as a model for low-energy nucleon-meson interactions. Moreover, comparing it to non-linear versions of the sigma model which share the same symmetries but look otherwise completely different shows that all models predict the same physics at low energies.

a.) First we note that \( \mathcal{L} = \mathcal{L}^* \) implies that the fields are real. Following then the hint, we calculate
\[ \Sigma \Sigma^\dagger = (\sigma + i \tau \cdot \pi)(\sigma - i \tau \cdot \pi) = \sigma^2 + (\tau \cdot \pi)^2 = \sigma^2 + \pi^2. \tag{2} \]

The last term is more precisely a matrix, \((\sigma^2 + \pi^2)1\), and thus we should take the trace to obtain a scalar. Thus the Lagrangian can be expressed as
\[ \mathcal{L} = \frac{1}{4} \text{tr} \left( \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \right) - \frac{1}{4} m^2 \text{tr} \left( \Sigma \Sigma^\dagger \right) - \frac{\lambda}{16} \left( \text{tr} \left( \Sigma \Sigma^\dagger \right) \right)^2. \]

Next we use \( \Sigma' = U \Sigma U^\dagger \), \( \Sigma'^\dagger = U \Sigma'^\dagger U^\dagger \) and \( \text{tr}(AB) = \text{tr}(BA) \), to obtain
\[ \text{tr} \left( \Sigma \Sigma^\dagger \right) \to \text{tr} \left( U \Sigma U^\dagger U \Sigma'^\dagger U^\dagger \right) = \text{tr} \left( \Sigma \Sigma^\dagger \right). \]

Since \( U \) is constant, the derivatives in the kinetic term do not harm. Thus \( \mathcal{L} \) is invariant under the (global) symmetry transformation (1).

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To find the conserved currents, we consider infinitesimal transformations $U = \exp(i\alpha \cdot \tau/2) \approx 1 + i\alpha \cdot \tau/2$ or

$$
\Sigma \rightarrow \Sigma' = U\Sigma U^\dagger \approx \left( 1 + \frac{i\alpha \cdot \tau}{2} \right) (\sigma + i\tau \cdot \pi) \left( 1 - \frac{i\alpha \cdot \tau}{2} \right)
$$

or

$$
\Sigma' = \sigma' + i\tau \cdot \pi' \approx \sigma + i\tau \cdot \pi - i\pi \cdot (\alpha \times \pi)
$$

Next we calculate the commutator using $[\tau^i, \tau^j] = 2i\varepsilon^{ijk} \tau^k$,

$$
\left[ \frac{\alpha \cdot \tau}{2}, \tau \cdot \pi \right] = i\alpha \times \pi \cdot \tau
$$

and find

$$
\Sigma' = \sigma' + i\tau \cdot \pi' \approx \sigma + i\tau \cdot \pi - i\pi \cdot (\alpha \times \pi)
$$

or

$$
\delta \sigma = 0 \quad \text{and} \quad \delta \pi = -\alpha \times \pi.
$$

(As expected, the isoscalar $\sigma$ is invariant, while $\pi$ transforms as a vector under rotation.)

The conserved (isospin vector) current following from Noether’s theorem is

$$
-\alpha \cdot V^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \pi)} \delta \pi = -\partial^\mu \pi \cdot (\alpha \times \pi).
$$

or

$$
V^\mu = \partial^\mu \pi \times \pi.
$$

b.) The $Z_2$ symmetry $\Sigma \rightarrow -\Sigma$ implies that one produces $\sigma$’s and $\pi^i$ always in pairs,

$$
\mathcal{L}_I = -\frac{\lambda}{4} (\pi_i \pi_i \pi_j \pi_j + 2\pi_i \pi_i \sigma^2 + \sigma^4).
$$

Their scalar propagators are diagonal

$$
\sigma \quad \sigma = \frac{i}{k^2 - m^2 + i\varepsilon}
$$

$$
\pi^i \quad \pi^j = \frac{i\delta^{ij}}{k^2 - m^2 + i\varepsilon}
$$

The vertex is the structure between the fields (in momentum space) from $i\mathcal{L}_I$. Since a scalar field (without normalisation factor) is one in momentum space, the vertex is simply $-i\lambda S$ with $S$ as the symmetry factor.

Formally, we can find the $\Sigma^i \Sigma^k \cdots \Sigma^n$ vertex by calculating $\partial^\mu \mathcal{L}/(\partial \Sigma^i \partial \Sigma^k \cdots \partial \Sigma^n)$ with $\Sigma^i = (\pi, \sigma)$. This gives a factor $4!/4 = 6$ for four identical particles and $2(2!)^2/4 = 2$ for two different pairs at the vertex. Thus a general $ijk$ vertex is $-2i\lambda[\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}]$, i.e. $-6i\lambda$ for four identical and $-2i\lambda$ for two pairs at the vertex.

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Projection operators.
Show that the inverse of the matrix \( A = \sum_i a_i P_i \), where the \( P_i \) are projection operators (i.e. \( \sum_i P_i = 1 \) and \( P_i P_j = \delta_{ij} P_j \)), is given by \( A^{-1} = \sum_i a_i^{-1} P_i \).

Determine
\[
A^{-1} A = \sum_{i,j} a_i^{-1} a_j P_i P_j = \sum_i a_i^{-1} a_i P_i = \sum_i P_i = 1.
\]

Dirac representation.
Show that the \( \gamma \) matrices in the Dirac representation,
\[
\gamma^0 = 1 \otimes \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
\[
\gamma^i = \sigma^i \otimes i\tau_2 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},
\]
where \( \sigma_i \) and \( \tau_i \) are the Pauli matrices, \( \otimes \) denotes the tensor product, satisfy \( \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu} \).

Straightforward, but simplifies expressing the gamma matrices as tensor product of Pauli matrices. Then e.g.,
\[
\gamma^i \gamma^j = (\sigma^i \otimes i\tau_2)(\sigma^j \otimes i\tau_2) = \sigma^i \sigma^j \otimes i^2 \tau_2 \tau_2 = -\sigma^i \sigma^j \otimes 1_2
\]
and thus
\[
\{ \gamma^i, \gamma^j \} = -\underbrace{\{ \sigma^i, \sigma^j \}}_{2\delta^{ij}1_2} \otimes 1_2 = -2\delta^{ij}1_2.
\]

Similarly \( \gamma^0 \gamma^i = (1 \otimes \tau_3)(\sigma^i \otimes i\tau_2) = i\sigma^i \otimes (\tau_3 \tau_2) = i\sigma^i \otimes (-i\tau_1) \) and thus \( \{ \gamma^0, \gamma^i \} = \{ \sigma^i, \sigma^0 \} \otimes \tau_1 = 0 \).

Traces of gamma’s.
Calculate
\[
\text{tr}[\gamma \gamma^\mu]\n\text{tr}[\gamma^\mu \gamma \gamma^\nu]\n\gamma^\mu \gamma^\nu
\]
Using the tensor method, we have to express \( \text{tr}[\gamma^\mu \cdots \gamma^\nu \cdots \gamma^\rho] \) as the sum of possible combinations of allowed tensors \( (\tau^\mu_{\nu\rho}) \) and possibly \( \epsilon^{\mu\nu\lambda\sigma} \) taking into account the cyclic

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property of the trace. Contracting \( \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu} \) with \( \eta_{\mu\nu} \) gives

\[
2\gamma^\mu \gamma^\nu = 2\eta^{\mu\nu} = 8
\]
or \( \gamma^\mu \gamma^\mu = 4 \). Thus

\[
\gamma^\lambda \gamma_\mu \gamma_\lambda = \gamma^\lambda (2\eta_{\mu\lambda} - \gamma_\lambda \gamma^\mu) = 2\gamma_\mu - 4\gamma_\mu = -2\gamma_\mu
\]
or \( \gamma^\mu \delta^\mu \gamma_\mu = -2\delta \).

We determine next \( \text{tr}(\gamma^\mu \gamma^\nu) \) using that this expression has to be proportional to the metric tensor, \( \text{tr}(\gamma^\mu \gamma^\nu) = A\eta^{\mu\nu} \). Contraction gives

\[
\text{tr}(4 \cdot 1) = 4A
\]
or \( A = 4 \) and \( \text{tr}[\delta^\mu \delta^\nu] = 4a \cdot b \).

Similarly, \( \text{tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa) \) has to be proportional to products \( \eta \eta \) of the metric tensor,

\[
\text{tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa) = C[\eta^{\mu\nu} \eta^{\lambda\kappa} + \eta^{\mu\kappa} \eta^{\nu\lambda}] + D\eta^{\mu\lambda} \eta^{\nu\kappa}.\]

The first pair combines the neighbouring indices, e.g. \( (\mu\nu) \) or \( (\kappa\mu) \), in one metric tensor, the second separated ones. Since the LHS is even under parity, \( \varepsilon^{\mu\nu\lambda\kappa} \) cannot appear on the RHS. (We could add \( E\varepsilon^{\mu\nu\lambda\kappa} \) to the RHS, deriving below \( E = 0 \).) Contracting first with \( \eta_{\mu\nu} \) gives

\[
\text{tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa) = 16\eta^{\lambda\kappa} = C[4 + 1]\eta^{\lambda\kappa} + D\eta^{\lambda\kappa}
\]
or \( 16 = 5C + D \). Contracting instead with \( \eta_{\mu\lambda} \) gives as second relation

\[
\text{tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa) = -8\eta^{\nu\kappa} = (2C + 4D)\eta^{\nu\kappa}
\]
or \( C = -4 - 2D \). Combined we find \( D = -4 = -C \).

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