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Løsning Øving 2

a) L' og L ekvivalente dersom

$$\frac{d}{dt}\frac{\partial}{\partial \dot q}\frac{d}{dt}F(q,t) - \frac{\partial}{\partial q}\frac{d}{dt}F(q,t) = 0. \qquad (1)$$

Deriverer : $\dfrac{dF}{dt} = \dfrac{\partial F}{\partial t} + \dfrac{\partial F}{\partial q}\dot q$

Venstre side av (1): $\dfrac{d}{dt}\dfrac{\partial}{\partial \dot q}\Big(\underbrace{\dfrac{\partial F}{\partial t}}_{\to 0} + \dfrac{\partial F}{\partial q}\dot q\Big) - \dfrac{\partial}{\partial q}\Big(\dfrac{\partial F}{\partial t} + \dfrac{\partial F}{\partial q}\dot q\Big) =$

$$= \frac{d}{dt}\frac{\partial F}{\partial q} - \frac{\partial^2 F}{\partial q \partial t} - \frac{\partial^2 F}{\partial q^2}\dot q =$$

$$= \left(\frac{\partial^2 F}{\partial q\partial t} + \frac{\partial^2 F}{\partial q^2}\dot q\right) - \frac{\partial^2 F}{\partial q \partial t} - \frac{\partial^2 F}{\partial q^2}\dot q = \underline{0},$$ som stemmer.

b) $$[\nabla\times(\nabla\times\vec A)]_i = \varepsilon_{ijk}\,\partial_j(\nabla\times\vec A)_k = \varepsilon_{ijk}\,\partial_j\,\varepsilon_{klm}\,\partial_l A_m$$
$$= \varepsilon_{kij}\,\varepsilon_{klm}\,\partial_j\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\partial_l A_m =$$
$$= \partial_i\partial_l A_l - \partial_j\partial_j A_i = \partial_i(\nabla\cdot\vec A) - \nabla^2 A_i, \Rightarrow$$
$$\underline{\nabla\times(\nabla\times\vec A) = \nabla(\nabla\cdot\vec A) - \nabla^2\vec A}$$

$$[\vec V\times(\nabla\times\vec V)]_i = \varepsilon_{ijk}\,V_j(\nabla\times\vec V)_k = \varepsilon_{ijk}\,V_j\,\varepsilon_{klm}\,\partial_l V_m$$
$$= \varepsilon_{kij}\,\varepsilon_{klm}\,V_j\,\partial_l V_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,V_j\,\partial_l V_m =$$
$$= V_j\,\partial_i V_j - V_j\,\partial_j V_i = \tfrac12\partial_i V^2 - (\vec V\cdot\nabla)V_i \Rightarrow$$
$$\underline{\vec V\times(\nabla\times\vec V) = \tfrac12\nabla V^2 - (\vec V\cdot\nabla)\vec V}$$

$$\nabla\cdot(\vec A\times\vec B) = \partial_i\,\varepsilon_{ijk}\,A_j B_k = \varepsilon_{ijk}\,(A_{j,i}B_k + A_j B_{k,i})$$

$$= B_k\,\varepsilon_{kij}\,A_{j,i} + A_j\,\varepsilon_{jki}\,B_{k,i}$$
$$= B_k(\nabla\times\vec A)_k - A_j\underbrace{\varepsilon_{jik}\,B_{k,i}}_{(\nabla\times\vec B)_j} = \vec B\cdot(\nabla\times\vec A) - \vec A\cdot(\nabla\times\vec B).$$

[Her er $\dfrac{\partial A_i}{\partial x_j} \equiv \partial_j A_i \equiv A_{i,j}$]

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Øving 2, forts.

c) Friksjonskraft $F_f = -\partial \mathcal{F}/\partial v$.

Systemets arbeid mot friksjonen, per tidsenhet, er

$\dot{W}_f = -F_f \cdot v = \frac{\partial \mathcal{F}}{\partial v} \cdot v$. Da $\mathcal{F} = C \cdot v^2$ blir

$\underline{\dot{W}_f} = 2Cv^2 = \underline{2\mathcal{F}}$

Lagrange: $\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial v} = 0$ gir, med

$L = T - V = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$ og $\mathcal{F} = 3\pi\mu a v^2$;

$m\dot{v} + kx + 6\pi\mu a v = 0$

$\therefore$

$$\underline{\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0, \quad \text{med } \lambda = \frac{3\pi\mu a}{m}, \quad \omega_0 = \sqrt{\frac{k}{m}}}$$

Løsning er $x(t) = x_0 e^{-\lambda t}\cos\omega_0 t$ når $\lambda/\omega_0 \ll 1$.

(Derivasjon $\dot{x}(t) \simeq -\omega_0 x_0 e^{-\lambda t}\sin\omega_0 t = 0$ når $t = 0$.)

Midlere energitap $\overline{\dot{W}_f}$ over én periode $2\pi/\omega_0$:

$\overline{\dot{W}_f} = 2\overline{\mathcal{F}}$.

Da $\mathcal{F} = 3\pi\mu a v^2 = m\lambda v^2$ blir $\overline{\dot{W}_f} = 2m\lambda\overline{v^2}$.

Her er $\overline{v^2} = \overline{(\omega_0 x_0)^2 e^{-2\lambda t}\sin^2\omega_0 t} \simeq \frac{1}{2}(\omega_0 x_0)^2 e^{-2\lambda t}$.

Altså $\underline{\overline{\dot{W}_f}} = 2m\lambda \cdot \frac{1}{2}(\omega_0 x_0)^2 e^{-2\lambda t} = \underline{m\lambda(\omega_0 x_0)^2 e^{-2\lambda t}}$