9 Linearized gravity and gravitational waves

9.1 Linearized gravity

9.1.1 Metric perturbation as tensor field

We are looking for small perturbations h_{ab} around the Minkowski[1] metric η_{ab},

$$g_{ab} = \eta_{ab} + h_{ab}\,, \qquad h_{ab} \ll 1\,. \tag{9.1}$$

These perturbations may be caused either by the propagation of gravitational waves through a detector or by the gravitational potential of a star. In the first case, current experiments show that we should not hope for h larger than $\mathcal{O}(h) \sim 10^{-22}$. Keeping only terms linear in h is therefore an excellent approximation. Choosing in the second case as application the final phase of the spiral-in of a neutron star binary system, deviations from Newtonian limit can become large. Hence one needs a systematic "post-Newtonian" expansion or even a numerical analysis to describe properly such cases.

We choose a Cartesian coordinate system x^a and ask ourselves which transformations are compatible with the splitting (9.1) of the metric. If we consider global (i.e. space-time independent) Lorentz transformations Λ_a^b, then $x'^a = \Lambda_b^a x^b$. The metric tensor transform as

$$g'_{ab} = \Lambda_a^c \Lambda_b^d g_{cd} = \Lambda_a^c \Lambda_b^d (\eta_{cd} + h_{cd}) = \eta_{ab} + \Lambda_a^c \Lambda_b^d h_{cd} = \eta'_{ab} + \Lambda_a^c \Lambda_b^d h_{cd}\,. \tag{9.2}$$

Thus Lorentz transformations respect the splitting (9.1) and the perturbation h_{ab} transforms as a rank-2 tensor on Minkowski space. We can view therefore h_{ab} as a symmetric rank-2 tensor field defined on Minkowski space that satisfies the linearized Einstein equations, similar as the photon field is a rank-1 tensor field fulfilling Maxwell's equations.

Although the splitting (9.1) is incompatible with general coordinate transformations, infinitesimal ones $\bar{x}^i = x^i + \varepsilon\xi(x^k)$ are of the same (linear) order. Hence the Killing equation simplifies to

$$h'_{ab} = h_{ab} - \partial_a \xi_b - \partial_b \xi_a\,, \tag{9.3}$$

because the term $\xi^c \partial_c h_{ab}$ is quadratic in the small quantities h and ξ and can be neglected.

It is more fruitful to view this equation not as coordinate but as a gauge transformation: Both h'_{ab} and h_{ab} describe the same physical situation, since the (linearized) Einstein equations do not fix uniquely h_{ab} for a given source.

Comparison with electromagnetism The photon field A_i is subject to gauge transformations,

$$A_i(x) \to A_i(x) + \partial_i \Lambda(x)\,. \tag{9.4}$$

[1] The same analysis could be performed for small perturbations around an arbitrary metric $g_{ab}^{(0)}$, adding however considerable technical complexity.

The Lagrange density for the photon field as well its interactions with other fields should be therefore not only Lorentz but also gauge invariant. Since the gauge transformations cancel in the anti-symmetric field-strength tensor $F^{\mu\nu}$, only the term $F_{\mu\nu}F^{\mu\nu}$ qualifies to enter $\mathcal{L}$. Next we consider the possible interaction terms for the example of a complex scalar field. Its global phase is not observable and thus

$$\phi(x) \to \phi(x)\exp[ie\Lambda(x)] \tag{9.5}$$

can compensate the change induced by (9.4) in the interaction term, if one chooses

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu \,. \tag{9.6}$$

Hence the way to include interactions in theories of free matter fields is both for electromagnetism and gravity very similar: Write down the free theory and replace then partial derivatives with gauge invariant and covariant derivatives, respectively.

Note that the change $\exp[ie\Lambda(x)]$ induced by the arbitrary function Λ results always in a complex phase, i.e. the gauge transformations of the charged field form an one-dimensional group, the Lie group U(1). By contrast, finite transformation of the type (9.3) applied to general tensors lead to an infinite dimensional transformation group. This difference explains why Noether's theorem leads either to normal (current conservation for gauge symmetries, energy-momentum conservation for Poincaré symmetry) or to "improper" (general covariance in general relatively) conservation laws.

9.1.2 Linearized Einstein equations in vacuum

From $\partial_a\eta_{bc} = 0$ and the definition

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \tag{9.7}$$

we find for the change of the connection linear in h

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}\eta^{ad}(\partial_b h_{dc} + \partial_c h_{bd} - \partial_d h_{bc}) = \frac{1}{2}(\partial_b h^a_c + \partial_c h^a_b - \partial^a h_{bc})\,. \tag{9.8}$$

Here we used η to raise indices which is allowed in linear approximation. Remembering the definition of the Riemann tensor,

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}\,, \tag{9.9}$$

we see that we can neglect the terms quadratic in the connection terms. Thus we find for the change

$$\begin{aligned}
\delta R^a{}_{bcd} &= \partial_c\delta\Gamma^a{}_{bd} - \partial_d\delta\Gamma^a{}_{bc} \\
&= \frac{1}{2}\left\{\partial_c\partial_b h^a_d + \partial_c\partial_d h^a_b - \partial_c\partial^a h_{bd} - (\partial_d\partial_b h^a_c + \partial_d\partial_c h^a_b - \partial_d\partial^a h_{bc})\right\} \\
&= \frac{1}{2}\left\{\partial_c\partial_b h^a_d + \partial_d\partial^a h_{bc} - \partial_c\partial^a h_{bd} - \partial_d\partial_b h^a_c\right\}\,.
\end{aligned} \tag{9.10}$$

The change in the Ricci tensor follows by contracting a and c,

$$\delta R_{bd} = \delta R^c{}_{bcd} = \frac{1}{2}\left\{\partial_c\partial_b h^c_d + \partial_d\partial^c h_{bc}) - \partial_c\partial^c h_{bd} - \partial_d\partial_b h^c_c\right\}\,. \tag{9.11}$$

Next we introduce $h \equiv h_c^c$, $\Box = \partial_c \partial^c$, and relabel the indices,

$$\delta R_{ab} = \frac{1}{2}\left\{\partial_a \partial_c h_b^c + \partial_b \partial_c h_a^c - \Box h_{ab} - \partial_a \partial_b h\right\} . \tag{9.12}$$

We now rewrite all terms apart from $\Box h_{ab}$ as derivatives of the vector

$$V_a = \partial_c h_a^c - \frac{1}{2}\partial_a h \,, \tag{9.13}$$

obtaining

$$\delta R_{ab} = \frac{1}{2}\left\{-\Box h_{ab} + \partial_a V_b + \partial_b V_a\right\} . \tag{9.14}$$

Looking back at the properties of h_{ab} under gauge transformations, Eq. (9.3), we see that we can gauge away the second and third term. Thus the linearized Einstein equation in vacuum becomes simply

$$\boxed{\Box h_{ab} = 0} \tag{9.15}$$

if the harmonic gauge,

$$\boxed{V_a = \partial_c h_a^c - \frac{1}{2}\partial_a h = 0 \,,} \tag{9.16}$$

is chosen. Thus the familiar wave equation holds for all ten independent components of h_{ab}, and the perturbations propagate with the speed of light c. Inserting plane waves $h = \exp(ikx)$ into the wave equation, one finds immediately that k is a null vector.

Alternative form of the Einstein equation We can express the Einstein equation, where the only geometrical term on the LHS is the Ricci tensor. Because of

$$R_a^a - \frac{1}{2}g_a{}^a(R - 2\Lambda) = R - 2(R - 2\Lambda) = -R + 4\Lambda = \kappa T_a{}^a \tag{9.17}$$

we can perform with $T \equiv T_a^a$ the replacement $R = 4\Lambda - \kappa T$ in the Einstein equation and obtain

$$R_{ab} = \kappa(T_{ab} - \frac{1}{2}g_{ab}T) + g_{ab}\Lambda \,. \tag{9.18}$$

Thus an empty universe with $\Lambda = 0$ is characterized by a vanishing Ricci tensor $R_{ab} = 0$.

9.1.3 Linearized Einstein equations with sources

We found $2\delta R_{ab} = -\Box h_{ab}$. By contraction follows $2\delta R = -\Box h$. Combining both terms gives

$$\begin{aligned} \Box\left(h_{ab} - \frac{1}{2}\eta_{ab}h\right) &= -2(\delta R_{ab} - \frac{1}{2}\eta_{ab}\delta R) \\ &= -2\kappa\delta T_{ab} \,. \end{aligned} \tag{9.19}$$

Since we assumed an empty universe in zeroth order, δT_{ab} is the complete contribution to the energy-momentum tensor. We omit therefore in the following the δ in δT_{ab}.

We introduce as useful short-hand notation the "trace-reversed" amplitude as

$$\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab}h \,. \tag{9.20}$$

The harmonic gauge condition becomes then

$$\partial_a \bar{h}_{ab} = 0 \tag{9.21}$$

and the linearized Einstein equation in harmonic gauge

$$\boxed{\Box \bar{h}_{ab} = -2\kappa T_{ab}\,.} \tag{9.22}$$

Newtonian limit The Newtonian limit corresponds to $v/c \to 0$ and thus the only non-zero element of the energy-momentum tensor becomes $T^{tt} = \rho$. We compare the metric

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{9.23}$$

to Eq. (9.1) and find as metric perturbations

$$h_{tt} = -2\Phi \qquad h_{ij} = -2\delta_{ij}\Phi \qquad h_{ti} = 0\,. \tag{9.24}$$

Thus $h = -4\Phi$ (remember $h_t^t = -h_{tt}$). In the static limit $\Box \to \Delta$ and $V = 0$, and thus

$$\Delta(h_{00} - \frac{1}{2}\eta_{00}h) = -4\Delta\Phi = -2\kappa\rho\,. \tag{9.25}$$

Hence the linearised Einstein equation has the same form as the Newtonian Poisson equation, and the constant κ equals $\kappa = 8\pi G$.

9.1.4 Polarizations states

TT gauge We consider a plane wave $h_{ab} = \varepsilon_{ab}\exp(ikx)$. The symmetric matrix ε_{ab} is called polarization tensor. Its ten independent components are constrained both by the wave equation and the gauge condition $\partial^a \bar{h}_{ab} = 0$.

Even after fixing the harmonic gauge $\partial^a \bar{h}_{ab} = 0$, we can still add four function ξ_a with $\Box\xi_a$. We can choose them such that four components of h_{ab} vanish. In the *transverse traceless* (TT) gauge, one sets ($\alpha = 1,2,3$)

$$h_{0\alpha} = 0, \qquad h = 0\,. \tag{9.26}$$

The harmonic gauge condition becomes $V_a = \partial_b h_a^b$ or

$$V_0 = \partial_b h_0^b = \partial_0 h_0^0 = -i\omega\varepsilon_{00}\mathrm{e}^{ikx} = 0 \tag{9.27}$$

$$V_\alpha = \partial_b h_\alpha^b = \partial_\beta h_\alpha^\beta = -ik^\beta\varepsilon_{\alpha\beta}\mathrm{e}^{ikx} = 0 \tag{9.28}$$

Thus $\varepsilon_{00} = 0$ and the polarization tensor is transverse, $k^\beta\varepsilon_{\alpha\beta} = 0$. If we choose the plane wave propagating in z direction, $\vec{k} = k\vec{e}_z$, the z raw and column of the polarization tensor vanishes too. Accounting for $h = 0$ and $\varepsilon_{ab} = \varepsilon_{ba}$, only two independent elements are left,

$$\varepsilon = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9.29}$$

In general, one can construct the polarization tensor in TT gauge by setting first the non-transverse part to zero and then subtracting the trace. The resulting two independent elements are (again for $\vec{k} = k\vec{e}_z$) then $\varepsilon_{11} = 1/2(\varepsilon_{xx} - \varepsilon_{yy})$ and ε_{12}.

Helicity We determine now how a metric perturbation h_{ab} transforms under a rotation with the angle α. We choose the wave propagating in z direction, $\vec{k} = k\vec{e}_z$, the TT gauge, and the rotation in the xy plane. Then the general Lorentz transformation Λ becomes

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{9.30}$$

Since $\vec{k} = k\vec{e}_z$ and thus $\Lambda_a^{\ b} k_b = k_a$, the rotation affects only the polarization tensor. We rewrite $\varepsilon'_{ab} = \Lambda_a^{\ c} \Lambda_b^{\ d} \varepsilon_{cd}$ in matrix notation,

$$\varepsilon' = \Lambda \varepsilon \Lambda^t \tag{9.31}$$

It is sufficient in TT gauge to perform the calculation for the xy sub-matrices; the result after introducing circular polarization states $\varepsilon_\pm = \varepsilon_{11} \pm i\varepsilon_{12}$ is

$$\varepsilon'_\pm = \exp(2i\alpha)\varepsilon'_\pm \,. \tag{9.32}$$

Thus gravitational waves have helicity two. Doing the same calculation in an arbitrary gauge, one finds that the remaining, unphysical degrees of freedom transform as helicity one and zero.

9.1.5 Detection principle

Consider the effect of a gravitational wave on a free test particle that is initially at rest, $u^a = (-1, 0, 0, 0)$. As long as the particle is at rest, the geodesic equation simplifies to $\dot{u}^a = \Gamma^a{}_{00}$. The four relevant Christoffel symbols are in linearized approximation, cf. Eq. (9.8),

$$\Gamma^a{}_{00} = \frac{1}{2}(\partial_0 h^a_0 + \partial_0 h^a_0 - \partial^a h_{00}) \,. \tag{9.33}$$

We are free to choose the TT gauge in which all component of h_{ab} appearing on the RHS are zero. Hence the acceleration of the test particle is zero and its coordinate position is unaffected by the gravitational wave. (TT gauge defines a "comoving" coordinate system.)

The physical distance l is given by integrating

$$\mathrm{d}l^2 = g_{\alpha\beta}\mathrm{d}\xi^\alpha \mathrm{d}\xi^\beta = (h_{\alpha\beta} - \delta_{\alpha\beta})\mathrm{d}\xi^\alpha \mathrm{d}\xi^\beta \tag{9.34}$$

where $g_{\alpha\beta}$ is the spatial part of the metric and $\mathrm{d}\xi$ the spatial coordinate distance between infinitesimal separated test particles. Hence the passage of a periodic gravitational wave, $h_{ab} \propto \cos(\omega t)$, results in a periodic change of the separation of freely moving test particles. The relative size of this change, $\Delta L/L$ is given by the amplitude h of the gravitational wave.

9.2 Energy-momentum pseudo-tensor for gravity

We consider again the splitting (9.1) of the metric, but we require now not that h_{ab} is small. We rewrite next the Einstein equation by bringing the Einstein tensor on the RHS and adding the linearized Einstein equation,

$$R^{(1)}_{ab} - \frac{1}{2} R^{(1)} \eta_{ab} = \kappa T_{ab} + \left(-R_{ab} + \frac{1}{2} R\, g_{ab} + R^{(1)}_{ab} - \frac{1}{2} R^{(1)} \eta_{ab} \right) . \tag{9.35}$$

The LHS of this equation is the usual gravitational wave equation, while the RHS now includes as source not only matter but also the gravitational field itself. It is therefore natural to define

$$R^{(1)}_{ab} - \frac{1}{2}\,R^{(1)}\,\eta_{ab} = \kappa\,(T_{ab} + t_{ab})\ . \tag{9.36}$$

with t_{ab} as the energy-momentum pseudo-tensor for gravity. If we expand all quantities,

$$g_{ab} = \eta_{ab} + h^{(1)}_{ab} + h^{(2)}_{ab} + \mathcal{O}(h^3)\,, \qquad R_{ab} = R^{(1)}_{ab} + R^{(2)}_{ab} + \mathcal{O}(h^3)\,. \tag{9.37}$$

we can set, assuming $h_{ab} \ll 1$, $R_{ab} - R^{(1)}_{ab} = R^{(2)}_{ab} + \mathcal{O}(h^3)$, etc. Hence we find as energy-momentum pseudo-tensor for the metric perturbations $h^{(1)}_{ab}$ at $\mathcal{O}(h^3)$

$$t_{ab} = -\frac{1}{\kappa}\left(R^{(2)}_{ab} - \frac{1}{2}\,R^{(2)}\,\eta_{ab}\right)\ . \tag{9.38}$$

This tensor is symmetric, quadratic in $h^{(1)}_{ab}$ and conserved because of the Bianchi identity. However, t_{ab} is not gauge-invariant, since it can be made at each point identically to zero by a coordinate transformation. In the case of gravitational waves we may expect that averaging t_{ab} over a volume large compared to the wave-length considered solves this problem. Moreover, such an averaging simplifies the calculation of t_{ab}, since all terms odd in kx cancel. Nevertheless, the calculation is messy, but gives a simple result. For the TT gauge one obtains

$$\langle t_{ab}\rangle = \frac{\pi}{\kappa} f^2 (h_+^2 + h_-^2) A_{ab}\,, \tag{9.39}$$

where for a wave travelling in z direction $A_{00} = A_{33} = 1$, $A_{03} = A_{30} = -1$ and zero otherwise.

9.3 Emission of gravitational waves

9.3.1 Quadrupol formula

Gravitational waves in the linearized approximation fulfill the superposition principle. Hence, if the solution for a point source is known,

$$\Box_x G(x - x') = \delta(x - x')\,, \tag{9.40}$$

the general solution can be obtained by integration,

$$\bar{h}_{ab}(x) = -2\kappa \int \mathrm{d}^4x' G(x - x') T_{ab}(x')\,. \tag{9.41}$$

The Green's function $G(x-x')$ is not completely specified by Eq. (9.40): We can add solutions of the homogenous wave equation and we have to specify how the poles of $G(x - x')$ have to be treated. In classical physics, one chooses the retarded Green's function $G(x - x')$ defined by

$$G(x - x') = -\frac{1}{4\pi(\vec{x} - \vec{x}')}\delta[|\vec{x} - \vec{x}'| - (t - t')]\vartheta(t - t')\,. \tag{9.42}$$

Inserting the retarded Green's function into Eq. (9.41), we can perform the $\mathrm{d}t$ integral using the delta function and obtain

$$\bar{h}_{ab}(x) = 4G \int \mathrm{d}^3x'\, \frac{T_{ab}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{\vec{x} - \vec{x}'}\,. \tag{9.43}$$

The retarded time $t_r \equiv t - |\vec{x} - \vec{x}'|$ denotes the time t_r on the past light-cone, when a signal had to be emitted at $\vec{x}'$ to reach $\vec{x}$ at time t propagating with the speed of light.

We perform now a Fourier transformation from time to angular frequency,

$$\bar{h}_{ab}(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}t\, \mathrm{e}^{i\omega t} \bar{h}_{ab}(t, \vec{x}) = \frac{4G}{\sqrt{2\pi}} \int \mathrm{d}t \int \mathrm{d}^3x'\, \mathrm{e}^{i\omega t} \frac{T_{ab}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{\vec{x} - \vec{x}'} \tag{9.44}$$

Next we change from the integeration varibale t to t_r,

$$\bar{h}_{ab}(\omega, \vec{x}) = \frac{4G}{\sqrt{2\pi}} \int \mathrm{d}t_r \int \mathrm{d}^3x'\, \mathrm{e}^{i\omega t} \mathrm{e}^{i\omega|\vec{x} - \vec{x}'|} \frac{T_{ab}(t_r, \vec{x}')}{\vec{x} - \vec{x}'} \tag{9.45}$$

and use the definition of the Fourier transform,

$$\bar{h}_{ab}(\omega, \vec{x}) = 4G \int \mathrm{d}^3x'\, \mathrm{e}^{i\omega|\vec{x} - \vec{x}'|} \frac{T_{ab}(\omega, \vec{x}')}{\vec{x} - \vec{x}'} \,. \tag{9.46}$$

We proceed using the same approximations as in electrodynamics: We restrict ourselved to slowly moving sources observed in the wave zone. Then most radiation is emitted at frequencies such that $|\vec{x} - \vec{x}'| \approx r$ and thus

$$\bar{h}_{ab}(\omega, \vec{x}) = 4G\, \frac{\mathrm{e}^{i\omega r}}{r} \int \mathrm{d}^3x'\, T_{ab}(\omega, \vec{x}') \,. \tag{9.47}$$

The calculation of $\bar{h}_{ab}(\omega, \vec{x})$ can be greatly simplified using the constraints implied by gauge and energy conservation. The harmonic gauge condition $\partial_a \bar{h}_{ab} = 0$ implies in Fourier space

$$h_{0b}(\omega, \vec{x}) = \frac{i}{\omega} h_{\alpha b}(\omega, \vec{x}) \tag{9.48}$$

Hence we need to calculate only the space-like components of $h^{ab}(\omega, \vec{x})$. Next we use (flat-space) energy-momentum conservation,

$$\frac{\partial}{\partial t} T^{00} + \frac{\partial}{\partial x^\beta} T^{0\beta} = 0 \tag{9.49}$$

$$\frac{\partial}{\partial t} T^{\alpha 0} + \frac{\partial}{\partial x^\beta} T^{\alpha\beta} = 0 \,. \tag{9.50}$$

We differentiate (9.49) wrt to time, and use again conservation law (9.50),

$$\frac{\partial^2}{\partial t^2} T^{00} = -\frac{\partial^2}{\partial x^\beta \partial t} T^{0\beta} = \frac{\partial^2}{\partial x^\alpha \partial x^\beta} T^{\alpha\beta} \tag{9.51}$$

Multiplying with $x^\alpha x^\beta$, integrating gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int d^3x x^\alpha x^\beta T^{00} = -\int d^3x x^\alpha x^\beta \frac{\partial^2}{\partial t \partial x^\sigma} T^{\sigma 0} = 2 \int d^3x T^{\alpha\beta} \tag{9.52}$$

we define as quadrupole moment tensor of the source energy-momentum

$$I_{\alpha\beta} = \int d^3x'\, x'^\alpha x'^\beta T^{00}(x') \tag{9.53}$$

Then we can rewrite the solution for $h_{0b}(\omega, \vec{x})$ as

$$\bar{h}_{\alpha\beta}(\omega, \vec{x}) = -4G\omega^2 \, \frac{\mathrm{e}^{i\omega r}}{r} I_{\alpha\beta}(\omega)\,. \tag{9.54}$$

Fourier-transforming back to time, the *quadrupole formula* for the emission of gravitational waves results,

$$\boxed{\bar{h}_{\alpha\beta}(t, \vec{x}) = \frac{2G}{r}\, \ddot{I}_{\alpha\beta}(t_r)\,.} \tag{9.55}$$

Binary system Consider a binary system with (for simplicity) circular orbits in the 12-plane. Then

$$x_a^1 = R\cos\Omega t\,, \qquad x_a^2 = R\sin\Omega t\,, \tag{9.56}$$

and

$$x_b^1 = -R\cos\Omega t\,, \qquad x_b^2 = -R\sin\Omega t\,. \tag{9.57}$$

The corresponding energy density is

$$T^{00} = M\delta(x^3)[\delta(x^1 - R\cos\Omega t)\delta(x^2 - R\sin\Omega t) + \delta(x^1 + R\cos\Omega t)\delta(x^2 + R\sin\Omega t)] \tag{9.58}$$

The quadropole moment follows as

$$I_{11} = 2MR^2\cos^2\Omega t = MR^2(1+\cos^2 2\Omega t) \tag{9.59}$$
$$I_{22} = 2MR^2\sin^2\Omega t = MR^2(1-\cos^2 2\Omega t) \tag{9.60}$$
$$I_{12} = I_{21} = 2MR^2\cos\Omega t\sin\Omega t = MR^2\sin 2\Omega t) \tag{9.61}$$
$$I_{\alpha 3} = 0 \tag{9.62}$$

Inserting these results into Eq. (9.55), we obtain as final result

$$\bar{h}_{\alpha\beta}(t, \vec{x}) = \frac{8GM}{r}\,(\Omega R)^2 \begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9.63}$$

9.4 Fourier-transformed energy-momentum tensor

Let us first consider a general scalar wave equation in flat spacetime,

$$\Box\,\varphi(\boldsymbol{x}, t) = -\,4\pi\,S(\boldsymbol{x}, t)\,, \tag{9.64}$$

and let us decompose the time variation of the source S in either a Fourier integral $S(\boldsymbol{x}, t) = \int (\mathrm{d}\omega/2\pi)\, e^{-i\omega t}\, S(\boldsymbol{x}, \omega)$ or (if the source motion is periodic) a Fourier series $S(\boldsymbol{x}, t) = \sum_n e^{-i\omega_n t}\, S(\boldsymbol{x}, \omega_n)$. Then we can concentrate on a single frequency ω (or ω_n). The corresponding decomposition of the solution, $\varphi(\boldsymbol{x}, t) = \sum_\omega e^{-i\omega t}\,\varphi(\boldsymbol{x}, \omega)$, leads to

$$(\Delta + \omega^2)\,\varphi(\boldsymbol{x}, \omega) = -\,4\pi\,S(\boldsymbol{x}, \omega)\,, \tag{9.65}$$

whose retarded Green function ($(\Delta + \omega^2)\,G_\omega(\boldsymbol{x}, \boldsymbol{x}') = -\,4\pi\,\delta(\boldsymbol{x} - \boldsymbol{x}')$) is $G_\omega(\boldsymbol{x}, \boldsymbol{x}') = \exp(+\,i\,\omega|\boldsymbol{x} - \boldsymbol{x}'|)/|\boldsymbol{x} - \boldsymbol{x}'|$ so that

$$\varphi(\boldsymbol{x}, \omega) = \int d^3\boldsymbol{x}'\; \frac{e^{i\omega\,|\boldsymbol{x} - \boldsymbol{x}'|}}{|\boldsymbol{x} - \boldsymbol{x}'|}\; S(\boldsymbol{x}', \omega)\,. \tag{9.66}$$

If the source is localized around the origin ($\boldsymbol{x}' = \boldsymbol{0}$) we can replace, in the local wave zone ($\omega\,|\boldsymbol{x}| \gg 1$), $|\boldsymbol{x} - \boldsymbol{x}'|$ by $r - \boldsymbol{n}\cdot\boldsymbol{x}'$ in the phase factor, and simply by r in the denominator. [Here $r \equiv |\boldsymbol{x}|$, and $\boldsymbol{n} \equiv \boldsymbol{x}/r$.] Let us define $\boldsymbol{k} \equiv \omega\,\boldsymbol{n}$ and the following spacetime Fourier transform of the source

$$S(k^a) = S(\boldsymbol{k},\omega) \equiv \int d^3\boldsymbol{x}'\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'}\, S(\boldsymbol{x}',\omega)\,. \tag{9.67}$$

With this notation the field φ in the local wave zone reads simply

$$\varphi(\boldsymbol{x},\omega) \simeq \frac{e^{i\omega r}}{r}\, S(k^a)\,, \tag{9.68}$$

$$\varphi(\boldsymbol{x},t) \simeq \frac{1}{r} \sum_{\omega} e^{-i\omega(t-r)}\, S(k^a)\,, \tag{9.69}$$

where $\sum_\omega$ denotes either an integral over ω (in the non-periodic case) or a discrete sum over ω_n (in the periodic, or quasi-periodic, case).

Let us now apply this general formula to the case of gravitational wave (GW) emission by any localized source. We can apply the previous formulas by replacing $\varphi \to \bar{h}_{ab}$, $S \to +\,4\,G\,T_{ab}$. Let us introduce the "renormalized" (distance-independent) asymptotic waveform κ_{ab}, such that (in the local wave zone)

$$\bar{h}_{ab}(\boldsymbol{x},t) = \frac{\kappa_{ab}(t-r,\boldsymbol{n})}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\,. \tag{9.70}$$

Note the dependence of κ_{ab} on the retarded time $t-r$ and the direction of emission $\boldsymbol{n}$. With this notation we have the simple formula valid for any source, at the linearized approximation

$$\kappa_{ab}(t-r,\boldsymbol{n}) = 4\,G \sum_{\omega} e^{-i\omega(t-r)}\, T_{ab}(\boldsymbol{k},\omega)\,, \tag{9.71}$$

where we recall that $\boldsymbol{k} \equiv \omega\,\boldsymbol{n}$. In the case of a periodic source with fundamental period T_1, the sum in the R.H.S. of Eq. (9.71) is a (two-sided) series over all the harmonics $\pm\,\omega_m = \pm\,m\,\omega_1$ with $m \in \mathbb{N}$ and $\omega_1 \equiv 2\pi/T_1$, and the spacetime Fourier component of T_{ab} is given by the following Fourier integral

$$T_{ab}(k) = T_{ab}(\boldsymbol{k},\omega) = \frac{1}{T_1}\int_0^{T_1} dt \int d^3\boldsymbol{x}\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}\, T_{ab}(\boldsymbol{x},t)\,. \tag{9.72}$$

Luminosity of gravitational radiation An accelerated system of electric charges emits dipole radiation with luminosity

$$L_{\rm em} = \frac{2}{3c^3}|\ddot{\mathbf{d}}|^2\,, \tag{9.73}$$

where the dipole moment of a system of N charges at position $\vec{x}_i$ is $\mathbf{d} = \sum_{i=1}^N q_i\vec{x}_i$. One might guess that for the emission of gravitational radiation the replacement $q_i \to Gm_i$ works. But since $\sum m_i\dot{\vec{x}}_i = \vec{p}_{\rm tot} = {\rm const.}$, momentum conservation means that there exists no gravitational dipole radiation. Thus one has to go to the next term in the multipole expansion, the quadrupole term,

$$Q_{ij} = \sum_{k=1}^N m^{(k)}(x_ix_j - \frac{1}{3}\delta_{ij}r^2)\,, \tag{9.74}$$

and finds then for the luminosity emitted into gravitational waves

$$L_{\rm gr} = \frac{G}{5c^5} \sum_{i,j} |\dddot{Q}_{ij}|^2 \,. \tag{9.75}$$