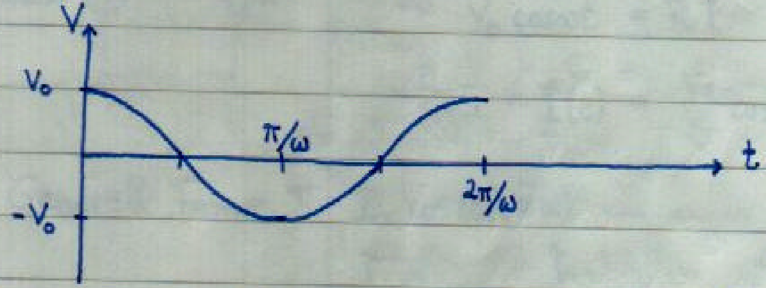
Vekselstrømkretser, impedans (LHL 27)

Vekselspenning: $V(t) = V_0 \cos \omega t$

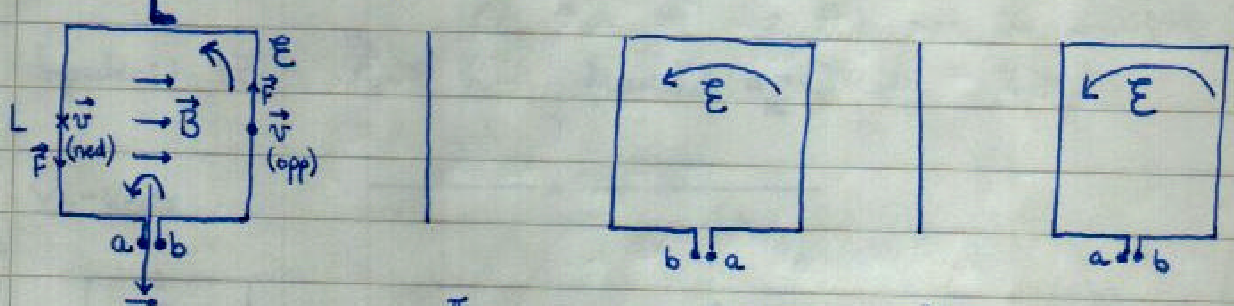
Kretssymbol: ⊙



Harmonisk svingning

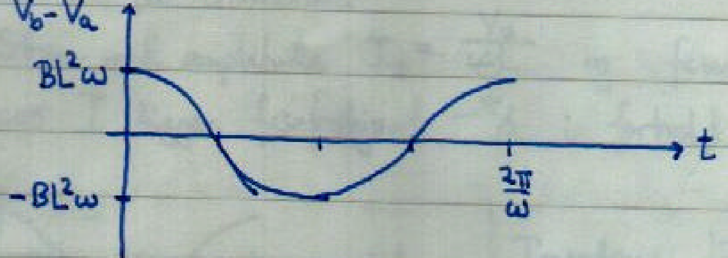
Frekvens: $f = \frac{\omega}{2\pi}$ Periode: $T = \frac{1}{f} = \frac{2\pi}{\omega}$ Amplitude: V_0

Hvordan lage vekselspenning?



$t=0$ $t = \frac{\pi}{2\omega}$ $t = \frac{\pi}{\omega}$ $t = \frac{3\pi}{2\omega}$ $t = \frac{2\pi}{\omega}$

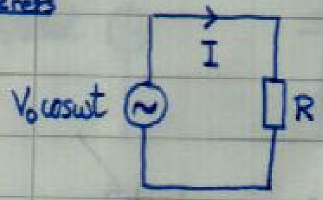
$$\mathcal{E} = -\frac{d\Phi_m}{dt} = -BL^2 \frac{d}{dt}\left(\cos(\omega t + \pi/2)\right) = BL^2 \omega \sin(\omega t + \pi/2)$$
$$= BL^2\omega \cos \omega t = V_b - V_a = V_0 \cos \omega t$$



(NTTT)

Impedans : "generalisert motstand"

• VR-krets



Ohms lov + Kirchhoff

$\Rightarrow V_0 \cos\omega t = RI$

$\Rightarrow I = I(t) = \frac{V_0}{R} \cos\omega t$

Generell form: $I = I_0 \cos(\omega t - \alpha)$

↑ amplitude ↑ fasevinkel, dvs: hvor langt "etter" ligger I i forhold til V ?

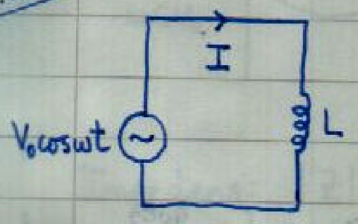
• Motstand : $I_0 = \frac{V_0}{R}$, $\alpha_R = 0$, dvs I og V "i fase"

Impedans: $|Z| = \frac{V_0}{I_0} = R$; fasevinkel $\alpha_R = 0$

Hit 14.11.02

Herfra 20.11.02

VL-krets



$V_0 \cos\omega t - L\dot{I} = 0$

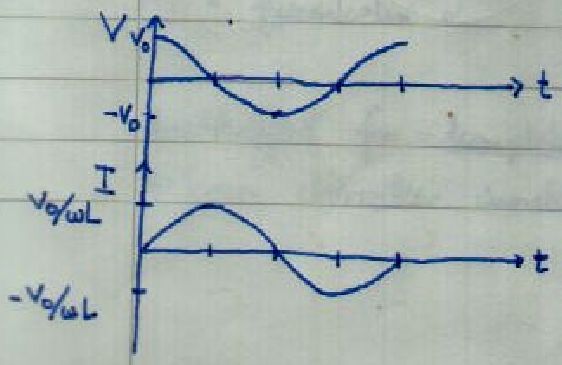
$\Rightarrow \frac{dI}{dt} = \frac{V_0}{L} \cos\omega t$

$\Rightarrow I = \frac{V_0}{\omega L} \sin\omega t = \frac{V_0}{\omega L} \cos(\omega t - \pi/2)$

Dermed, for induktans :

Strøm med amplitude $I_0 = \frac{V_0}{\omega L}$ og fasevinkel $\alpha_L = \pi/2$,

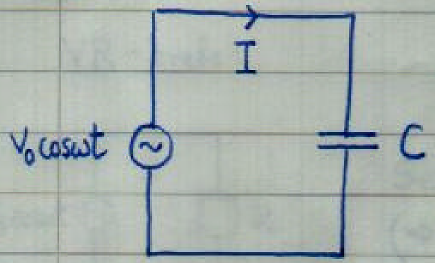
dvs I ligger faseforskjøvet $-\pi/2$ i forhold til V



Impedans: $|Z| = \frac{V_0}{I_0} = \omega L$;

fasevinkel $\alpha_L = \pi/2$

<u>VC-krets</u>



$$V_0 \cos\omega t = Q/C, \quad I = \frac{dQ}{dt}$$

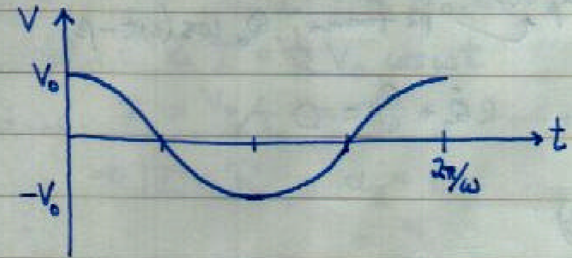
$$\Rightarrow Q = V_0 C \cos\omega t$$

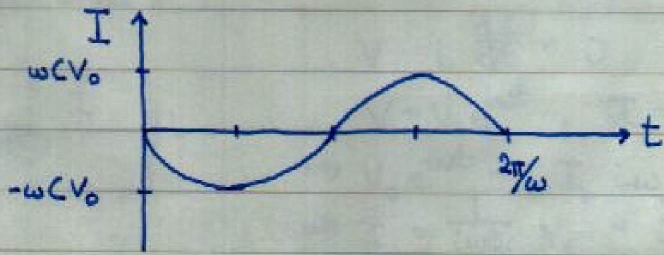
$$I = -V_0 \omega C \sin\omega t$$

$$= V_0 \omega C \cos(\omega t + \pi/2)$$

Dermed, for kapasitans:

$I_0 = \omega C V_0$, $\alpha_C = -\pi/2$, dvs I ligger $\pi/2$ <u>foran</u> V





Impedans: $|Z| = \frac{V_0}{I_0} = \frac{1}{\omega C}$; faseinkel $\alpha_c = -\pi/2$

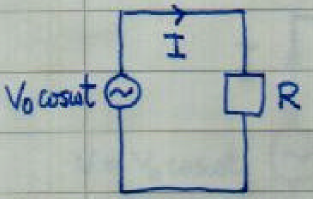
Altså <u>to</u> interessante størrelser:

- $|Z| = V_0/I_0$ („hvor stor strøm-<u>amplitude</u> I_0 for gitt V_0?")
- faseinkelen α („hvor langt etter ligger I i forhold til V?")

Skreddersydd for <u>kompleks</u> notasjon : $Z = |Z| e^{i\alpha}$

→ <u>begge</u> størrelser inneholdt i den komplekse impedansen Z !

VR-krets:



$$V - RI = 0$$

Setter $V = V_0 e^{i\omega t}$, $I = I_0 e^{i\omega t}$

($\Rightarrow$ fysisk ems $= \mathrm{Re}\, V = V_0 \cos\omega t$)

Dermed:

$$V_0 e^{i\omega t} - R I_0 e^{i\omega t} = 0$$

$$\Rightarrow I_0 = \frac{1}{R} V_0$$

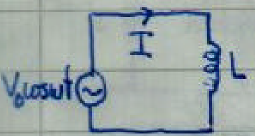
$$\Rightarrow I = \frac{1}{R} V_0 e^{i\omega t}$$

Fysisk strøm: $\mathrm{Re}\, I = \frac{1}{R} V_0 \cos\omega t$ (ok!)

Impedans: $Z = V_0/I_0 = \underline{R}$

$\Rightarrow |Z| = R, \quad \alpha_R = 0$ (ok!)

VL-krets:



$$V - L\frac{dI}{dt} = 0$$

$$V = V_0 e^{i\omega t}, \quad I = I_0 e^{i\omega t}$$

$$\Rightarrow V_0 e^{i\omega t} - L I_0\, i\omega e^{i\omega t} = 0$$

$\Rightarrow I_0 = \frac{1}{i\omega L} V_0 = \frac{V_0}{\omega L} e^{-i\pi/2}$ (kompleks; inneholder fasevinkelen α_L)

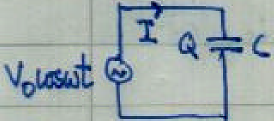
$$\Rightarrow I = \frac{V_0}{\omega L} e^{-i\pi/2} e^{i\omega t}$$

Fysisk strøm: $\mathrm{Re}\, I = \frac{V_0}{\omega L} \cos(\omega t - \pi/2)$ (ok!)

Impedans: $Z = V_0/I_0 = \underline{i\omega L} = \omega L e^{i\pi/2}$

$\Rightarrow |Z| = \omega L, \quad \alpha_L = \pi/2$ (ok!)

VC-krets:



$$V = \frac{Q}{C} \Rightarrow V_0 e^{i\omega t} = \frac{1}{C} Q_0 e^{i\omega t}$$

$$\Rightarrow I = I_0 e^{i\omega t} = \dot{Q} = V_0 C\, i\omega e^{i\omega t} = \omega C V_0 e^{i\pi/2} e^{i\omega t}$$

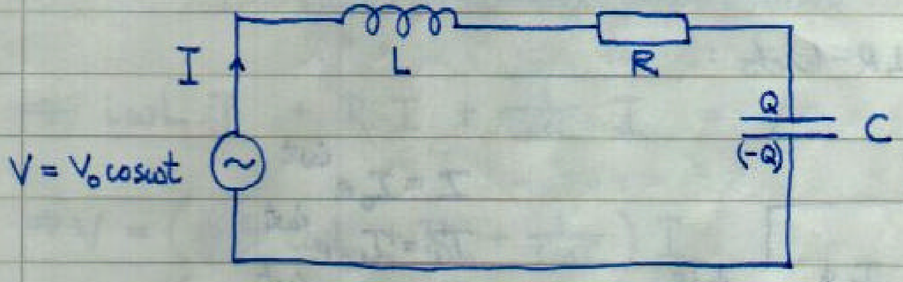
Fysisk strøm: $\mathrm{Re}\, I = \omega C V_0 \cos(\omega t + \pi/2)$ (ok!)

Impedans: $Z = V_0/I_0 = \underline{1/i\omega C} = \frac{1}{\omega C} e^{-i\pi/2}$

$\Rightarrow |Z| = \frac{1}{\omega C}, \quad \alpha_C = -\pi/2$ (ok!)

4.2.04

RCL-krets, drevne elektriske svingninger (A&F 27.10)

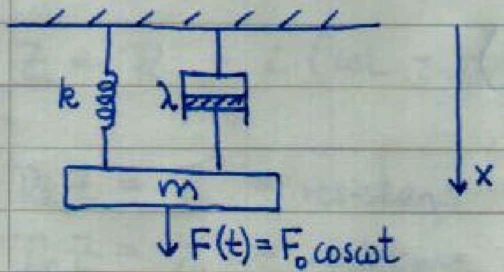


Kirchhoff: $V - L\dot{I} = RI + \frac{Q}{C}$

$I = \dot{Q}$

$\Rightarrow \; L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_0 \cos\omega t$

Mekanisk analogi:



Newton: $F = ma = m\ddot{x}$

$\Rightarrow F_0 \cos\omega t - \lambda\dot{x} - kx = m\ddot{x}$

$\Rightarrow m\ddot{x} + \lambda\dot{x} + kx = F_0 \cos\omega t$

$\Rightarrow$ analogt RCL-krets med

$m \leftrightarrow L$ ("treghet")

$\lambda \leftrightarrow R$ ("motstand")

$k \leftrightarrow \frac{1}{C}$

$x \leftrightarrow Q$

$F_0 \leftrightarrow V_0$ (drivende kraft)

Løsning

Ser på stasjonær løsning, dvs ikke evt. "innsvingningsforløp"

$\Rightarrow$ kan sette $Q = Q_0 e^{i\omega t}$ og $I = I_0 e^{i\omega t}$ når $V = V_0 e^{i\omega t}$

Dermed: $I = \dot{Q} = i\omega Q \quad , \quad Q = \frac{1}{i\omega} I$

$\dot{I} = i\omega I$ ~~[illegible]~~

$$\Rightarrow i\omega L\, I + R I + \frac{1}{i\omega C} I = V$$

$$\Rightarrow V = \left(i\omega L + R + \frac{1}{i\omega C} \right) I$$
$$= (Z_L + Z_R + Z_C) I$$
$$= Z I$$

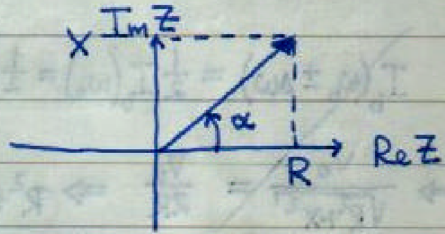
(Seriekobling av komplekse impedanser : $Z = \sum_i Z_i$ } dvs som for motstander
Parallellkobling —— " —— : $Z^{-1} = \sum_i Z_i^{-1}$ } i likestrømkretser)

Hit 20-11-02

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right) \equiv R + iX$$

$\mathrm{Re}\, Z = R$ = resistans

$\mathrm{Im}\, Z = X$ = reaktans



$Z = |Z| e^{i\alpha}$

$$\Rightarrow |Z| = \sqrt{(\mathrm{Re}\,Z)^2 + (\mathrm{Im}\,Z)^2} = \sqrt{R^2 + X^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\tan\alpha = \frac{X}{R} = \frac{\omega L - \frac{1}{\omega C}}{R} \quad , \quad \alpha = \arctan\left(\frac{X}{R}\right)$$

Strømmen:

$$I = V/Z = \frac{V_0}{R + iX} e^{i\omega t} = \frac{V_0}{\sqrt{R^2 + X^2}} e^{-i\alpha} e^{i\omega t}$$

): fysisk strøm $= \mathrm{Re}\, I = \frac{V_0}{\sqrt{R^2 + X^2}} \cos(\omega t - \alpha)$

): $|I_0| = \frac{V_0}{|Z|} = \frac{V_0}{\sqrt{R^2 + X^2}} \quad , \quad \alpha = \arctan\left(\frac{X}{R}\right)$

Effekttap i AC-kretser

Tilført energi pr tidsenhet ved tidspunkt t: $P(t) = V(t)I(t)$

$V(t) = V_0 \cos\omega t$, $I(t) = I_0 \cos(\omega t - \alpha)$

$\Rightarrow P(t) = V_0 I_0 \cos\omega t \cos(\omega t - \alpha)$

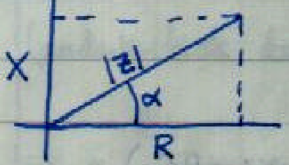
Midlere effekt over en periode $T = \frac{1}{f} = \frac{2\pi}{\omega}$:

$$\langle P \rangle = \frac{1}{T}\int_0^T P(t)\,dt \qquad \cos a \cos b = \tfrac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$$

$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V_0 I_0 \tfrac{1}{2}\left(\underbrace{\cos(2\omega t - \alpha)}_{} + \cos\alpha\right) dt = \frac{\omega}{2\pi} V_0 I_0 \tfrac{1}{2}\cos\alpha \,\frac{2\pi}{\omega}$$

null bidrag: enkel harmonisk funksjon integrert over 2 hele perioder

$$= \tfrac{1}{2} V_0 I_0 \cos\alpha$$



$$\cos\alpha = \frac{R}{|Z|} = \frac{R I_0}{V_0}$$

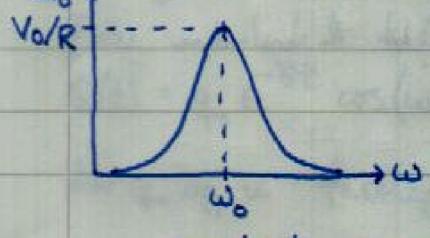
$$\Rightarrow \langle P \rangle = \tfrac{1}{2} R I_0^2 = R I_{eff}^2$$

effektivverdier: $I_{eff} = \frac{1}{\sqrt{2}} I_0$

$V_{eff} = \frac{1}{\sqrt{2}} V_0$

Resonans: $I_0 = \frac{V_0}{\sqrt{R^2 + X^2}}$ maksimal når $X = 0$

For RCL-kretsen: $X(\omega) = \omega L - \frac{1}{\omega C} = 0$ for $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$



Ved resonans:

$\alpha(\omega = \omega_0) = 0 \Rightarrow I(t)$ i fase med $V(t)$

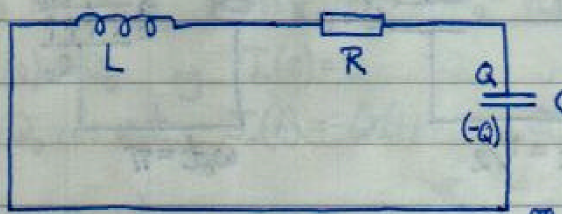
$V_L + V_C = 0$, $V = RI$

$Z = R$, $\langle P \rangle = \frac{1}{2} V_0^2/R$ (maksimal)

ω_0 = kretsens resonansfrekvens

KOMHIT 21.11.

RCL-krets, fri elektriske svingninger (A&F 27.9)

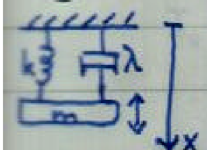


Startbetingelser $(t=0)$: $I(0) = I_0$ og $Q(0) = Q_0$

Kirchhoff:

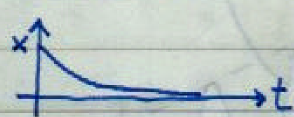
$$-L\dot{I} = RI + \frac{Q}{C}, \quad I = \dot{Q}$$

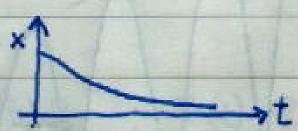
$$\Rightarrow L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0 \qquad (\text{evt: } L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0)$$



Mekanisk analogi: $(m \leftrightarrow L,\ \lambda \leftrightarrow R,\ k \leftrightarrow \frac{1}{C},\ x \leftrightarrow Q)$ har løsning

$$x(t) = \begin{cases} x_0 e^{-\gamma t} \cos(\omega t + \alpha) & \gamma = \frac{\lambda}{2m},\ \omega = \sqrt{\omega_0^2 - \gamma^2},\ \omega_0 = \sqrt{\frac{k}{m}} \\ \text{(underkritisk demping, } \gamma < \omega_0) & \\ x_1 e^{-\gamma_1 t} + x_2 e^{-\gamma_2 t} & \gamma_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \\ \text{(overkritisk demping, } \gamma > \omega_0) & \\ (x_1 + x_2 t) e^{-\gamma t} & \\ \text{(kritisk demping, } \gamma = \omega_0) & \end{cases}$$





Startbetingelser, $x(0)$ og $\dot{x}(0)$ fastlegger integrasjonskonstantene x_0, α (evt. x_1, x_2)

Tilsvarende løsning for $Q(t)$ i RCL-kretsen!

F.eks. underkritisk tilfelle:

$$Q(t) = q_0 e^{-\gamma t} \cos(\omega t + \alpha) \qquad \gamma = \frac{R}{2L},\ \omega = \sqrt{\omega_0^2 - \gamma^2},\ \omega_0 = \frac{1}{\sqrt{LC}}$$

der q_0 og α fastlegges ved at $Q(0) = Q_0$ og $I(0) = \dot{Q}(0) = Q_0$

Energi forhold:

LC-krets: $\{R=0,\ \gamma=0\} \Rightarrow \omega=\omega_0=\frac{1}{\sqrt{LC}}$

Anta $Q(0)=Q_0$ og $I(0)=0$

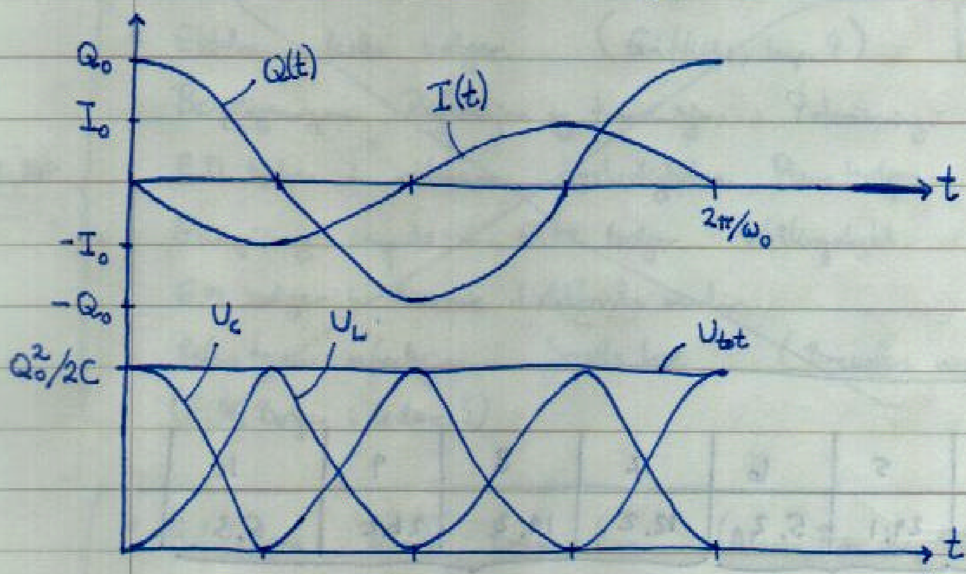
$\Rightarrow Q(t)=Q_0\cos\omega_0 t,\quad I(t)=\dot{Q}(t)=-\omega_0 Q_0\sin\omega_0 t=-I_0\sin\omega_0 t$

Energi:

$U_C=\frac{1}{2}CV^2=\frac{1}{2}\frac{Q^2}{C}$ = elektrisk energi, "potensiell energi" $(\hat{=}\frac{1}{2}kx^2)$

$U_L=\frac{1}{2}LI^2=\frac{1}{2}L\dot{Q}^2$ = magnetisk energi, "kinetisk energi" $(\hat{=}\frac{1}{2}m\dot{x}^2)$

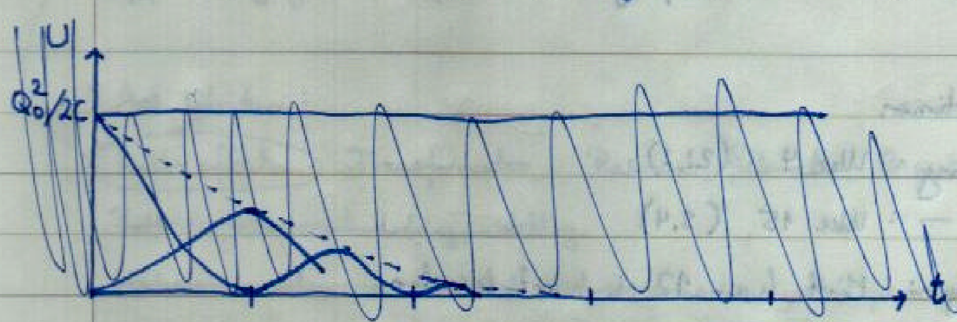
$$\Rightarrow U_{tot}=U_C+U_L=\frac{1}{2}\frac{Q_0^2}{C}\cos^2\omega_0 t+\frac{1}{2}\underbrace{L\omega_0^2}_{1/C}Q_0^2\sin^2\omega_0 t=\underline{Q_0^2/2C}$$

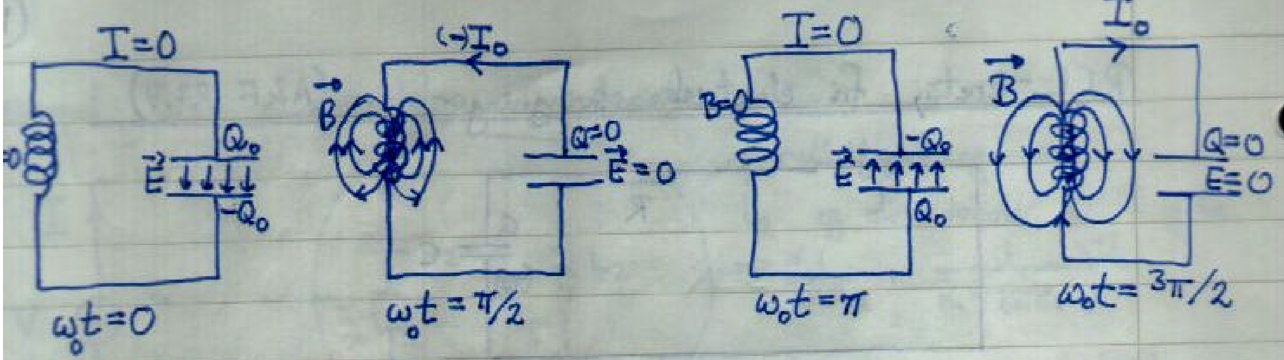


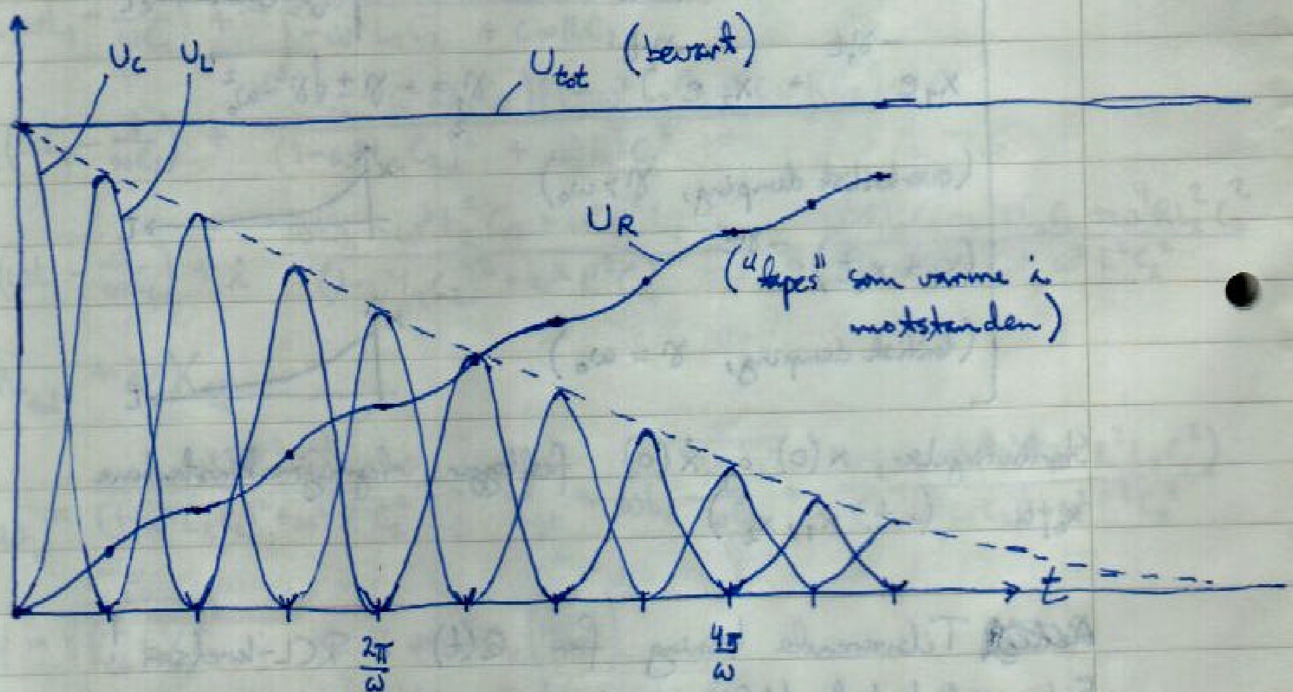
RCL-krets:

$$U_{tot}=U_C+U_L+U_R=\frac{Q^2}{2C}+\frac{1}{2}LI^2+\int_0^t P(t)\,dt\ ;\quad P(t)=\underbrace{RI(t)}_{V_R(t)}\cdot I(t)$$

$$t\to\infty: Q,I\to 0 \Rightarrow U_{tot}=U_R=\int_0^\infty RI(t)^2\,dt=U(0)=\frac{Q_0^2}{2C}$$




I=0
Q₀
E
-Q₀
ω₀t=0
(→)I₀
B
Q=0
E=0
ω₀t=π/2
I=0
B=0
-Q₀
E
Q₀
ω₀t=π
I₀
B
Q=0
E=0
ω₀t=3π/2


U_C
U_L
U_tot (bevart)
U_R
("tapes" som varme i motstanden)
t
2π/ω
4π/ω