Exercise sheet 8

1. Maximal energy in shock acceleration.

a.) The maximal energy achievable in shock acceleration may be limited by the time t the accelerator operates. Use a dimensional argument to derive a relation $t = t(D, v)$, where D is the diffusion coefficient responsible for the random walk of the CR and u the shock/fluid velocity.

b.) Assume $D = cR_L/3$ with R_L the Larmor radius (why?), and use 3μG, 1000 yr and 10.000km/s for a numerical estimate for the maximal energy achievable in a SNR.

a.) From the general definition $D = vl/3$, it is $[D] = \mathrm{cm}^2/\mathrm{s}$. Thus we obtain a time from

$$t = \alpha \frac{D}{u^2},$$

with the unknown dimensionless constant α which we set to one in the following.

b.) Setting $l = R_L$ as suggested, it is

$$t_{\max} = \frac{cR_L(E_{\max})}{3u^2} = \frac{cE_{\max}}{3eBu^2},$$

leading to $E_{\max} \simeq 30\,\mathrm{TeV}$. Although our estimate is optimistic, this is far below the energy of the CR knee at 3 PeV, the energy to which at least a (dominant) subset of CR sources should be able to accelerate. A solution to this puzzle requires probably that the magnetic field gets amplified by the action of CRs

Bonus: Why $l = R_L$?
In our case $v = c$, and a possible energy dependence arise from $D(E) = cl(E)/3$. For diffusion in a turbulent magnetic field, the scattering length can depend only on two length scales: the Larmor radius $R_L(E)$ and the injection scale $L_{\max}$ of the turbulence. Thus we can set

$$l = f(R_L, L_{\max}) = L_{\max} g(R_L/L_{\max}),$$

where the dimensionless function g should depend only on the ratio $R_L/L_{\max}$. It is natural to assume a power law, $g(R_L/L_{\max}) = (R_L/L_{\max})^\alpha$. For larger R_L, the particle should diffuse faster and hence $\alpha > 0$. On the other hand $\alpha = 1$ seems to be an upper limit, since we cannot expect to change the trajectory faster than within R_L. In this limit, $l = cR_L/3$.

2. Spectrum of synchrotron photons.

Assume that ultrarelativistic electrons with energy spectrum $\mathrm{d}N/\mathrm{d}E \propto E^{-\alpha}$ emit synchrotron radiation with the power ($c = \hbar = 1$)

$$P_{\mathrm{syn}} = \frac{2}{3}\,\alpha m^2 \left(\frac{p_\perp}{m}\frac{eB}{m^2}\right)^2$$

Solutions are discussed Thursday, 02.11.23

and that an electron with energy E emits photons with $\omega_s = \omega_{\rm cr} = 3\gamma^2\omega_0/2$ and $\omega_0 = eB/m$. What is the slope of the resulting synchrotron intensity I_ν, how does it scale with the magnetic field strength B?

The emitted intensity is proportional to

$$I_\nu \mathrm{d}\nu \propto P_{\rm syn}\frac{\mathrm{d}N}{\mathrm{d}E}\mathrm{d}E.$$

For ultrarelativistic electrons, we can use $P_{\rm syn} \propto E^2B^2$, and thus

$$I_\nu \mathrm{d}\nu \propto E^{2-\alpha}B^2\mathrm{d}E.$$

With $\gamma = E/m$, we obtain a relation between E and $\omega = 2\pi\nu$, or differentiated

$$\mathrm{d}\omega_s \propto 2E\mathrm{d}E\,B$$

Inserting everything gives

$$I_\nu \mathrm{d}\nu \propto E^{2-\alpha}B^2\frac{\mathrm{d}\nu}{EB} = E^{1-\alpha}B\mathrm{d}\nu = \nu^{\frac{1-\alpha}{2}}B^{\frac{1+\alpha}{2}}.$$

Thus the slope of the synchrotron spectrum $I_\nu \propto \nu^\beta$ is $\beta = (1-\alpha)/2$, and it scales with the magnetic field strength as $B^{\frac{1+\alpha}{2}}$.

3. Interaction length of electrons on CMB.
Estimate the interaction length of an electrons scattering on CMB photons with temperature 2.7 K. in the Thomson limit.

The mean-free path is $l_{\rm int} = 1/(n_\gamma\sigma_{\rm Th}) \sim 1\,\mathrm{kpc}$.

4. Greisen-Zatsepin-Kuzmin cutoff.
a.) Calculate the threshold energy (the "Greisen-Zatsepin-Kuzmin cutoff") of protons for the reaction $p+\gamma \to p+\pi^0$ on CMB photons with temperature 2.7 K. Estimate the mean-free path of a proton above the threshold using $\sigma = 0.1$ mbarn.
b.) What reaction could be important at smaller energies as energy-loss mechanism for protons? Estimate the corresponding cross-section.

a.) The mean energy of BB photons with $T = 2.7\,\mathrm{K}$ is $\varepsilon \simeq 2.7kT \simeq 6.3\times10^{-4}\,\mathrm{eV}$. At threshold, the final state is produced at rest,

$$s = (p_1+p_2)^2 \geq (m_N+m_\pi)^2.$$

Thus

$$m_N^2 + 2E_p\varepsilon(1-\cos\vartheta) \geq m_N^2 + 2m_Nm_\pi + m_\pi^2$$

where we assumed that the proton is ultrarelativistic. For a head-on collision, $1 - \cos\vartheta = 2$, and thus

$$E_p \geq \frac{2m_N m_\pi + m_\pi^2}{4E_p\varepsilon} \simeq 10^{20}\,\mathrm{eV}.$$

The mean-free path is $l_{\mathrm{int}} = 1/(n_\gamma\sigma)$ with $n_\gamma \simeq 411/\mathrm{cm}^3$. Since protons are not destroyed, but only loss energy, the more interesting quantity is the energy-loss length

$$\frac{1}{E}\frac{\mathrm{d}E}{c\mathrm{d}t} \simeq \langle y \rangle n_\gamma \sigma \simeq (17\,\mathrm{Mpc})^{-1}$$

with $\langle y \rangle = (E - E')/E \simeq m_\pi/m_p \simeq 0.1$ as the energy lost per interaction.

b.) For a smaller threshold energy, the mass of the final state should be reduced, as in $p + \gamma \to p + e^+ + e^-$. This replaces also a strong interaction (which creates the pion) with a electromagnetic interaction (which creates the e^+e^- pair). Thus one expects that the cross section is reduced by a factor $\alpha_s/\alpha_{\mathrm{em}}$. With $\alpha_s \sim 1$ and $\alpha_{\mathrm{em}} \sim 1/137$, the cross section is factor of order 100 smaller.