Exercise sheet 8

0. Green function of the wave equation.
Recall from your electrodynamics lectures the derivation of the Green function of the wave equation.

1. Hyperbolic plane H^2.
Calculate the Riemann (or curvature) tensor $R^a{}_{bcd}$ and the scalar curvature R for the hyperbolic plane H^2.

b. We calculate e.g.

$$\begin{aligned} R^y{}_{xyx} &= \partial_y \Gamma^y{}_{xx} - \partial_x \Gamma^y{}_{xy} + \Gamma^y{}_{ey}\Gamma^e{}_{xx} - \Gamma^y{}_{ex}\Gamma^e{}_{xy} \\ &= -1/y^2 + 0 + \Gamma^y{}_{yy}\Gamma^y{}_{xx} - \Gamma^y{}_{xx}\Gamma^x{}_{xy} \\ &= -1/y^2 + 0 - 1/y^2 + 1/y^2 = -1/y^2 \,. \end{aligned}$$

Next we remember that the number of independent components of the Riemann tensor in $d = 2$ is one, i.e. we are already done: All other components follow by the symmetry properties.
The scalar curvature is (diagonal metric with $g^{xx} = g^{yy} = y^2$)

$$R = g^{ab}R_{ab} = g^{xx}R_{xx} + g^{yy}R_{yy} = y^2(R_{xx} + R_{yy})\,.$$

Thus we have to find only the two diagonal components of the Ricci tensor $R_{ab} = R^c{}_{acb}$. With

$$\begin{aligned} R_{xx} &= R^c{}_{xcx} = R^x{}_{xxx} + R^y{}_{xyx} = 0 + R^y{}_{xyx} = -1/y^2 \\ R_{yy} &= R^c{}_{ycy} = R^x{}_{yxy} + R^x{}_{yxy} = R^x{}_{yxy} + 0 = -R^y{}_{xxy} = R^x{}_{yxy} = -1/y^2\,, \end{aligned}$$

the scalar curvature follows as $R = -2$. Hence the hyperbolic plane H^2 is a space of constant curvature, as $\mathbb{R}^2$ and S^2.
[If you wonder that $R = -2$, not -1: in $d = 2$, the Gaussian curvature K is connected to the "general" scalar curvature R via $K = R/2$. Thus $K = \pm 1$ means $R = \pm 2$ for spaces of constant unit curvature radius, S^2 and H^2. You may also check that the Riemann and Ricci tensor satisfy the relations for maximally symmetric spaces, $R_{ab} = Kg_{ab}$ and $R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc})$.]

2. Spin and helicity of tensor fields.
A plane wave ψ which is transformed into $\psi' = \mathrm{e}^{-\mathrm{i}h\alpha}\psi$ by a rotation with angle α around its propagation axis is said to have helicity h. Specifically, consider a photon and a graviton propagating into the z direction, $\boldsymbol{k} = k\boldsymbol{e}_z$, and choose the rotation in the xy plane. Start with linear polarised states,

$$A^\mu = \varepsilon^\mu \mathrm{e}^{-\mathrm{i}kx}, \qquad h^{\mu\nu} = \varepsilon^{\mu\nu}\mathrm{e}^{-\mathrm{i}kx},$$

where the two polarisation vectors are $\varepsilon_\mu^{(1)} = \delta_\mu^1$ and $\varepsilon_\mu^{(2)} = \delta_\mu^2$ for the photon, and the polarisation tensors for the graviton are

$$\varepsilon_1^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon_2^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Rotate the states, introduce circular polarised states and find their helicity. [Remark: Helicity is the projection of spin on the momentum: Thus this exercises shows that the photon (or generally a vector field) is a spin-1, while the graviton is a spin-2 particle.]

For a rotation in the xy plane, the general Lorentz transformation $\Lambda_\mu^{\ \nu}$ becomes

$$\Lambda_\mu^{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Since $\boldsymbol{k} = k\boldsymbol{e}_z$ and thus $\Lambda_\mu^{\ \nu} k_\nu = k_\mu$, the rotation affects only the polarisation vector and tensor, respectively. Moroever, it is sufficient to perform the calculation for the xy sub-matrices. In the case of the photon,

$$\begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha e_1 - \sin\alpha e_2 \\ \sin\alpha e_1 + \cos\alpha e_2 \end{pmatrix}$$

Next we change to circularly polarised states,

$$\begin{aligned} \tilde{e}_+ &= \tilde{e}_1 + \mathrm{i}\tilde{e}_2 = \cos\alpha e_1 - \sin\alpha e_2 + \mathrm{i}(\sin\alpha e_1 + \cos\alpha e_2) = (\cos\alpha + \mathrm{i}\sin\alpha)e_1 + (\mathrm{i}\cos\alpha - \sin\alpha)e_2 \\ &= (\cos\alpha + \mathrm{i}\sin\alpha)e_1 + \mathrm{i}(\cos\alpha + \mathrm{i}\sin\alpha)e_2 = \mathrm{e}^{\mathrm{i}\alpha}(e_1 + \mathrm{i}e_2) = \mathrm{e}^{\mathrm{i}\alpha}e_+ \end{aligned}$$

Thus the photon has helicity $h = 1$. For gravitons, we rewrite $\varepsilon'_{\mu\nu} = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma}\varepsilon_{\rho\sigma}$ in matrix notation, $\varepsilon' = \Lambda\varepsilon\Lambda^T$. Performing the same calculation for the xy sub-matrices gives

$$\begin{aligned} \tilde{\varepsilon}_1 &= (\cos^2\alpha - \sin^2\alpha)e_1 - 2\cos\alpha\sin\alpha e_2 \\ \tilde{\varepsilon}_2 &= (\cos^2\alpha - \sin^2\alpha)e_2 + 2\cos\alpha\sin\alpha e_1 \end{aligned}$$

Introducing again ciruclarly polarised states gives

$$\tilde{\varepsilon}_+ = (\cos^2\alpha - \sin^2\alpha + 2\mathrm{i}\cos\alpha\sin\alpha)(e_1 + \mathrm{i}e_2) = \exp(2\mathrm{i}\alpha)\varepsilon_+\,.$$

Thus a GW (or in the particle picture the graviton) has helicity $h = 2$. Geometrically, we can visulaize the symmetric tensor ε_{ij} as ellipsoid which is mapped by a rotatation by 180° on itself.