Physics 106a/196a – Problem Set 4 – Due Oct 27, 2006
Solutions

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Version 2: Corrections to typos in Problem 4 and Problem 6. They were probably obvious.

Problem 1

(a) sphere

First, we show that any parallelepiped inscribed in a sphere must be a right parallelepiped. We use a sphere of unit radius without lack of generality. We first choose $\vec{r} = (x, y, z)$, the position of one of the vertices. It must hold that

$$x^2 + y^2 + z^2 = 1 \tag{1}$$

Now, let the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ define the directions to the nearest three vertices from $\vec{R}$, placing those vertices at $\vec{R} + \vec{A}$, $\vec{R} + \vec{B}$, and $\vec{R} + \vec{C}$. Then we also have the equations

$$(x + x_A)^2 + (y + y_A)^2 + (z + z_A)^2 = 1 \tag{2}$$

$$(x + x_B)^2 + (y + y_B)^2 + (z + z_B)^2 = 1 \tag{3}$$

$$(x + x_C)^2 + (y + y_C)^2 + (z + z_C)^2 = 1 \tag{4}$$

Since the sphere defines a full 2-dimensional surface, there is an infinite number of possible choices of $\vec{A}$, $\vec{B}$, and $\vec{C}$ that stay on the sphere. But now the remainder of the vertices are fully determined: they are at $\vec{R} + \vec{A} + \vec{B}$, $\vec{R} + \vec{A} + \vec{C}$, $\vec{R} + \vec{B} + \vec{C}$, and $\vec{R} + \vec{A} + \vec{B} + \vec{C}$. We thus have four more equations that must be satisfied:

$$(x + x_A + x_B)^2 + (y + y_A + y_B)^2 + (z + z_A + z_B)^2 = 1 \tag{5}$$

$$(x + x_A + x_C)^2 + (y + y_A + y_C)^2 + (z + z_A + z_C)^2 = 1 \tag{6}$$

$$(x + x_B + x_C)^2 + (y + y_B + y_C)^2 + (z + z_B + z_C)^2 = 1 \tag{7}$$

$$(x + x_A + x_B + x_C)^2 + (y + y_A + y_B + y_C)^2 + (z + z_A + z_B + z_C)^2 = 1 \tag{8}$$

Now, if one subtracts from the $\vec{R} + \vec{A} + \vec{B}$ equation the $\vec{R} + \vec{A}$ and the $\vec{R} + \vec{B}$ equations and adds the $\vec{R}$ equation, and does similarly for the $\vec{R} + \vec{A} + \vec{C}$ and $\vec{R} + \vec{B} + \vec{C}$ equations, one obtains

$$2\, x_A\, x_B + 2\, y_A\, y_B + 2\, z_A\, z_B = 0 \tag{9}$$

$$2\, x_A\, x_C + 2\, y_A\, y_C + 2\, z_A\, z_C = 0 \tag{10}$$

$$2\, x_B\, x_C + 2\, y_B\, y_C + 2\, z_B\, z_C = 0 \tag{11}$$

These are just dot-product equations, indicating that $\vec{A}$, $\vec{B}$, and $\vec{C}$ must be mutually perpendicular. Hence, the parallelepiped must be a right parallelepiped.

Given the above, let us find the optimal side length for the parallelepiped. Since a sphere is rotationally symmetric, and the choice of the first vertex (x, y, z) is arbitrary, we may without loss of generality assume that the faces of the right parallelepiped are parallel to the coordinate axes. Then the volume of the parallelepiped is $8xyz$. Therefore we are to solve the following constrained optimization problem:

$$\text{maximize} \quad 8xyz$$

$$\text{subject to} \quad x^2 + y^2 + z^2 = 1$$

At a maximum point, the function

$$F(x, y, z, \lambda) = 8xyz + \lambda\left(x^2 + y^2 + z^2 - 1\right) \tag{12}$$

must have vanishing partial derivatives with respect to x, y, z, and to the Lagrange multiplier λ. This yields

$$\frac{\partial F}{\partial x} = 8yz + \lambda(2x) = 0 \tag{13}$$

$$\frac{\partial F}{\partial y} = 8xz + \lambda(2y) = 0 \tag{14}$$

$$\frac{\partial F}{\partial z} = 8xy + \lambda(2z) = 0 \tag{15}$$

$x\times$Eq. (13)-$y\times$Eq. (14) gives us

$$\lambda\left(2x^2 - 2y^2\right) = 0$$

Similarly we have

$$\lambda\left(2x^2 - 2z^2\right) = 0$$

$$\lambda\left(2y^2 - 2z^2\right) = 0$$

We find either $\lambda = 0$ or

$$x^2 = y^2 = z^2 \tag{16}$$

Obviously $\lambda \neq 0$ as otherwise at least two of x, y, z would have to be zero. Substituting Eq. (16) back into the equation of the ellipsoid, we obtain

$$x = \frac{1}{\sqrt{3}} \qquad y = \frac{1}{\sqrt{3}} \qquad z = \frac{1}{\sqrt{3}}$$

So the dimensions of the parallelepiped are $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and the volume is $\frac{8}{3\sqrt{3}}$. This result can of course be scaled to an arbitary radius sphere.

(b) ellipsoid

An ellipsoid of semimajor axes a, b, c can be mapped to a sphere by

$$x \to x' = x/a \qquad y \to y' = y/b \qquad z \to z' = z/c \tag{17}$$

The mapping is one-to-one. The Jacobian matrix of the mapping is $1/(abc)$, so all volumes are scaled by a factor $1/(abc)$ in this mapping. It can also be seen that parellelepipeds remain parallelepipeds. A parallelepiped is defined by one vertex $\vec{r} = (x, y, z)$ and three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ describing the vectors from $\vec{r}$ to the nearest three vertices; the remaining four vertices are built up from sums of $\vec{r}$ and combinations of $\vec{A}$, $\vec{B}$, and $\vec{C}$. If we scale the components of $\vec{r}$, $\vec{A}$, $\vec{B}$, and $\vec{C}$ by a, b, and c as in the above transformation, then all the vertices of an inscribed parallelepiped on the ellipsoid will map onto the sphere, and the same defining characteristics of the parallelepiped will remain with the transformed vectors $\vec{r}'$, $\vec{A}'$, $\vec{B}'$, and $\vec{C}'$.

Because a) the mapping is one-to-one; b) all volumes are scaled by a common factor set only by the shape of the ellipsoid; and c) parallelepipeds remain parallelpipeds, we may therefore transform our sphere result that the solution is a cube back to the ellipsoid. Put another way, if we take our cube and transform it back, all other possible parallelepipeds that can be inscribed in the ellipsoid must have volume less than or equal to the transformed cube, because the parallelepipeds they map to in the sphere case must be of equal or lesser volume than the cube. Now, they may map to rotated cubes, which is fine – whatever they are, they can have volume no larger than the solution we have already found.

So, the result for the ellipsoidal case is the vertices satisfy

$$|x| = \frac{a}{\sqrt{3}} \qquad |y| = \frac{b}{\sqrt{3}} \qquad |z| = \frac{c}{\sqrt{3}} \tag{18}$$

the side lengths are $\left(\frac{2a}{\sqrt{3}}, \frac{2b}{\sqrt{3}}, \frac{2c}{\sqrt{3}}\right)$ and the volume is $\frac{8abc}{3\sqrt{3}}$.

Problem 2

(a) Let c and b subscripts denote the car and the pendulum bob. The spatial position vectors of the box and the pendulum bob are

$$\overrightarrow{r}_c = X\widehat{x} \tag{19}$$
$$\overrightarrow{r}_b = (X + l\sin\theta)\,\widehat{x} - l\cos\theta\widehat{y} \tag{20}$$

The velocity of the bob in the lab frame is

$$\begin{aligned}\overrightarrow{v}_b = \dot{\overrightarrow{r}}_b &= \left(\dot{X} + l\cos\theta\dot{\theta}\right)\widehat{x} + D\sin\theta\dot{\theta}\widehat{y} \\ &= \left(v_0 + at + l\cos\theta\dot{\theta}\right)\widehat{x} + D\sin\theta\dot{\theta}\widehat{y}\end{aligned} \tag{21}$$

where we use $\dot{X} = v_0 + at$. The kinetic energy is

$$\begin{aligned}T &= \frac{1}{2}m_b\overrightarrow{v}_b\cdot\overrightarrow{v}_b \\ &= \frac{1}{2}m_b\left[\left(\dot{X} + D\cos\theta\dot{\theta}\right)^2 + D^2\sin^2\theta\dot{\theta}^2\right] \\ &= \frac{1}{2}m_b\left[\left(v_0 + at + D\cos\theta\dot{\theta}\right)^2 + D^2\sin^2\theta\dot{\theta}^2\right] \\ &= \frac{1}{2}m_b\left[(v_0 + at)^2 + 2\,(v_0 + at)\,D\cos\theta\dot{\theta} + D^2\dot{\theta}^2\right]\end{aligned}$$

(b) The potential is

$$V = -m_b g D \cos\theta \tag{22}$$

The Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m_b\left[(v_0+at)^2 + 2(v_0+at)D\cos\theta\dot{\theta} + D^2\dot{\theta}^2\right] + m_b g D\cos\theta \end{aligned}$$

which depends explicitly on the time.

(c) The Euler-Lagrange equation is

$$\begin{aligned} \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0 \qquad (23) \\ -m_b(v_0+at)D\sin\theta\dot{\theta} - m_b g D\sin\theta - \frac{d}{dt}\left(m_b\left[(v_0+at)D\cos\theta + D^2\dot{\theta}\right]\right) &= 0 \\ -gD\sin\theta - aD\cos\theta - D^2\ddot{\theta} &= 0 \\ \ddot{\theta} + \frac{g}{D}\sin\theta + \frac{a}{D}\cos\theta &= 0 \end{aligned}$$

(d) When the pendulum remains at rest in the equilibrium, $\ddot{\theta} = 0$ which implies

$$\begin{aligned} \frac{g}{D}\sin\theta_{eq} + \frac{a}{D}\cos\theta_{eq} &= 0 \\ \tan\theta_{eq} &= -\frac{a}{g} \end{aligned}$$

To check that the equilibrium is stable or not, we need to expand the EOM around $\theta = \theta_{eq}$. We first note that

$$\begin{aligned} \cos(\theta_{eq}+\eta) &= \cos\theta_{eq} - \eta\sin\theta_{eq} + O(\eta^2) \\ \sin(\theta_{eq}+\eta) &= \sin\theta_{eq} + \eta\cos\theta_{eq} + O(\eta^2) \end{aligned}$$

So around θ_{eq}, EOM becomes up to $O(\eta^2)$

$$\ddot{\eta} + \frac{g}{D}(\sin\theta_{eq} + \eta\cos\theta_{eq}) + \frac{a}{D}(\cos\theta_{eq} - \eta\sin\theta_{eq}) = 0 \tag{24}$$

$$\ddot{\eta} + \frac{g\cos\theta_{eq}}{D}\eta\left(1 - \frac{a}{g}\tan\theta_{eq}\right) = 0 \tag{25}$$

$$\ddot{\eta} + \frac{g\cos\theta_{eq}}{D}\left(1 + \frac{a^2}{g^2}\right)\eta = 0 \tag{26}$$

Since $\frac{g\cos\theta_{eq}}{D}\left(1+\frac{a^2}{g^2}\right) > 0$, $\theta = \theta_{eq}$ is a stable equilibrium.

(e)

$$\begin{aligned} \cos\theta_{eq} &= \frac{1}{\sqrt{1+\tan^2\theta_{eq}}} \\ &= \frac{g}{\sqrt{g^2+a^2}} \end{aligned}$$

So from Eq. (26), we find

$$\begin{aligned}
\omega^2 &= \frac{g\cos\theta_{eq}}{D}\left(1+\frac{a^2}{g^2}\right) \\
&= \frac{g^2}{D\sqrt{g^2+a^2}}\left(1+\frac{a^2}{g^2}\right) \\
&= \frac{\sqrt{g^2+a^2}}{D}
\end{aligned}$$

Problem 3

(a) We start with the Lagrangian

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}\left(\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\psi^*}{\partial x}\right) - V\psi^*\psi \tag{27}$$

We treat ψ and ψ^* as two independent variables and the corresponding Euler-Lagrange equations are

$$\psi^* : \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial\psi^*}{\partial t}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial\psi^*}{\partial x}}\right) - \frac{\partial\mathcal{L}}{\partial\psi^*} = 0 \tag{28}$$

$$-\frac{i\hbar}{2}\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) + V\psi - \frac{i\hbar}{2}\frac{\partial\psi}{\partial t} = 0 \tag{29}$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}\right) + V\psi = i\hbar\frac{d\psi}{dt} \tag{30}$$

$$\psi : \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial\psi}{\partial t}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial\psi}{\partial t}}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \tag{31}$$

$$\frac{i\hbar}{2}\frac{\partial\psi^*}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}\right) + V\psi^* + \frac{i\hbar}{2}\frac{\partial\psi^*}{\partial t} = 0 \tag{32}$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi^*}{\partial x^2}\right) + V\psi^* = -i\hbar\frac{d\psi^*}{dt} \tag{33}$$

(b)

$$
\begin{aligned}
&\mathcal{L}\left(\psi', \psi'^*, \frac{\partial \psi'}{\partial t}, \frac{\partial \psi'^*}{\partial t}, \frac{\partial \psi'}{\partial x}, \frac{\partial \psi'^*}{\partial x}, x, t\right) \\
&= -\frac{i\hbar}{2}\left(\psi'^* \frac{\partial \psi'}{\partial t} - \psi' \frac{\partial \psi'^*}{\partial t}\right) - \frac{\hbar^2}{2m}\left(\frac{\partial \psi'}{\partial x}\right)\left(\frac{\partial \psi'^*}{\partial x}\right) - V\psi'^*\psi' \\
&= -\frac{i\hbar}{2}\left(\left(\psi e^{i\phi}\right)^* \frac{\partial\left(\psi e^{i\phi}\right)}{\partial t} - \left(\psi e^{i\phi}\right)\frac{\partial \left(\psi e^{i\phi}\right)^*}{\partial t}\right) \\
&\quad - \frac{\hbar^2}{2m}\left(\frac{\partial\left(\psi e^{i\phi}\right)}{\partial x}\right)\left(\frac{\partial \left(\psi e^{i\phi}\right)^*}{\partial x}\right) - V\left(\psi e^{i\phi}\right)^*\left(\psi e^{i\phi}\right) \\
&= -\frac{i\hbar}{2}\left(\psi^* e^{-i\phi}\frac{e^{i\phi}\partial\psi}{\partial t} - \psi e^{i\phi}\frac{e^{-i\phi}\partial\psi^*}{\partial t}\right) \\
&\quad - \frac{\hbar^2}{2m}\left(\frac{\partial \psi}{\partial x}\right) e^{i\phi}e^{-i\phi}\left(\frac{\partial \psi^*}{\partial x}\right) - V\psi^* e^{-i\phi}e^{i\phi}\psi \\
&= -\frac{i\hbar}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}\left(\frac{\partial \psi}{\partial x}\right)\left(\frac{\partial \psi^*}{\partial x}\right) - V\psi^*\psi \\
&= \mathcal{L}\left(\psi, \psi^*, \frac{\partial \psi}{\partial t}, \frac{\partial \psi^*}{\partial t}, \frac{\partial \psi}{\partial x}, \frac{\partial \psi^*}{\partial x}, x, t\right)
\end{aligned} \tag{34}
$$

From Eq. (34), we note $\mathcal{L}\left(\psi', \psi'^*, \frac{\partial \psi'}{\partial t}, \frac{\partial \psi'^*}{\partial t}, \frac{\partial \psi'}{\partial x}, \frac{\partial \psi'^*}{\partial x}, x, t\right)$ is independent of ϕ so

$$
\frac{d}{d\phi}\mathcal{L}\left(\psi', \psi'^*, \frac{\partial \psi'}{\partial t}, \frac{\partial \psi'^*}{\partial t}, \frac{\partial \psi'}{\partial x}, \frac{\partial \psi'^*}{\partial x}, x, t\right) = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial \psi'}\frac{\partial \psi'}{\partial \phi} + \frac{\partial \mathcal{L}}{\partial \psi^{*\prime}}\frac{\partial \psi^{*\prime}}{\partial \phi} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi'}{\partial t}}\frac{\partial \frac{\partial \psi'}{\partial t}}{\partial \phi} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi^{*\prime}}{\partial t}}\frac{\partial \frac{\partial \psi^{*\prime}}{\partial t}}{\partial \phi} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi'}{\partial x}}\frac{\partial \frac{\partial \psi'}{\partial x}}{\partial \phi} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi^{*\prime}}{\partial x}}\frac{\partial \frac{\partial \psi^{*\prime}}{\partial x}}{\partial \phi} = 0
$$

If we apply the Euler-Lagrange equation, we find

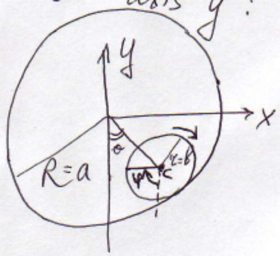
$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi'}{\partial t}}\right)\frac{\partial \psi'}{\partial \phi} + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*'}{\partial t}}\right)\frac{\partial \psi^{*\prime}}{\partial \phi} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi'}{\partial x}}\right)\frac{\partial \psi'}{\partial \phi} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*'}{\partial x}}\right)\frac{\partial \psi^{*\prime}}{\partial \phi} \\
&\quad + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi'}{\partial t}}\frac{\partial \frac{\partial \psi'}{\partial \phi}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*'}{\partial t}}\frac{\partial \frac{\partial \psi^{*\prime}}{\partial \phi}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi'}{\partial x}}\frac{\partial \frac{\partial \psi'}{\partial \phi}}{\partial x} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*'}{\partial x}}\frac{\partial \frac{\partial \psi^{*\prime}}{\partial \phi}}{\partial x} \\
0 &= \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi'}{\partial t}}\frac{\partial \psi'}{\partial \phi}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*'}{\partial t}}\frac{\partial \psi^{*\prime}}{\partial \phi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi'}{\partial x}}\frac{\partial \psi'}{\partial \phi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*'}{\partial x}}\frac{\partial \psi^{*\prime}}{\partial \phi}\right) \\
0 &= \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial t}}\frac{\partial \psi'}{\partial \phi}|_{\phi=0}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*}{\partial t}}\frac{\partial \psi^{*\prime}}{\partial \phi}|_{\phi=0}\right) \\
&\quad + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial x}}\frac{\partial \psi'}{\partial \phi}|_{\phi=0}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*}{\partial x}}\frac{\partial \psi^{*\prime}}{\partial \phi}|_{\phi=0}\right) \\
0 &= \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial t}}\psi - \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi*}{\partial t}}\psi^*\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial x}}\psi - \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi^*}{\partial x}}\psi^*\right) \\
0 &= i\hbar\frac{\partial}{\partial t}\left(\psi\psi^*\right) - \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial \psi^*}{\partial x}\psi - \frac{\partial \psi}{\partial x}\psi^*\right) \\
0 &= \frac{\partial}{\partial t}\left(\psi\psi^*\right) + \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\frac{\partial \psi^*}{\partial x}\psi - \frac{\partial \psi}{\partial x}\psi^*\right)
\end{aligned}
$$

1

Problem #4 Solution.

(A) The constraint between coordinates θ and φ depends on the way you choose the reference frame for φ.

For example, if one chooses the absolute reference for φ, when φ is referenced to axis $\hat{y}$:



$$(R-r)\cdot d\theta = r\cdot d\varphi;$$

OR

$$(a-b)\cdot d\theta = b\cdot d\varphi;$$

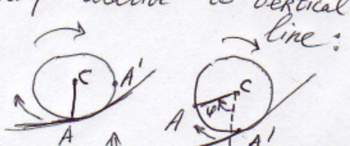
$$\boxed{(a-b)\cdot\theta = b\cdot\varphi} \quad (1)$$

Condition of rolling without slipping

Comment:

Angle θ determines the position of the center of mass of the hoop

Angle φ determines the position of an arbitrary line inside the hoop relative to vertical line:



at some moment t_0: point A is on the vertical line from the center of the hoop

At a later time t_1: point A moved in clockwise direction

2

Note that center of mass of the rolling hoop has coordinates (x, y) given by:

$$\left((a-b)\cdot\sin\theta,\ -(a-b)\cdot\cos\theta\right)$$

Velocity of the center of mass:

$$\left((a-b)\cdot\dot{\theta}\cdot\cos\theta,\ (a-b)\cdot\dot{\theta}\cdot\sin\theta\right)$$

Therefore, kinetic energy of this rolling hoop can be written as

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\phi}^2 =$$

$$= \frac{1}{2}m(a-b)^2\dot{\theta}^2 + \frac{1}{2}\cdot mb^2\cdot\dot{\phi}^2,$$

where $I = mb^2 \rightarrow$ moment of inertia of hoop with mass m, radius b

$\dot{\phi}$ is the angular velocity of rotation of the hoop.

In general when a rigid body participate in progressive and in rotational motions,

(3)

the kinetic energy of the rigid body is <u>the sum</u> of the energy of progressive motion $\boxed{T_{PROGR} = \frac{1}{2} m v^2}$

AND

the energy of rotational motion

$\boxed{T_{ROTAT} = \frac{1}{2} I \omega^2}$ (in our case $T_{ROTAT} = \frac{1}{2}(m b^2) \dot{\varphi}^2$)

We will use these results in part (b) when we will write Lagrangian.

As it has been mentioned, constraints on ~~equation~~ parameters θ and φ depend on the reference for φ. If one chooses the <u>Rolling reference</u> for φ, when angle φ is <u>not</u> related to axis $\hat{y}$, one gets:

$a \cdot d\theta = b \cdot d\varphi$

$\boxed{a \cdot \theta = b \cdot \varphi}$ (2)

(4)

<u>PART (b)</u> LAGRANGIAN $L = T - U$

IN the absolute reference for φ:

$$\boxed{L(\theta,\varphi) = \frac{1}{2} m (a-b)^2 \dot{\theta}^2 + \frac{1}{2} m b^2 \dot{\varphi}^2 + mg(a-b)\cos\theta;} \quad \text{(eq. 3)}$$

IN the <u>Rolling reference</u> for φ:

$$\boxed{L(\theta,\varphi) = \frac{1}{2} m (a-b)^2 \dot{\theta}^2 + \frac{1}{2} m b^2 (\dot{\varphi} - \dot{\theta})^2 + mg(a-b)\cos\theta;} \quad \text{(eq. 4)}$$

$$\frac{\partial L(\theta,\varphi)}{\partial \theta} = -mg(a-b)\cdot\sin\theta;$$

$$\frac{\partial L(\theta,\varphi)}{\partial \varphi} = 0;$$

$$\frac{\partial L(\theta,\varphi)}{\partial \dot{\theta}} = \begin{cases} m(a-b)^2 \dot{\theta} & \underline{\text{FOR (eq. 3)}} \\ m(a-b)^2 \dot{\theta} - m b^2 (\dot{\varphi} - \dot{\theta}) & \underline{\text{for (eq. 4)}} \end{cases}$$

$$\frac{\partial L(\theta,\varphi)}{\partial \dot{\varphi}} = \begin{cases} \frac{2}{2} m b^2 \dot{\varphi} = m b^2 \dot{\varphi} & \underline{\text{for (eq. 3)}} \\ -m b^2 (\dot{\theta} - \dot{\varphi}) = m b^2 (\dot{\varphi} - \dot{\theta}) & \underline{\text{for (eq. 4)}} \end{cases}$$

(5)

Parts c, d Find the Euler-Lagrange Equations with Lagrange multipliers.

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right) - \frac{\partial \tilde{L}}{\partial q} = 0 \quad \text{where } \tilde{L} = \underbrace{L}_{\text{FROM part } b} + \lambda \underbrace{G}_{\text{FROM constraint equations}}$$

By now we have two possible expressions for Lagrangian, because we could initially solve the problem in 2 different references for φ.

Now we can choose function G in several ways, so there is even more ambiguity in the solution of the problem!

For example, if I use (eq. 1) $\Rightarrow (a-b)\theta = b\varphi$ and Lagrangian from (eq. 3), I can chose

$$G(\theta,\varphi) = \varphi - \frac{a-b}{b}\theta = 0 = \text{const},$$

which will give the following Euler-Lagrange eq.:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\theta}}\right) - \frac{\partial \tilde{L}}{\partial \theta} = 0 &\rightarrow \left\{ m(a-b)^2\ddot{\theta} + mg(a-b)\sin\theta + \lambda\frac{a-b}{b} = 0 \right. \\ \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \tilde{L}}{\partial \varphi} = 0 &\rightarrow \left\{ m b^2\ddot{\varphi} - \lambda \cdot 1 = 0 \right. \qquad \text{(eq.5)} \end{aligned}$$

where $\theta(t)$, $\varphi(t)$ and $\lambda(t)$ are unknown

(6)

But even if I use (eq. 1) and (eq. 3) I can chose different function $G(\theta,\varphi)$:

$$G(\theta,\varphi) = b\varphi - (a-b)\theta.$$

Then $\frac{\partial G}{\partial \theta} = -(a-b)$ and $\frac{\partial G}{\partial \varphi} = b$, and instead of (eq. 5) we have the following Euler-Lagrange equations;

$$\boxed{\begin{cases} m(a-b)^2\ddot{\theta} + mg(a-b)\sin\theta + \lambda(a-b) = 0, \\ m b^2\ddot{\varphi} - \lambda b = 0 \end{cases} \qquad \text{(eq.6)}}$$

Therefore, Lagrange multipliers can have different units (!) in the same problem; but actual physical constraint forces do not depend on this ambiguity in choosing Lagrange multipliers.

If we start from (eq. 2) and Lagrangian (eq. 4), we can define $G(\theta,\varphi) = a\theta - b\varphi = 0 = \text{const}$,

(7)

and we arrive at the following Euler-Lagrange equations:

$$\boxed{\begin{cases} m(a-b)^2\ddot{\theta} - mb^2(\ddot{\varphi}-\ddot{\theta}) + mg(a-b)\sin\theta - \lambda a = 0 \\ mb^2(\ddot{\varphi}-\ddot{\theta}) + b\lambda = 0 \end{cases} \quad \text{eq}(7)}$$

Let's solve this system of equations (eq. 7).

$$a\theta = b\varphi \rightarrow a\ddot{\theta} - b\ddot{\varphi} = 0; \quad \underline{\ddot{\varphi} = \frac{a}{b}\ddot{\theta}}$$

substiture it into the upper equation of (eq. 7)

$$m(a-b)^2\cdot\ddot{\theta} - mb^2\left(\frac{a}{b}-1\right)\ddot{\theta} + mg(a-b)\sin\theta - \lambda a = 0;$$

$$\ddot{\theta}\left(a^2+b^2-2ab-ab+b^2\right) - g(a-b)\sin\theta - \frac{\lambda}{m}a = 0;$$

$$\ddot{\theta}\cdot\left(a^2+2b^2-3ab\right) + g(a-b)\sin\theta - \frac{\lambda}{m}a = 0; \quad \underline{(\text{eq. }8)}$$

Use the second equation in (eq. 7): $b^2\left(\frac{a}{b}-1\right)\ddot{\theta} + b\cdot\frac{\lambda}{m} = 0;$

$\frac{\lambda}{m} = -b\left(\frac{a}{b}-1\right)\cdot\ddot{\theta}$, and substitute it into $\underline{\text{eq. }8}$:

$$\ddot{\theta}\left(a^2+2b^2-3ab\right) + g(a-b)\cdot\sin\theta + \left(a^2-ab\right)\ddot{\theta} = 0;$$

$$2\ddot{\theta}\underbrace{\left(a^2-2ab+2ab+b^2\right)}_{(a-b)^2} + g(a-b)\cdot\sin\theta = 0;$$

(8)

So, we arrive at the following equation:

$$2\ddot{\theta}(a-b) + g\cdot\sin\theta = 0,$$

or

$$\boxed{2\ddot{\theta} + \left(\frac{g}{a-b}\right)\cdot\sin\theta = 0} \quad (\text{eq. } 9)$$

Every student should be able to carry out similar algebra for another function $G(\theta,\varphi)$ and another constraint (eq. 1), which leads to different Lagrangian (eq. 3). Every student should be able to show that both (eq. 5) and (eq. 6) lead to the same equation for θ;

$$\underline{\ddot{\theta} + \frac{1}{2}\left(\frac{g}{a-b}\right)\sin\theta = 0.}$$

Part (e) Generalized constraint force N_φ:

$$N_\varphi = \lambda\cdot\frac{\partial G}{\partial\varphi} = -\lambda b = -\left(-mb\left(\frac{a}{b}-1\right)\ddot{\theta}\right)\cdot b = mb^2\left(\frac{a}{b}-1\right)\cdot\frac{g}{2(a-b)}\cdot\sin\theta = \underline{\frac{1}{2}mgb\cdot\sin\theta}$$

(f)

Regarding finding the radial constraint force – the key would be to introduce a radial coordinate ρ for the center of mass of the rolling hoop and to write down a constraint equation $\rho - (a - b) = 0$. One would have to let ρ be a completely undetermined variable, with the appropriate kinetic energy term, until its value is set by the Lagrange multiplier equation. For a problem in this style, see Problem 6.

Part g, If $\theta \ll 1 \Rightarrow$

$\sin\theta \approx \theta$;

(Eq. 9) becomes $\ddot{\theta} + \frac{1}{2}\frac{g}{a-b}\theta = 0$

"Standart" equation for harmonic oscillations: $\ddot{\theta} + \omega^2\theta = 0$

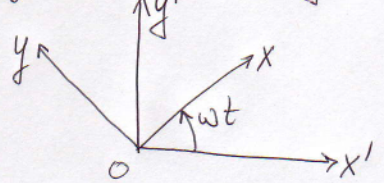
$$\Rightarrow \omega = \sqrt{\frac{g}{2(a-b)}} \rightarrow \text{angular frequency}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2(a-b)}{g}} \rightarrow \text{period}$$

Problem 5

(1)

Let (x', y', z') be inertial coordinate frame;
(x, y, z) be rotating coordinate frame.



$$\begin{cases} x = x'\cdot\cos\omega t + y'\sin\omega t \\ y = -x'\sin\omega t + y'\cos\omega t \\ z = z' \end{cases}$$

$$\begin{cases} x' = x\cdot\cos\omega t - y\cdot\sin\omega t \\ y' = x\cdot\sin\omega t + y\cdot\cos\omega t \\ z' = z \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}' = \dot{x}\cos\omega t - \dot{y}\sin\omega t - \\ \quad - x\cdot\omega\cdot\sin\omega t - y\omega\cdot\cos\omega t; \\ \dot{y}' = \dot{x}\cdot\sin\omega t + \dot{y}\cdot\cos\omega t + \\ \quad + x\cdot\omega\cdot\cos\omega t - y\cdot\omega\cdot\sin\omega t; \\ \dot{z}' = \dot{z} \end{cases}$$

Kinetic energy: $T = \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2) + \frac{1}{2}m\omega^2(x^2+y^2) =$

(2)

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}m\omega^2(x^2+y^2) +$$

$$+ m\omega\Big[-\dot{x}y\cdot\cos^2\omega t - \dot{x}x\cdot\cos\omega t\cdot\sin\omega t + \dot{y}x\sin^2\omega t +$$

$$+ \dot{y}y\cdot\sin\omega t\cdot\cos\omega t + \dot{x}x\cdot\cos\omega t\cdot\sin\omega t -$$

$$- \dot{x}y\cdot\sin^2\omega t + \dot{y}x\cos^2\omega t - \dot{y}y\cdot\cos\omega t\cdot\sin\omega t\Big] =$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}m\omega^2(x^2+y^2) + m\omega(x\dot{y} - \dot{x}y);$$

$$\boxed{\text{LAGRANGIAN: } L = T - V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \frac{1}{2}m\omega^2(x^2+y^2) + m\omega(x\dot{y} - \dot{x}y) - V;}$$

Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0;$

$$\begin{cases} m\ddot{x} - 2m\omega\dot{y} - m\omega^2 x + \frac{\partial V}{\partial x} = 0; \\ m\ddot{y} + 2m\omega\dot{x} - m\omega^2 y + \frac{\partial V}{\partial y} = 0; \\ m\ddot{z} + \frac{\partial V}{\partial z} = 0 \end{cases}$$

(3)

Remark. We showed how to express velocity components $(\dot{x}', \dot{y}', \dot{z}')$ in inertial frame in terms of velocity components $(\dot{x}, \dot{y}, \dot{z})$ in rotating frame.

You will see in the future that velocity $\vec{V}'$ in inertial frame is related to velocity $\vec{V}$ in rotating frame as

$$\boxed{\vec{V}' = \vec{V} + [\vec{\omega} \times \vec{r}]}$$

In our case: $\vec{\omega} = (0, 0, \omega)$; $\vec{r} = (x, y, z)$; $\vec{V} = (\dot{x}, \dot{y}, \dot{z})$

$$V'^2 = V^2 + 2(\vec{V}[\vec{\omega} \times \vec{r}]) + |[\vec{\omega} \times \vec{r}]|^2 =$$
$$= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2\omega(x\dot{y} - \dot{x}y) + \omega^2(x^2 + y^2);$$

(4)

Lagrange's equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

(б) If a particle m moves in a fixed reference frame (x, y, z) under a force $-\text{grad}\, V \equiv -\nabla V$ and an additional velocity-dependent potential U, the Lagrangian is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V - U$

Comparing with Lagrangian in (a) $\Rightarrow$

$$U = -m\omega(x\dot{y} - \dot{x}y) - \frac{1}{2}m\omega^2(x^2 + y^2).$$

Components of the force can be found according to:

$$F_x^U = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}} =$$

$$= m\omega^2 x + m\omega\dot{y} + m\omega\dot{y} =$$

$$= m\omega^2 x + 2m\omega\dot{y} \; ;$$

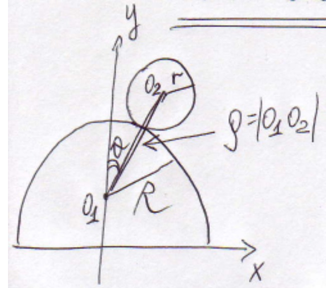
$$F_y^U = -\frac{\partial U}{\partial y} + \frac{d}{dt}\frac{\partial U}{\partial \dot{y}} = m\omega^2 y - m\omega\dot{x} - m\omega\dot{x} =$$

$$= m\omega^2 y - 2m\omega\dot{x} \, .$$

5

Problem 6.

1



θ is polar angle determining the position of the center mass of the hoop.

φ is rotation angle of hoop.

ρ = distance at which the center mass of the hoop is from the fixed cylinder's center.

Constraints:

$G_1 = \rho - (r+R) = 0$ hoop keeps rolling on the surface of fixed cylinder

$G_2 \Rightarrow R\theta = r\varphi$ Rolling without slipping (see discussion in problem 4)

$$\begin{cases} T = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\theta}^2 + \frac{1}{2}mr^2(\dot{\theta}+\dot{\varphi})^2 & \text{kinetic energy} \\ V = mg\rho\cdot\cos\theta & \text{Potential energy} \end{cases}$$

We can use the first constraint G_1 to eliminate φ.

(2)

$$\boxed{\begin{aligned} T &= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\theta}^2 + \frac{1}{2}m\left[r+R\right]^2\dot{\theta}^2 = \\ &= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\theta}^2 + \frac{1}{2}m(R+r)^2\dot{\theta}^2 ; \end{aligned}} \quad \text{(eq.1)}$$

$$\frac{\partial L}{\partial \rho} = m\rho\dot{\theta} - mg\cdot\cos\theta \; ; \qquad \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} ;$$

$$\frac{\partial L}{\partial \theta} = mg\rho\cdot\sin\theta \; ; \qquad \frac{\partial G_1}{\partial \rho} = 1 ;$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\left(\rho^2 + (r+R)^2\right)\dot{\theta} \; ; \qquad \frac{\partial G_1}{\partial \theta} = 0 ;$$

LAGRANGE equations:

$$\begin{cases} m\rho\dot{\theta}^2 - mg\cdot\cos\theta - m\ddot{\rho} + \lambda = 0 ; & (a) \\ mg\rho\cdot\sin\theta - \dfrac{d}{dt}\left[m\left(\rho^2 + (R+r)^2\right)\dot{\theta}\right] = 0 ; & (b) \\ \rho - (r+R) = 0 ; & (c) \end{cases} \quad \text{(eq.2)}$$

Substituting ρ FROM (c) into (a) AND (b):

(3)

$$\begin{cases} m(r+R)\cdot\dot{\theta} - mg\cdot\cos\theta + \lambda = 0 ; \\ mg(r+R)\sin\theta - 2m(R+r)^2\ddot{\theta} = 0 ; \end{cases}$$

$$\boxed{\frac{\lambda}{m} = g\cdot\cos\theta - (r+R)\dot{\theta}^2 ;} \quad \text{(eq. 3)}$$

$$\ddot{\theta} = \frac{g\cdot\sin\theta}{2(R+r)} ;$$

Let's remove $\dot{\theta}^2$ from (eq.3) using the law of conservation of energy:

$$\begin{cases} E = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\theta}^2 + \frac{1}{2}m(R+r)^2\dot{\theta}^2 - mg\rho\cos\theta \\ \dot{\rho} = 0 ; \quad \rho^2 = (R+r)^2 \end{cases}$$

$$(R+r)^2\dot{\theta}^2 = \frac{E}{m} - g(R+r)\cos\theta ;$$

$$\dot{\theta}^2 = \frac{\frac{E}{m} - g(R+r)\cdot\cos\theta}{(R+r)^2} ;$$

$$E = V(\theta=0) = mg(R+r);$$

$$\Rightarrow \boxed{\dot{\theta}^2 = g\,\frac{1-\cos\theta}{R+r}} \qquad (eq.\ 4)$$

Substituting (eq. 4) into (eq. 3) $\Rightarrow$

$$\frac{\lambda}{m} = g\cdot\cos\theta - g\cdot(1-\cos\theta) = g\cdot(2\cos\theta - 1).$$

$\lambda = 0$ when $\cos\theta = \frac{1}{2}$,

$\boxed{\text{i.e. hoop will fall off the cylinder when } \theta_0 = 30^\circ\text{, or } \cos\theta_0 = \tfrac{1}{2}}$

Remark about problem 6

We cannot use just one constraint equation instead of two constraint equations. What I mean is the following: we used two constraint functions

$$G_1(\rho,\theta,\varphi) \equiv \rho - (r+R) = 0;$$

$$G_2(\rho,\theta,\varphi) \Rightarrow R\theta = r\varphi;$$

It seems that we can substitute expression for R from G_1 into G_2:

$$R = \rho - r \rightarrow \quad \widetilde{G_2}(\rho,\theta,\varphi) \equiv (\rho - r)\theta - r\varphi \equiv 0.$$

Unfortunately, such constraint

$$\widetilde{G_2}(\rho,\theta,\varphi) \equiv (\rho - r)\theta - r\varphi = 0$$

will lead to incorrect answer for θ_0, at which the hoop tears off the cylinder.

Indeed,

$$T = \underbrace{\tfrac{1}{2}m\dot{\rho}^2}_{\text{Radial motion of } \sout{\text{cylinder}}\text{ hoop}} + \underbrace{\tfrac{1}{2}m\rho^2\dot{\theta}^2}_{\text{circular motion of CM}} + \underbrace{\tfrac{1}{2}mr^2(\dot{\theta}+\dot{\varphi})^2}_{\text{Rotation around CM}}$$

(7)

$V = mg\rho\cdot\cos\theta;$

Eleminating φ, we have:

$$T = \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\dot\theta^2 + \frac{1}{2}m\left[r+(\rho-r)\right]^2\dot\theta^2 =$$

$$= \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\dot\theta^2 + \frac{1}{2}m\rho^2\dot\theta^2 =$$

$$= \frac{1}{2}m\dot\rho^2 + m\rho^2\dot\theta^2;$$

$$\frac{\partial L}{\partial\rho} = 2m\rho\cdot\dot\theta^2 - mg\cdot\cos\theta; \qquad \frac{\partial L}{\partial\dot\rho} = m\dot\rho;$$

$$\frac{\partial L}{\partial\theta} = m\rho g\cdot\sin\theta; \qquad \frac{\partial G_1}{\partial\rho} = 1;$$

$$\frac{\partial L}{\partial\dot\theta} = 2m\rho^2\dot\theta; \qquad \frac{\partial G_1}{\partial\theta} = 0.$$

Therefore, three Euler-Lagrange equations take the form:

(A) $2m\rho\dot\theta^2 - mg\cos\theta - m\ddot\rho + \lambda = 0;$

(B) $m\rho g\sin\theta - \frac{d}{dt}(2m\rho^2\dot\theta) = 0;$

(C) $\rho - (r+R) = 0$ or $\ddot\rho = 0$.

(8)

Therefore, Euler-Lagrange equations become:

$$\begin{cases} (a') \; 2(r+R)\dot\theta^2 - g\cdot\cos\theta + \lambda = 0; \\ (b') \; g(r+R)\cdot\sin\theta - 2\cdot(r+R)^2\ddot\theta = 0; \end{cases}$$ which gives

$$\begin{cases} \lambda = g\cdot\cos\theta - 2(r+R)\dot\theta^2; \\ \ddot\theta = \frac{1}{2}\cdot\frac{g}{r+R}\cdot\sin\theta; \end{cases}$$

$\dot\theta^2$ in the expression for λ can be expressed in terms of energy E (which obeys the law of conservation)

$$\Rightarrow \cancel{\frac{1}{2}m\dot\rho^2}_{\,0} + m\rho^2\dot\theta^2 + mg\rho\cdot\cos\theta = E;$$

$$\dot\theta^2 = \frac{E - m\rho g\cdot\cos\theta}{m\rho^2} = \frac{E - mg(r+R)\cos\theta}{m(r+R)^2};$$

From the other hand,

$E = E_{initial} = mg(r+R)$ when $\theta = 0$, $\dot\theta = 0$ at $t=0$

So, $\lambda = g\cdot\cos\theta - 2\,\frac{E - mg(r+R)\cos\theta}{m(r+R)} = g\cos\theta - 2g(1-\cos\theta)$

$\boxed{\lambda = g\cdot(3\cos\theta - 2)}$, $\lambda = 0$ when $\cos\theta = \frac{2}{3}$.