Exercise sheet 9

**Higgs decay into fermions and the optical theorem.**
In the Standard Model, the Higgs particle $h$ is a scalar particle that interacts with all fermions via a Yukawa coupling $y$ proportional to the fermion mass $m$, $y = \frac{1}{2} gm/m_W$.

![Diagram of Higgs decay](image)

a.) Look up the formula for $1 \rightarrow 2$ decay and calculate the decay width $\Gamma(h \rightarrow \bar{f}f)$ of a Higgs particle with mass $M$ into a antifermion-fermion pair (at tree-level).

b.) The contribution of (coloured) fermions to the self-energy $\Sigma(p^2)$ of the Higgs is given by

$$\Sigma^f = N_c \frac{12 g^2}{(4\pi)^2} \left[ \mu^2 \left( \frac{m^2 - q^2}{6} \right) \frac{1}{\varepsilon} + \left( -\frac{m^2}{3} + \frac{q^2}{18} + \int_0^1 d\alpha \alpha^2 \ln(\alpha^2/\mu^2) \right) \right]$$

where we absorbed the constants in $\mu$ and $a^2 = m^2 - p^2z(1 - z) - i\varepsilon$. Determine the imaginary part $\Im \Sigma^f$ of the self-energy and show that the optical theorem holds, i.e. that $\Im \Sigma^f = M \Gamma(h \rightarrow \bar{f}f)$ for $p^2 = M^2$.

c.) Obtain $\Im \Sigma^f$ directly by “cutting the self-energy”: Consider

$$\ii \Sigma^f(p^2) = \int \frac{d^4 q}{(2\pi)^4} \cdots$$

for $p = (M, 0)$; find the poles and apply the identity

$$\frac{1}{x \pm i\varepsilon} = P \left( \frac{1}{x} \right) \mp i\pi \delta(x)$$

to the $q^0$ integral in order to obtain the imaginary part.

a.) The Feynman amplitude is

$$\ii \mathcal{A} = -iy \bar{u}(q_1, s_1) v(q_2, s_2).$$

Squaring and summing over final spins gives

$$\sum_{s_1, s_2} |\mathcal{A}|^2 = y^2 \sum_{s_1, s_2} \bar{u}(q_1, s_1)v(q_2, s_2)\bar{u}(q_2, s_2)v(q_1, s_1) = y^2 \text{tr}[(\not{q}_1 + m)(\not{q}_2 - m)]$$

with $\text{tr}[(\not{q}_1 + m)(\not{q}_2 - m)] = 4(q_1q_2 - m^2)$. In the cms frame, the momenta are

$$p^\mu = (M, 0), \quad q_1^\mu = (M/2, q_{\text{cms}}), \quad q_2^\mu = (M/2, -q_{\text{cms}}).$$

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with
\[ q_{\text{cms}} = \frac{M}{2} \left( 1 - \frac{4m^2}{M^2} \right)^{1/2} = \frac{M}{2} \beta^{1/2} \quad \text{and} \quad 4q_1q_2 - m^2 = 4 \left( \frac{M^2}{4} + q_{\text{cms}}^2 - m^2 \right) = 2M^2 \beta. \]

Here, \( \beta \) denotes the velocity of the fermions in the cms frame. Since \(|\mathcal{A}|^2\) is a scalar, the remaining angular integration in \(d\Phi^{(3)}\) gives a trivial factor \(4\pi\). Assembling everything, the total decay width in one fermion type follows as
\[ \Gamma = N_c \frac{g^2}{8\pi} M \left( 1 - \frac{4m^2}{M^2} \right)^{3/2} = N_c \frac{g^2}{32\pi} \frac{M^3}{m_W^2} \beta^{3/2}, \]
where \(N_c = \{1, 3\}\) for leptons/quarks takes into account the 3 quark color.

b.) Only the log term can generate an imaginary part for \( q^2 \geq 4m^2 \),
\[ F = \int_0^1 dz \left[ m^2 - q^2 z(1 - z) \right] \ln \left[ m^2 - q^2 z(1 - z) \right]. \tag{1} \]
The argument of the logarithm becomes negative for
\[ z_{1/2} = \frac{1}{2} \left[ 1 \pm \sqrt{1 - 4m^2/q^2} \right]. \tag{2} \]
Using now \(\Im[\ln(-x - i\epsilon)] = -\pi\), the imaginary part follows as
\[ \Im(F) = -\pi \int_{z_1}^{z_2} dz \left[ m^2 - q^2 z(1 - z) \right] = \frac{\pi}{6} \sqrt{1 - \frac{4m^2}{q^2}} (q^2 - 4m^2) = \frac{\pi}{6} \left( 1 - \frac{4m^2}{q^2} \right)^{3/2}. \tag{3} \]
Adding the prefactor \(N_c \frac{12g^2}{(4\pi)^2}\), we find
\[ \Im(\Sigma) = N_c \frac{g^2}{8\pi} q^2 \left( 1 - \frac{4m^2}{q^2} \right)^{3/2}, \tag{4} \]
what agrees for \( q^2 = M^2 \) with the prediction of the optical theorem, \(M\Gamma = \Im\Sigma\).

c.) Consider in
\[ i\Sigma = -y^2 \int \frac{d^4q}{(2\pi)^4} \frac{4[q^2 - qp + m^2]}{(q^2 - m^2 + i\epsilon)(q^2 - m^2 + i\epsilon)\left[(q - p)^2 - m^2 + i\epsilon\right]} \]
the denominator. Setting \( p = (M, 0) \) and \( E_q = +\sqrt{q^2 + m^2} \), we find as poles of the integrand \( q^0 = E_q - i\epsilon, \ q^0 = -E_q + i\epsilon, \ q^0 = M + E_q - i\epsilon, \) and \( q^0 = M - E_q + i\epsilon. \) We can choose the integration contour either in the upper or lower half-plane. Choosing the lower one, we pick up the two residues at \( q^0 = E_q - i\epsilon \) and \( q^0 = M + E_q - i\epsilon. \) Hence we obtain
\[ \Sigma = -4y^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2ME_q} \left( \frac{ME_q - 2m^2}{M - 2E_q + i\epsilon} + \frac{A}{M + 2E_q - i\epsilon} \right). \]

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The second denominator never vanishes and thus gives no contribution to the imaginary part. For the first one, we obtain using the given identity

$$\text{Im}\Sigma = 4y^2\pi \int \frac{d^3q}{(2\pi)^3} \frac{ME_q - 2m^2}{2ME_q} \delta(M - 2E_q)$$

As $E_q = +\sqrt{q^2 + m^2} \geq m$, the argument of the delta function is never zero for $M \leq 2m$ and the imaginary part of the amplitude vanishes thus. For $M > 2m$, we can set $E_q = M/2$ and perform then the integral,

$$\text{Im}\Sigma = \frac{y^2}{8\pi} \sqrt{1 - \frac{4m^2}{M^2}}$$

Thus we confirmed again the relation $\Gamma = \text{Im}\Sigma$.

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