Exercise sheet 9

1. Spacetime transformations in Minkoswki space are linear.
Show that spacetime transformations in Minkoswki space are linear, i.e. that

$$\frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu \partial\tilde{x}^\rho} = 0$$

for a Poincaré transformation. [Hint: Start from the general transformation law for rank-2 tensor applied to $\eta_{\mu\nu}$.]

As recommended, we start from

$$\tilde{\eta}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}\,\eta_{\alpha\beta}(x).$$

We differentiate to get an expression with second-order derivatives,

$$\frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu \partial\tilde{x}^\rho}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}\,\eta_{\alpha\beta}(x) + \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial^2 x^\beta}{\partial\tilde{x}^\nu \partial\tilde{x}^\rho}\,\eta_{\alpha\beta}(x) = 0.$$

Then we add the equation with μ and ρ interchanged, and subtract the equation with ν and ρ interchanged, resulting in

$$2\frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu \partial\tilde{x}^\rho}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}\,\eta_{\alpha\beta}(x) = 0.$$

This implies

$$\frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu \partial\tilde{x}^\rho} = 0$$

Thus the transformation is linear.

2. Decomposition of $h_{\mu\nu}$ in irreducible components
An object which contains invariant subgroups with respect to a symmetry operation is called reducible. As an example, consider the reducible subgroups of a symmetric tensor $h^{\mu\nu}$ of rank two with respect to spatial rotations.
a.) Split $h^{\mu\nu}$ in a first step into a 3-scalar, a 3-vector, and a 3-tensor.
b.) Recognise next that the 3-vector and the 3-tensor are still reducible, recalling the Helmholtz decomposition theorem ("Any vector in $\mathbb{R}^3$ can be written as the sum of a divergence-free and a rotation-free vector"). Use this and the tensor method to find the irreducible components of $h_{\mu\nu}$

a.) The decomposition is the same for all symmetric tensors of rank 2. Knowing $T^{\mu\nu}$, we set.

$$h^{\mu\nu} = \begin{pmatrix} h^{00} & h^{0i} \\ h^{i0} & h^{ij} \end{pmatrix}.$$

or, changing notation,

$$h_{\mu\nu} = \begin{pmatrix} 2A & B_i \\ B_i & -C_{ij} \end{pmatrix}. \tag{1}$$

b.) The function A is already a scalar, while we have to find the irreducible components of B_i and C_{ij}. Any vector in $\mathbb{R}^3$ can be written as the sum of a divergence-free and a rotation-free vector; the latter is the gradient of a scalar. Thus we can perform the replacements

$$B_i = -\partial_i B + V_i \tag{2}$$

and

$$C_{ij} = 2D\delta_{ij} + 2\partial_i\partial_j E + (\partial_i E_j + \partial_j E_i) + h_{ij}. \tag{3}$$

The six degrees of freedom of the reducible tensor C_{ij} are decomposed into two scalar (D, E), two vector (E_i with the constraint $\partial_i E_i = 0$) and two tensor (h_{ij}) degrees of freedom. The tensor h_{ij} corresponds to gravitational waves in the TT gauge, with $h_{ii} = 0$ and $\partial_i h_{ij} = 0$. he three degrees of freedom of the reducible vector B_i are decomposed into one scalar (B) and two vector (B_i with the constraint $\partial_i B_i = 0$) degrees of freedom. Thus altogether, $h_{\mu\nu}$ contains four scalar, four vector and two tensor degrees of freedom.
This decomposition is uesful, since at linear order in Minkowski and FLRW spaces scalar, vector and tensor degrees of freedom do not mix. Decomposing in the same way the stress tensor, the scalar perturbation of the stress tensors sources the scalar components of $h_{\mu\nu}$, and vece versa, and similar for the vector and tensor perturbations.

3. Binary system on circular orbits

Consider a binary system of two stars with equal masses on circular orbits.
a.) Calculate the quadrupole moments I_{ab}.
b.) Find the amplitude of the gravitational wave $\bar{h}_{\alpha\beta}(t, \boldsymbol{x})$ from (10.70).
c.) Find the emitted luminosity of gravitational waves from (10.86) and (10.87).
d.) Estimate the strength for a Galactic neutron star neutron star binary with a separation of $r = 0.1\,\mathrm{AU}$.

We choose a binary system with circular orbits centered at the origin in the xy-plane. Then

$$x_1 = R\cos\Omega t\,, \qquad y_1 = R\sin\Omega t\,, \tag{4}$$

and

$$x_2 = -R\cos\Omega t\,, \qquad y_2 = -R\sin\Omega t\,. \tag{5}$$

The corresponding energy density is

$$T^{00} = M\delta(z)[\delta(x - R\cos\Omega t)\delta(y - R\sin\Omega t) + \delta(x + R\cos\Omega t)\delta(y + R\sin\Omega t)] \tag{6}$$

The quadrupole moment follows as

$$
\begin{aligned}
I_{xx} &= 2MR^2\cos^2\Omega t = MR^2[1+\cos(2\Omega t)] && \text{(7a)}\\
I_{yy} &= 2MR^2\sin^2\Omega t = MR^2[1-\cos 2\Omega t)] && \text{(7b)}\\
I_{xy} &= I_{yx} = 2MR^2\cos\Omega t\sin\Omega t = MR^2\sin 2\Omega t && \text{(7c)}\\
I_{iz} &= 0 && \text{(7d)}\\
I &= 2MR^2 && \text{(7e)}
\end{aligned}
$$

b.) Differentiating twice gives adds a factor $-4\Omega^2$ in font of the sin and cos functions. The GW amplitude follows as

$$
\bar{h}_{ij}(t,\boldsymbol{x}) = \frac{8GM}{r}\,(\Omega R)^2 \begin{pmatrix} \cos 2\Omega t_r & \sin 2\Omega t_r & 0 \\ \sin 2\Omega t_r & -\cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}
$$

Thus the frequency ω of the GW is twice the rotation frequency, $\omega = 2\Omega$. For a wave travelling in the z direction, the expression is already in the TT gauge. In general, the projection operator defined in the appendix has to be applied to obtain the physical states of the GW.

b.) We differentiate once more, obtaining

$$
\dddot{Q}_{ij}\dddot{Q}^{ij} = \dddot{I}^{\,2}_{xx}+2\,\dddot{I}_{xy}+\dddot{I}_{yy}-\frac{1}{3}\,\dddot{I}^{\,2} = (8MR^2\omega^3)^2\left[\sin^2(2\Omega t_r)+2\cos^2(2\Omega t_r)+\sin^2(2\Omega t_r)\right] = 128M^2R^4\Omega^6. \tag{9}
$$

Using $\Omega^2 = GM/a^3 = 4\omega^2$ into the qudrupol formula, we obtain as final result

$$
L = \frac{128}{5}M^2R^4\Omega^6 = \frac{2}{5}G^4M^5/R^5. \tag{10}
$$

d.) Neglecting the oscillating factor, the amplitude is

$$
h = \frac{8GM}{r}\,(\Omega R)^2
$$

For a numerical estimate, it is convinient to eleminate $(\Omega R)^2$ using the Keplerian velocity and to replace $2GM$ by R_S,

$$
h \simeq 2\frac{R_s^2}{ar} \simeq 10^{-33}
$$