Exercise sheet 8

1. Binary system on circular orbits

Consider a binary system of two stars with equal mass $M$ on circular orbits.

a.) Calculate the quadrupole moments $I_{ab}$ using (9.68).

b.) Find the amplitude of the gravitational wave $\bar{h}_{\alpha\beta}(t, \mathbf{x})$ using (9.69).

c.) Estimate the strength for a Galactic neutron star-neutron star binary with a separation of $r = 0.1$ AU.

We choose a binary system with circular orbits centers at the origin in the xy-plane. Then

$$x_1 = R \cos \Omega t, \quad y_1 = R \sin \Omega t,$$

and

$$x_2 = -R \cos \Omega t, \quad y_2 = -R \sin \Omega t.$$  

The corresponding energy density is

$$T^{00} = M \delta(z)[\delta(x - R \cos \Omega t)\delta(y - R \sin \Omega t) + \delta(x + R \cos \Omega t)\delta(y + R \sin \Omega t)]$$

The quadrupole moment follows as

$$I_{xx} = 2MR^2 \cos^2 \Omega t = MR^2(1 + \cos^2 2\Omega t)$$

$$I_{yy} = 2MR^2 \sin^2 \Omega t = MR^2(1 - \cos^2 2\Omega t)$$

$$I_{xy} = I_{yx} = 2MR^2 \cos \Omega t \sin \Omega t = MR^2 \sin 2\Omega t$$

$$I_{zz} = 0$$

b.) Differentiating twice gives $\ddot{I}_{ij} = 4\Omega^2 I_{ij}$ and thus the GW amplitude follows as

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{8GM}{r} (\Omega R)^2 \begin{pmatrix} \cos 2\Omega t_r & \sin 2\Omega t_r & 0 \\ \sin 2\Omega t_r & -\cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Thus the frequency $\omega$ of the GW is twice the rotation frequency, $\omega = 2\Omega$. For a wave travelling in the $z$ direction, the expression is already in the TT gauge. In general, the projection operator defined in the appendix has to be applied to obtain the physical states of the GW.

c.) Neglecting the oscillation factor, the amplitude is

$$h = \frac{8GM}{r} (\Omega R)^2$$

For a numerical estimate, it is convenient to eliminate $(\Omega R)^2$ using the Keplerian velocity and to replace $2GM$ by $R_S$,

$$h \simeq \frac{2R_S^2}{ar} \simeq 10^{-24}$$

Solutions are discussed Friday, 01.04.22
2. Energy flux of a GW
The energy flux $F$ of a GW is

$$F = \frac{c^3}{32\pi G} \omega^2 (a^2 + b^2).$$

where $a$ and $b$ are the amplitudes of the two polarisation states.

a.) Estimate the energy flux for the binary system in 1.

b.) Estimate how much energy is dissipated if a GW crosses the interstellar or intergalactic medium: Which processes might be relevant? Use simple dimensional analysis for your estimate.

a.) We calculate first

$$\frac{c^3}{32\pi G} \approx 10^{32} \frac{\text{erg}}{s \text{ cm}^2}$$

Thus the energy flux of a GW with amplitude $h \approx 10^{-22}$ is $10^{12} \frac{\text{erg}}{s \text{ cm}^2}$.

b.) In a fluid with friction, the relative oscillations of fluid elements induced by the passage of a GW will lead to dissipation of energy. We estimate the absorption rate $\Gamma$ of a GW by dimensional analysis: The effect of friction in a fluid is described the viscosity $\nu = \eta/\rho$ with $[\nu] = \text{cm}^2/\text{s}$. The absorption rate should be proportional to $G\rho$ and thus we use as ansatz

$$\Gamma \propto G\nu \rho c^n$$

where we added an arbitrary power of $c$ to obtain the the correct dimension of $\Gamma$. Defining $\Gamma$ as absorption rate per length, $|h|^2 = |h_0|^2 \exp(-\Gamma L)$, it is

$$\frac{1}{\text{cm}} = \frac{\text{cm}^3}{\text{g s}} \frac{\text{cm}^2}{\text{s}} \frac{\text{g}}{\text{s}} \left(\frac{\text{cm}}{\text{s}}\right)^n$$

Thus it is $n = -3$; the exact result is

$$\Gamma = 16\pi G \frac{\nu \rho}{c^3}$$

For the ISM of the Milky Way, $\nu \sim 10^{18}\text{cm}^2/\text{s}$, $n_h \approx 1/\text{cm}^3$ and thus $\Gamma \approx 10^{-43}/\text{cm}$. Using $L = 10\text{ kpc}$, $\Gamma L \approx 10^{-22}$ and absorption is completely negligible.

Solutions are discussed Friday, 01.04.22