The Coriolis Effect – a conflict between common sense and mathematics

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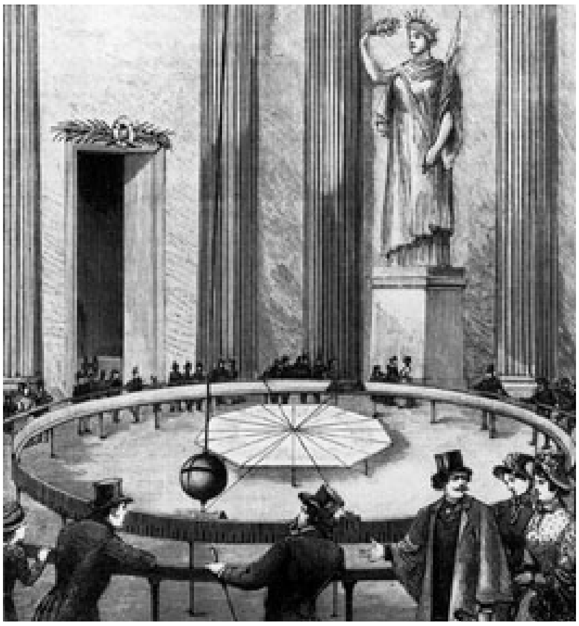
1. Introduction: The 1905 debate

Hundred years ago the German journal "Annalen der Physik", the same 1905 volume where Albert Einstein published his first five ground breaking articles, provided a forum for a debate between three physicists, Denizot, Rudzki and Tesař on the correct interpretation of the Coriolis effect, in particular how it manifested itself in the Foucault pendulum experiment. The debate was complicated by many side issues, but the main problem was this: if the pendulum's plane of swing was fixed relative to the stars, as it was often said, why then was not its period of rotation the same, one sidereal day (23 hours and 56 minutes), everywhere on earth and not only at the poles?

Instead the period was 28 hours in Helsinki, 30 hours in Paris and 48 hours in Casablanca, i.e. the sidereal day divided by the sine of latitude. At the equator the period was infinite; there was no deflection. This could only mean that the plane of swing indeed was turning relative the stars. But how could then, as it was also said, a 'fictitious' inertial force be responsible for the turning?

Hundred years later, Einstein's five 1905 "Annalen der Physik" papers are common ground in the elementary physics education whereas teachers and students, just like Denizot, Rudzki and Tesař, struggle to come to terms with the Coriolis effect. This article will try to explain the complex and contradictory understanding of the deflective mechanism in rotating systems. But first it might be appropriate to remind us what is generally agreed on.

In the 18th century the problems of finding the longitudinal position at sea was of prime importance. One method demanded very accurate time keeping. In 1847 the French mathematician Poisson stated that the movement of a simple pendulum would not be affected by the rotation of the earth. Only four years later another French scientist Foucault could show that this was indeed the case. Although the deflection in one swing was minute, successive swings would accumulate and make the swing change substantially over time. Since the period Foucault measured was 30 hours he more or less that it would be proportional to the inverse of the sine of latitude. This was actually in conflict with his own physical explanation according to which the plane of swing would remain fixed versus the fix stars while the earth was rotating under it. The fact that it turns versus the starts indicates that a real force is doing work, and this real force is the component of gravitation perpendicular to the rotational axis. Only at the poles is this component zero, and there, but only there, does the plane of swing preserve its orientation relative to the stars. Foucault's experiment was hailed as the definite poof that Galileo had been right and the Church wrong about the rotation of earth. However, had the experiment been conducted in the Tropics where the period exceeds three days, the link to the earth rotation would have been less obvious and the propagandistic value highly reduced.



2. The Coriolis effect – the basics

A mass particle (m) that is *stationary* in a rotating system ($\mathbf{\Omega}$) at a distance $\mathbf{R}$ from the centre of rotation, appears to an observer taking part in the rotation, to be affected by a fictitious force $\mathbf{C} = m(\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{R})$, the so called centrifugal force. If the particle is not stationary but moves ($\mathbf{V_r}$) relative to the rotating system it appears to be affected by an additional fictitious force $\mathbf{F} = -2m\mathbf{\Omega}\times\mathbf{V_r}$, the so called Coriolis force.

The cross product ($\times$) indicates that $\mathbf{F}$ is perpendicular not only to the rotational axis but also to the relative motion[1]. A moving body is therefore driven into a circular path, "inertia circle", with radius $R=V_r/2\Omega$ and a period of $\tau=\pi/\Omega$. In contrast to "normal" inertia, which resists changes in a body's motion, the Coriolis inertial force resists displacements by trying to return the motion to the origin. The clearest example in nature of the Coriolis effect is inertia oscillations in the oceans (fig.1).

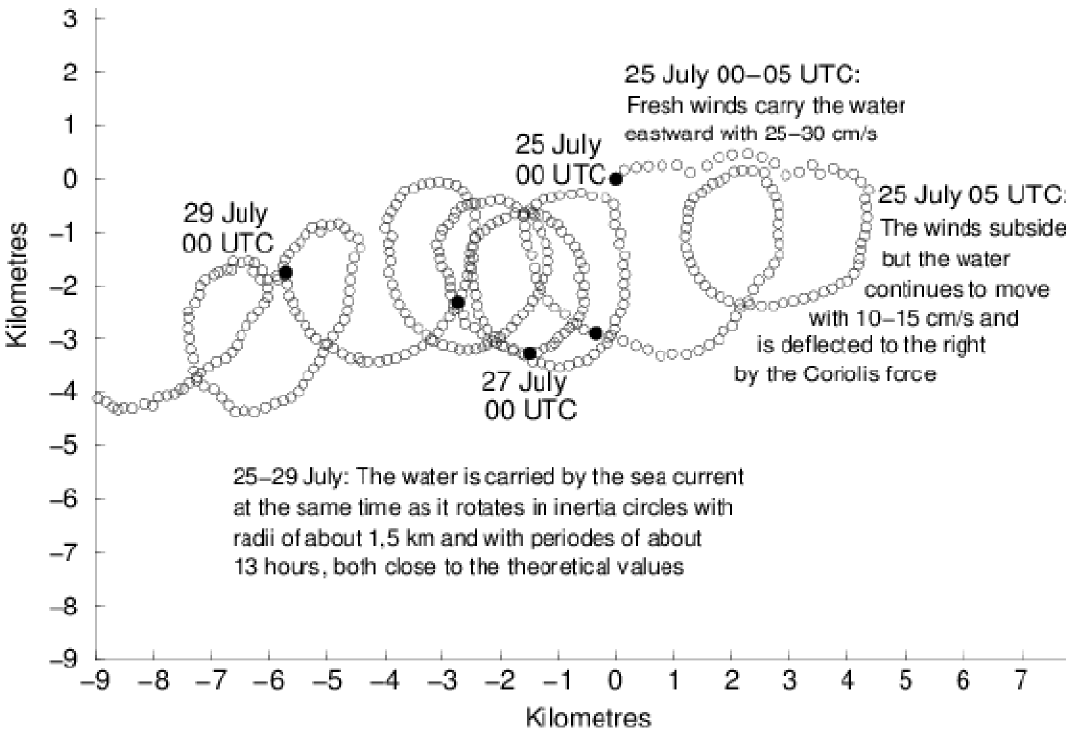


Fig.1: A drifting buoy set in motion by strong westerly winds in the Baltic Sea in July 1969. The uppermost water layers of the oceans tend to, when the wind has decreased, to move under inertia and follow approximately inertia circles. This is reflected in the motions of drifting buoys. In the case there are steady ocean currents the trajectories will become cycloides (Courtesy Barry Broman, SMHI) .

Any mathematical derivation or intuitive explanations of the Coriolis force, which is in conflict with the notion of the inertia circle motion, is therefore misleading, incomplete or wrong.

The cross product formulation also tells us that the Coriolis force takes its largest value when the motion is perpendicular to the rotational axis, and vanishes for all motions parallel to it. Only motions, or components of motions, perpendicular to $\mathbf{\Omega}$ are deflected (fig.2). Vertical motions at the poles are not deflected, but at the equator fully deflected. On the other hand horizontal motion at the poles are fully deflected, but on the equator only if they are in the east-west direction. They are then deflected vertically (see the Eötvös effect below).

[1] Also for this reason, and not only because the force is inertial, the Coriolis force does not do any work on the body, i.e. it does not change its speed (kinetic energy), only the direction of its motion. The statement that the Coriolis force "does not do any work" should not be misunderstood that it "doesn't do anything".

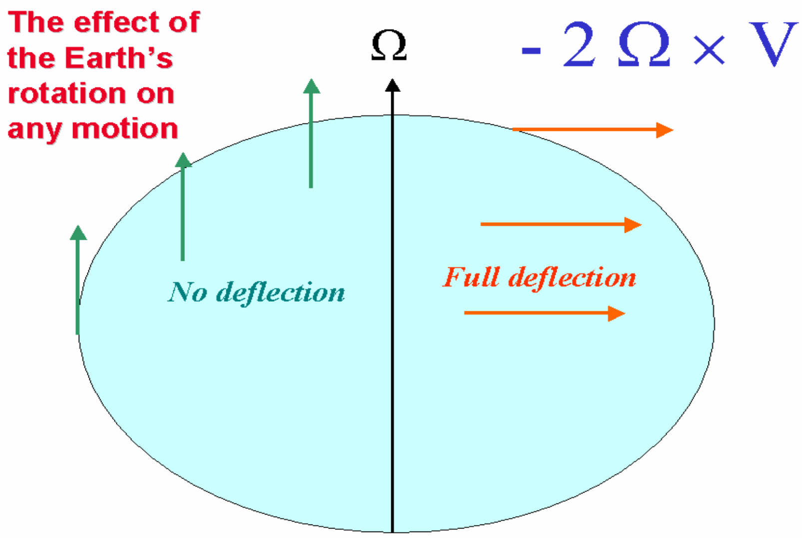


Fig.2: The cross-product formulation means that all motions which are perpendicular to the earth's axis are deflected, those parallel to it are not.

A motion parallel to a *latitude* (u) is always perpendicular to the rotational axis and is totally deflected ($2\Omega u$) since its direction is straight out from the axis, the component along the earth's surface at latitude φ is $2\Omega u\sin\varphi$. A motion along a *longitude* (v) has one component, $v\cos\varphi$, which is parallel to the axis of rotation and is not affected. The other component ($v\sin\varphi$) is completely deflected which yields $2\Omega v\sin\varphi$. This explains why the Coriolis force on a rotating planet varies with the sine of latitude φ, $F= -2m\Omega\sin\varphi V_r$, the "sine law". So for example, at latitude 43º (of central Italy) where $2\Omega\sin\varphi$ is approximately equal to $10^{-4}s^{-1}$, a motion of 10 m/s would move in an inertia circle of 100 km radius completing an orbit in almost 14 hours.

But the Coriolis effect is only one part of a three dimensional deflective mechanism[2]. We can summarise the three-dimensional Coriolis deflections for different motions in a an array where the mathematical terms have, for simplicity, been indicated only by their signs:

	Northward motion	Eastward motion	Downward motion
Northward deflection	0	**-1**	0
Eastward deflection	**1**	0	**1**
Downward deflection	0	**-1**	0

Table 1: The three-dimensional relation between the motion on a rotating planet and the Coriolis deflection. The number **0** means no deflection, **1** means deflection in the indicated direction and **–1** the opposite direction. (For example the **–1** in the upper row represents both eastward motions deflected southward, and westward motion deflected northward.) The deflections, which involve vertical motions are proportional to the *cosine* of the latitude, while those, which do not involve the vertical motions are proportional to *sine* of the latitudes.

[2] The three-dimensional Coriolis terms or, as Lord Kelvin called them, *gyroscopic terms*, play an important role in general laws concerning the stability of all rotating system, for example *it is the three-dimensional Coriolis effect, which provides a "gyroscopic resistance" to children's spinning tops.*

The three-dimensional deflection mechanism was discovered and discussed at separate historical epochs: the horizontal deflection of vertical motion in the 17th early 19th century, the vertical deflection of horizontal motion in the late 19th century and horizontal deflection of horizontal motion was discussed from the early 18th century until now.

3. The horizontal deflection of vertical motion: How Newton almost discovered the Coriolis effect

In the 1600's the possible deflection of falling objects was one of the prominent scientific problems. It was considered as a way, perhaps the ultimate way to prove or disprove the Copernican theory that Earth rotates and not the stars. The anti-Copernicans claimed that, if the earth was spinning around its axis, an object dropped from a tower would be "left behind", i.e. deflected to the west. Galileo argued that this was wrong since the object would take part in the earth's rotation. But, he added, since the rotational velocity at the top of the tower would be slightly larger than at the surface, the falling object would actually overtake the tower and land slightly to the *east* of it (fig.3)

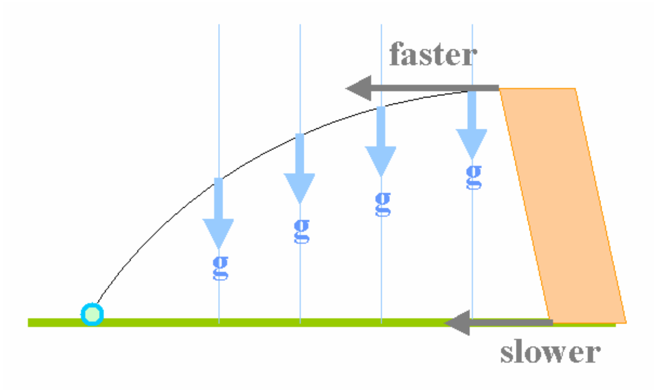


Fig.3: A tower at the Equator of height h rotating with the earth (with radius R and angular velocity Ω) has a velocity of ΩR at the base and Ω(R+h) at the top. An object falling from the top of the tower with an acceleration g will have a horizontal velocity excess of Ωh, which, over the time of the fall $\sqrt{\frac{2h}{g}}$ will carry the object a horizontal distance $S_1 = \frac{\Omega}{2}\sqrt{\frac{8h^3}{g}}$ Away from the Equator the deflection is proportional to the cosine of the latitude.

If we put Galileo's reasoning into mathematics, we will find that an object dropped from 100 m will, as seen from outside the earth, follow a parabolic path and be deflected 3 cm. Such small values were at that time difficult to confirm by measurements. Actually, the deflection according to Galileo's method is not quite correct and yields results, which are 50% too large.

We can understand this in two ways, one by treating the deflection as a consequence of the Coriolis effect. The velocity w=gt of a falling body can be split up into one component wsinφ parallel to the Earth's axis, another component, wcosφ, perpendicular to the Earth's axis. The first will not be deflected since it is parallel to the rotation axis, the second deflected to the right (east), by a Coriolis force -2Ωwcosφ (per unit mass). Integrating this over the time of the fall from a height h yields a deflection $S = \frac{\Omega\cos\varphi}{3}\sqrt{\frac{8h^3}{g}}$.

Another way is to start from Galileo's approach, but to take into account that during the fall gravity will not point in the same direction. Due to the shape of the earth it will change with a component pointing increasingly back towards the starting point (fig. 4).

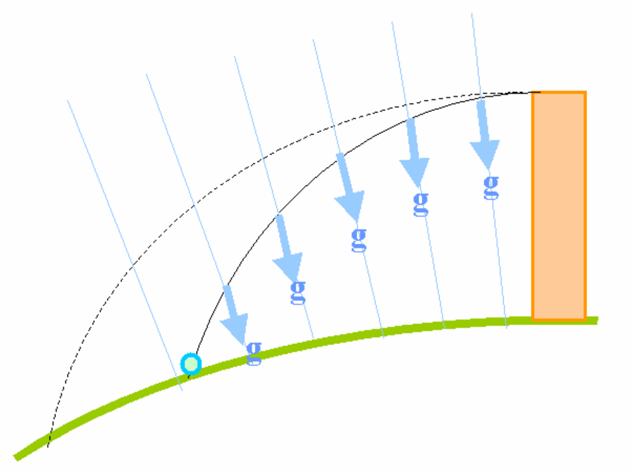


Fig.4: The trajectory of a falling object, seen from outside the earth. Due to the curvature of the earth the object will be affected by a component of gravity g pointing towards the centre of the earth. This slightly backward directed acceleration can be written $a = -g\sin\Omega t \approx g\Omega t$, which, integrated over t, the time of the fall, yields $-\frac{\Omega}{6}\sqrt{\frac{8h^3}{g}}$ which added to $\frac{\Omega}{2}\sqrt{\frac{8h^3}{g}}$ yields the correct deflection $\frac{\Omega}{3}\sqrt{\frac{8h^3}{g}}$.

The ”backward” acceleration reduces the 3 cm deflection by 1 cm to 2 cm, just as given by the Coriolis effect. But more interesting, retarding in its eastward motion, the object will, seen from outside the Earth, follow an elliptic path. The first to realize this was two famous British scientists, Robert Hooke and Isaac Newton.

In November 1679 Robert Hooke, in his capacity as newly elected Secretary of the Royal Society, tried to draw Isaac Newton into a discussion on the motions of the planets and comets. But Newton had just returned from a long vacation at his family home in Lincolnshire - where he incidentally might have watched apples fall in the garden. Perhaps inspired by these falling apples he had something else on his mind, "a fancy of my own", the horizontal deflection of objects dropped from a high altitude as proof of the Earth's rotation.

The exchange of letters that followed during the winter 1679-80 between Newton and Hooke shows that it was thanks to Hooke, they came to realise that that the fall of the body must be treated as an *elliptic* orbit with the centre of the Earth in one of its focii (fig. 5).

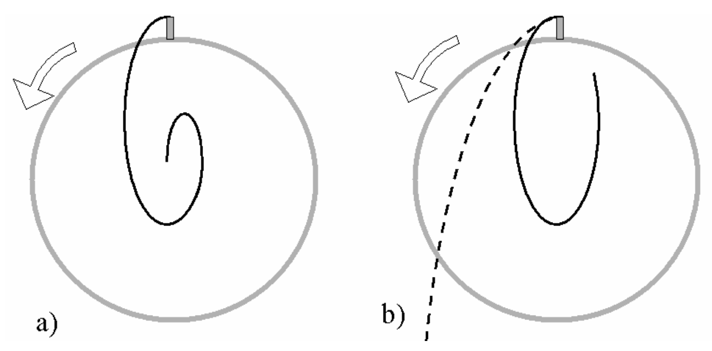


Fig.5: a) Newton’s first intuitive idea was that the trajectory of a falling object would spiral towards the centre of the earth, b) just considering conservation of absolute velocity would result in a parabolic path (dashed line), while the true trajectory would be an ellips (full line)

From the insight that a falling object in absolute space follows the same type of orbit as any of the planets or comets around the Sun, it was possible for Newton to infer that the motions of all terrestrial and extra-terrestrial bodies might be controlled by the same mechanism, *universal gravitation*. He never discovered the Coriolis effect, but looking for it found the laws of motion.

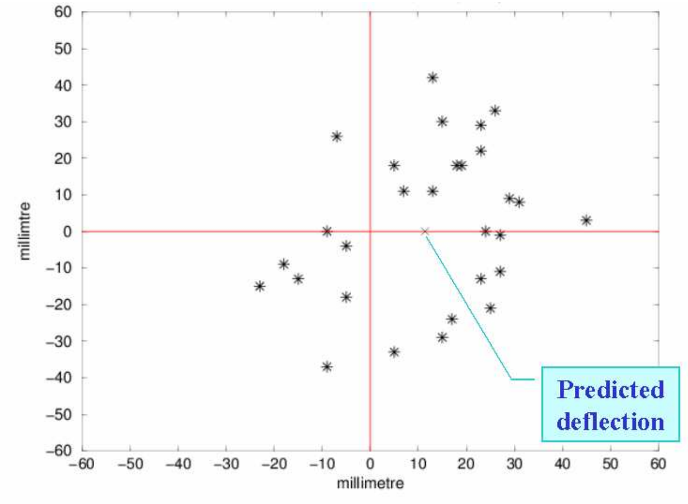


Fig.6: More than a century after Newton, in 1803, an experiment was conducted in Schlebusch, Germany. Twenty-nine iron pebbles were dropped into a 90 metre deep mineshaft. The average deflection was estimated to 8.5 mm compared to the theoretically expected value 8.8 mm.

4. The vertical deflection of horizontal motion: The Eötvös effect

In the early 1900:s a German team from the Institute of Geodesy in Potsdam carried out gravity measurements on moving ships in the Atlantic, Indian and Pacific Oceans. While studying their results the Hungarian nobleman and physicist Lorand Roland Eötvös (1848-1919) noticed that the readings were lower when the boat moved eastwards, higher when it moved westward (fig.7). He identified this as primarily a consequence of the rotation of the earth. In 1908 new measurements were made in the Black Sea on two ships, one moving eastward and one westward. The results substantiated Eötvös' claim. Since then geodesists use the correction formula

$$a_r = 2\Omega u \cos\varphi + \frac{u^2 + v^2}{R}$$

where u is the eastward velocity, v the northward and R the radius of the earth. The first term is the vertical Coriolis effect[3], the second term reflects the upward centrifugal effect of moving over any, even non-rotatig spherical surface.

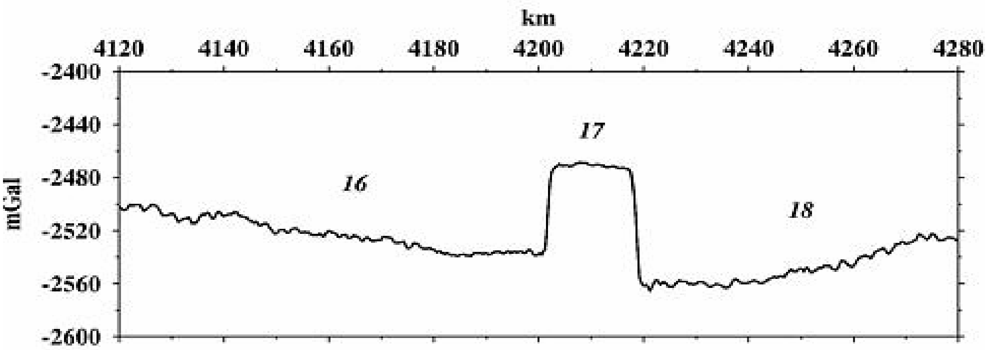


Fig. 7: Fig. 1.3.1: Example of the Eötvös effect measured by a French research vessel Samudra in the South Indian Ocean. The ship is first moving slowly in a westerly direction (16), then faster westward (17) and finally slowly eastward (18). The unit on the y-axis measures gravity per unit mass (mGal measures acceleration and is 1:1000th of 1 cm/s^2) and is proportional to the ship's weight (Courtesy Hélène Hébert, 1999).

[3] The reason why only east-west motion contributes to the first term is for the same reasons as laid out above: v has two components, one pointing parallel to Ω and not deflected, the other perpendicular to Ω and fully deflected. But since the deflection is parallel to the earth's surface it cannot change g.

To understand why the weight of a body on the earth is dependent of its motion we must understand why the earth is not a perfect sphere.

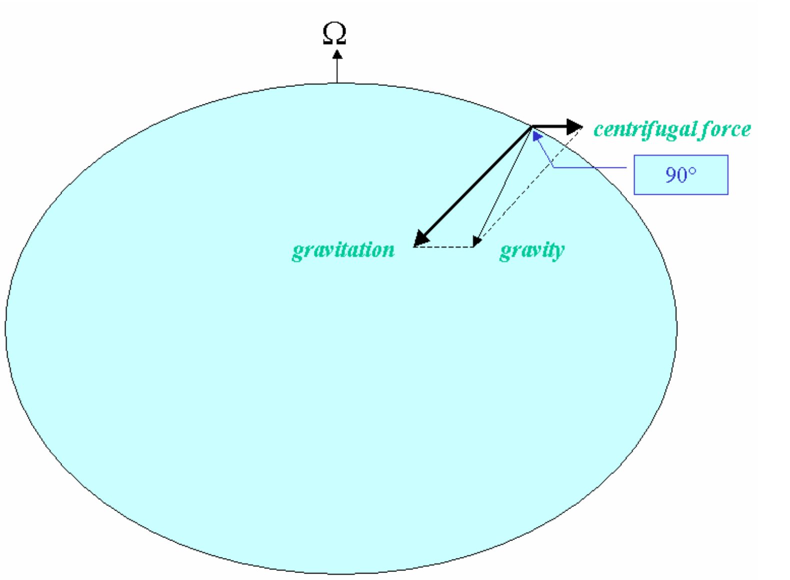


Fig.8: On a slightly flattened rotating planet the gravitational force is not directed perpendicularly to the earth's surface, but with an angle slightly pointing towards the poles. The component pointing towards the axis balances the outward directed centrifugal force, or in other words, the sum of gravitational attraction and the centrifugal force, gravity, is perpendicular to the earth's surface.

The earth is quite a fast-rotating planet. Its radius of about 6370 km and rotational speed at the equator of 465 m/s yields a centrifugal acceleration of 3.4 x 10 $m^{-2}s^{-2}$. As first recognised by Newton, during the course of the earth's early evolution, when it was a spinning deformable ball, the centrifugal force moved a substantial amount of mass from higher to lower latitudes to form a slightly flattened ball (or, to be more precise, an oblate ellipsoid) with a radius 21 km greater at the equator than at the poles. The combined effect of the gravitational force g* and the centrifugal force, we get the force of gravity or effective gravity, g, which determines how much a body weighs. Any stationary body on the earth's surface remains stationary because effective gravity points perpendicularly to the surface. However, this is only valid as long as the body is stationary. For a moving body the gravitational attraction g* remains the same but the centrifugal force changes in magnitude and/or direction. The gravity will change and with it the weight of the body – *the Eötvös effect*.

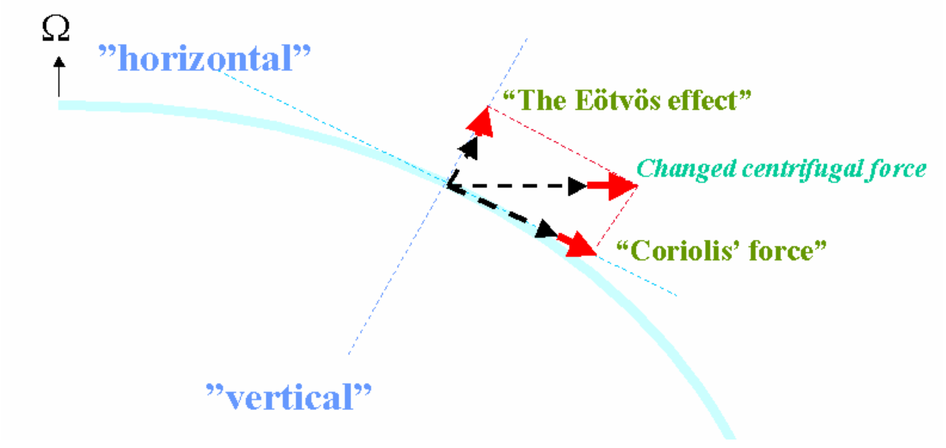
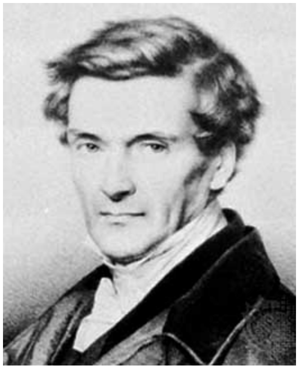


Fig. 9: Any motion of a body on the earth's surface affects the centrifugal force in direction and/or magnitude. The balance with the gravitational attraction is broken and the unbalance manifest itself as vertical and horizontal accelerations, the Eötvös and Coriolis effects respectively.

But not only the vertical component of effective gravity is changing. The other component parallel to the earth's surface also changes - and this is the *Coriolis force*.

5. The horizontal deflection of horizontal motion: the Coriolis effect

At the start of the Industrial Revolution a radical and patriotic movement developed in France to promote technical development by educating workers, craftsmen and engineers in 'mechanique rationelle'. Gaspard Gustave Coriolis (1792-1843), a well-respected teacher at l'Ecole Polytechnique in Paris, published in 1829 a textbook which presented mechanics in a way that could be used by industry. Here we find for the first time the correct expression for kinetic energy, $mv^2/2$. Two years later he established the relation between potential and kinetic energy in a rotating system.



Gaspard Gustave Coriolis was born on 21 May 1792 in Paris to a small aristocratic family that was impoverished by the French Revolution. The young Gaspard early showed remarkable mathematical talents. At sixteen he was admitted to the Ecole Polytechnique where he later became a teacher. He was regarded as a very good teacher and teaching inspired his work. The education of mechanics at the time was dominated by statics, which was suited only for problems related to constructional work, not for machines driven by water or wind. Coriolis was among the first to promote a reform in the education and 1829 he published a textbook in mechanics suited for the construction industry. Here for the first time kinetic energy is defined as $mV^2/2$. During the following years Coriolis became interested in rotating systems, first the relation between kinetic and potential energy in such a system, later (in 1835) the centrifugal effect on a body that is moving within a rotating system. In 1836 he was elected into the Academie de Science and in 1838 he became deputy director at the Polytechnique. In 1843 his health deteriorated and he died while revising his 1829 book.

In 1835 came the paper that would make his name famous: "Sur les equations du mouvement relatif des systemes de corps", where the 'deflective force' explicitly appears. The problem Coriolis set out to solve was related to the design of certain types of machines with separate parts, moving relative to the rotation. Coriolis showed that the total inertial force is the sum of two inertial forces, the common centrifugal force Ω^2R and the "compound centrifugal force" $2\Omega V_r$, was later became known as the "Coriolis force"[4].

Coriolis' way to explain the Coriolis effect can be qualitatively understood from the simple reformulation of the common centrifugal force into $C = \frac{mV^2}{R}$ where V is the absolute velocity and R the radius of curvature of the trajectory of the mass element m. A relative *tangential* motion V_r will increase or decrease the absolute velocity V and thereby the centrifugal force depending on if V_r is directed with or against the rotation. A relative *radial* motion, when Vr is directed inward or outward from the centre of rotation, will yield a trajectory which is an inward or outward directed (Archimedian) spiral. Since the centrifugal force is always perpendicular to the absolute motion, it will no longer be directed radially, inward or outward from the centre of rotation, but with some angle. Again, the difference between this centrifugal force and the common centrifugal force constitutes the Coriolis force.

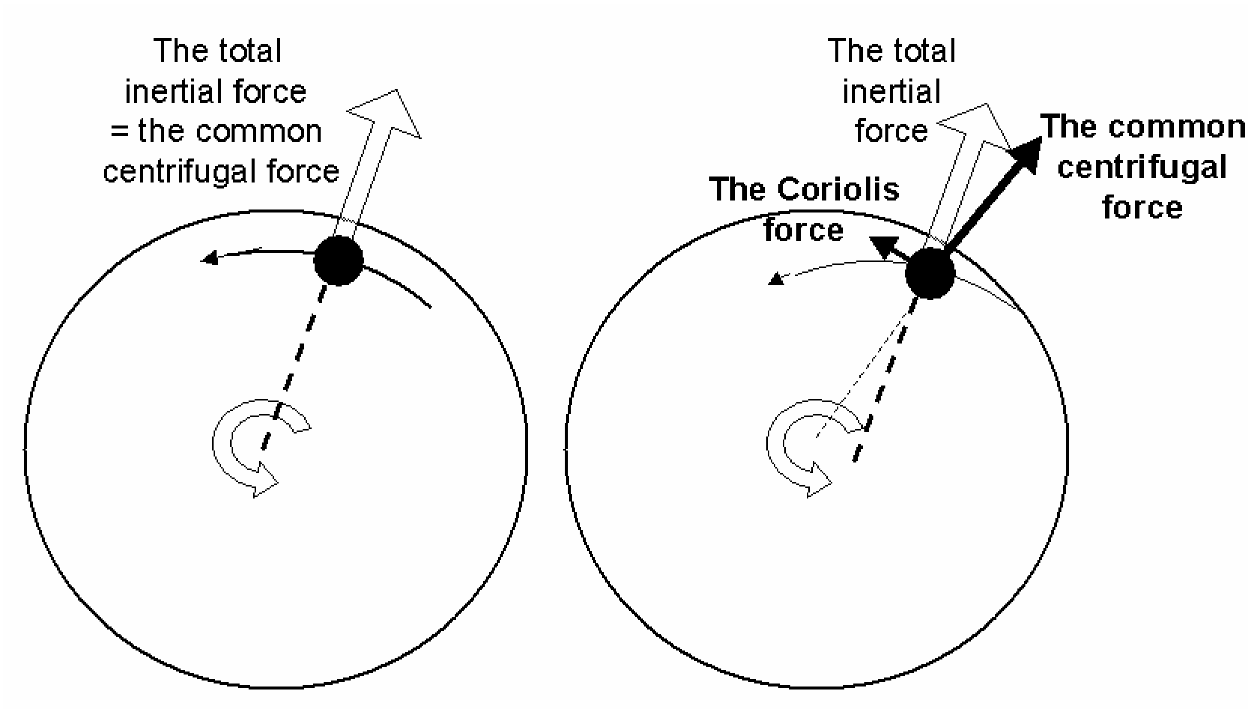


Fig. 10: a) An object fixed to a rotating platform follows a curved trajectory and is affected by a total inertial force, which we call the common centrifugal force. b) The body can move along the same trajectory, also due to a combination of the rotation and the motion relative the platform. The total inertial force is the same, but is now the sum of the common centrifugal force and the"Coriolis force".

Coriolis was not interested in "his" force as much as we are. He only valued it the part of the total inertial force, which is not explained by the common centrifugal force.

[4] To consider the Coriolis force as an extension to the centrifugal force is in agreement with the standard equation $ma_r = ma - 2m\Omega\times V_r - m\Omega\times(\Omega\times R)$ where the last two terms represent the total inertial force.

6. The Coriolis effect on a rotating planet

Coriolis reasoning can easily be applied to motions on a rotating planet. The main direction of the centrifugal force due to the earth's rotation is perpendicular to the earth's axis and will remain so although its magnitude might vary depending on the motion of an object on the earth's surface. The vertical component of this variation, affecting the weight of the object, is the Eötvös effect. But there is also a horizontal component, which is nothing else than the Coriolis effect.

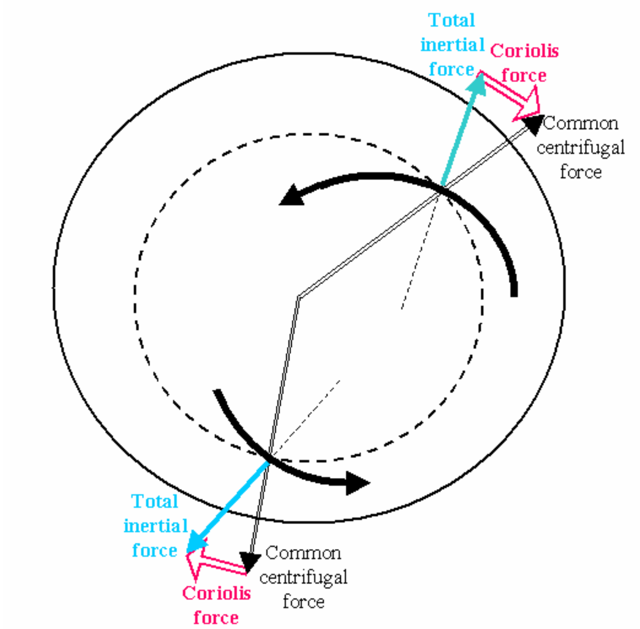


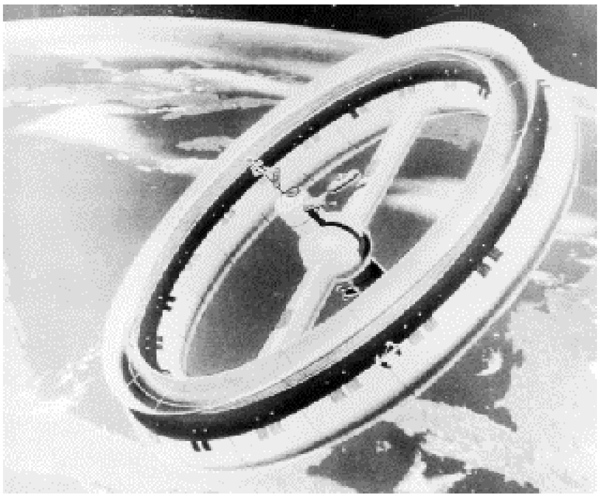
Fig.11: A further clarification of fig.10: inward and outward relative motions appear for an outside observer as absolute motions with trajectories spiralling inwards respectively outwards. The corresponding centrifugal force is different to the common centrifugal force (for a statuionary object) and this difference is the "force" Coriolis discovered in 1835.

Note that mathematically neither the Eötvös nor the Coriolis forces contain an expression for the earth's radius or eccentricity. On a rotating but perfect spherical earth there would still be a Coriolis force, but no Coriolis effect since it would be completely overshadowed by the general centrifugal effect of accelerating all movable objects towards the equator. The importance of the (non-spherical) shape of the earth lies in the cancelling (or balancing out) of the common centrifugal force by the radial component of gravitation. This leaves the "extra" centrifugal component to express the "Coriolis effect".

It has turned out, quite surprisingly, that by far the simplest way to qualitatively, intuitively, understand the Coriolis effect on the earth is to consider our planet as it actually is, a rotating oblate ellipsoid where the dominating forces are gravitation and the common centrifugal force.

It has not been possible to find out to what degree Coriolis' 1835 paper influenced the technology of the period. However, a direct application of his work emerged in the 1960's when American and Russian engineers planned for future space stations. They should be made as gigantic wheels, slowly rotating to provide a centrifugal force that would serve as an artificial gravity and give a comfortable life on board. An example of this can be seen in the 1969 Stanley Kubric science fiction film"2001 - A Space Odyssey". However, at about the time of the release of the film the engineers soon realised that their space stations would not be very comfortable places because of the Coriolis effect.

A space wheel with a radius of 100 meter would need a rotation period of 20 seconds to generate a centrifugal force of the same magnitude terrestrial gravity. But in this rotating environment the Coriolis forces would be almost 4000 times stronger than on the earth! Machinery with moving or rotating parts, like centrifuges and washing machines might break down. The crew would suffer from physiologically or psychologically uncomfortable Coriolis effects when they made any movement. Modern space technology is therefore trying to solve the problem of artificial gravity along other lines.



7. The flat and parabolic turntable

A common way to illustrate the Coriolis deflection is to throw or roll a ball over a turntable rotating anticlockwise. From outside the ball is seen to move in a straight line; seen from the anticlockwise rotating turntable, the ball appears to be deflected to the right (fig. 11).

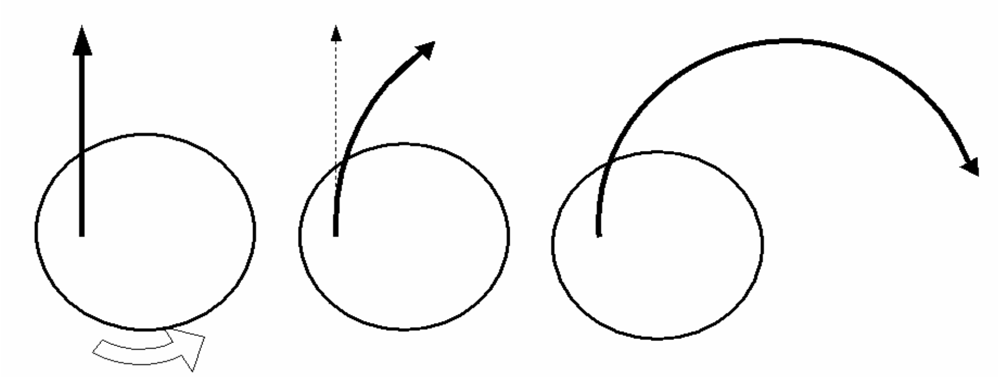


Fig. 11: A ball rolling straight seen from outside (a) will for an observer in the rotating system appear to be deflected (b), but will not follow an inertia circle, but an ever-widening expanding spiral.

However, most of what we see here is not the Coriolis force, but the centrifugal force. If only the Coriolis force had been present the ball would return after having performed a circular orbit (“inertia circle”) instead of disappearing out of sight in an ever-widening spiral. This explains why one does not simply experience the Coriolis force by walking across a merry-go-round.

However, we can cancel out the centrifugal force by deforming the turntable into a slightly concave parabola. A parabolid is a surface normal to the resultant of the centrifugal and gravity forces. It means that a small marble remains at rest at any point on the surface. For a certain rotation a stationary object at a distance from the centre of rotation the components of the centrifugal force, and gravity parallel to the surface, balance each other (fig. 12).

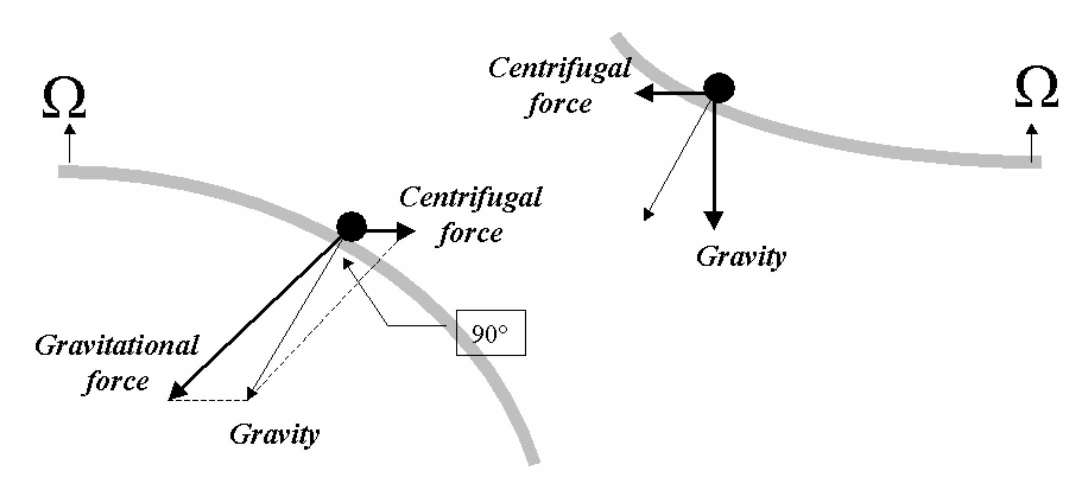


Fig. 12: For a stationary object on a rotating planet (a) or a rotating parabolic surface (b) the horizontal components of the centrifugal force is balanced by the horizontal component of the gravitational force of the planet and, on the turntable, the horizontal component of the weight of the body. In both cases the component of gravitation and gravity, perpendicular to the rotational axis, equals the centrifugal force.

If the object is set in relative motion by some impulsive force this balance will be disturbed and the object perform inertia oscillations, with twice the angular velocity of the pan (fig. 13).

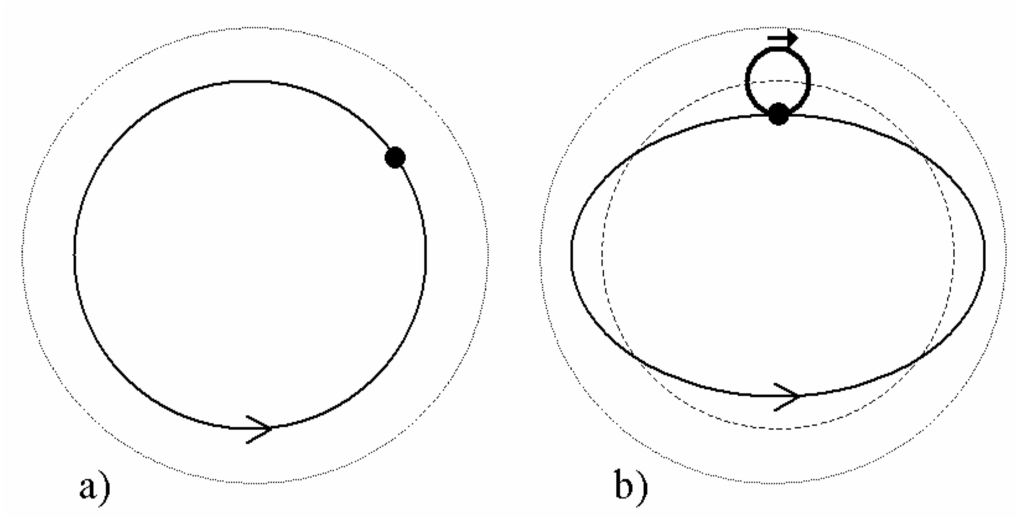


Fig.13: Absolute and relative motion of a ball in a parabolic shaped turntable rotating anti-clockwise. a) A ball, stationary in the rotating system, appears from outside to be moving in an anticlockwise circle (full line); b) the ball has been given an impetus and is from outside it appears to be moving anticlockwise in an elliptic orbit, in the rotating system moving in a clockwise inertia circle. The vertical movement of the ball is neglected since it introduces a slow anticlockwise precession of the ellipse.

But there is still a paradoxical discovery to be made. We have all the time assumed that there is no friction between the moving object and the rotating surface of the planet. But, in a frictionless environment, how does the moving object "feel" the rotation of the underlying surface? How does the object "know" that it is moving within a rotating system?

On a flat merry-go-round, making one revolution in 2 seconds and 3 m from the centre of rotation, a body moving with a speed of 1 m/s, will experience a centrifugal acceleration that is about five times stronger than the Coriolis acceleration. Only at a distance of 0.6 m from the centre of rotation are the two accelerations of equal strength. To cancel the centrifugal effect the merry-go-round must have a slightly concave, parabolic surface.

Such a parabolic dish was in use by professor Norman A. Phillips in his meteorological lecturers at the Massachusetts Institute of Technology in the 1950's. A largish flat pan (similar to a pie pan) was filled with a slightly liquid cement mixture and run overnight and was set up to rotate on a turntable driven by a small constant speed motor (with a period of about 2 seconds). After a night's rotation the cement hardened into a parabolic surface. The surface was then smoothed, with water-soluble valve grinding compound and a small slightly rounded disc.

Anybody who actually begins using the concave turntable, will realise a paradox. If there is no frictional coupling between the puck and the underlying parabolic dish there is no exchange of momentum, it would not matter whether or not the parabolic dish was rotating or not. We can perform two identical experiments: one with the turntable and camera rotating together with the ball at rest on the dish or the turntable is at rest with *the ball moving* with a tangential velocity at a constant distance from the centre of rotation. The camera will still rotate together with the ball. From the point of view of the camera the results will be identical.

The "role" of the turntable is therefore not to provide a rotational impetus on the ball, but to define a parabolic shape that will act as a constraint. The same applies to the earth. If it suddenly stopped rotating, the atmosphere would for some short time continue due to inertia. An observer on a previously geostationary, but now orbiting, satellite would see the Coriolis effect still in action and weather systems develop. The reason is that the non-rotating earth would not attain a spheroidal shape.

8. Coriolis effects in outer space – the stable Lagrange's points

The fact that a body affected by the Coriolis effect does not have to be in direct contact with a rotating foundation/base leads us to one of the most spectacular manifestations of the Coriolis effect in outer space, the so-called stable Lagrange points derived mathematically in 1772 by the French mathematician J. L. Lagrange (1736-1813). A Lagrange point is a position in space where the gravitational fields of two celestial bodies, **M** and **m**, of substantial but differing mass, combine to form a point at which a third body of negligible mass would be stationary relative to the two bodies (fig. 14a).

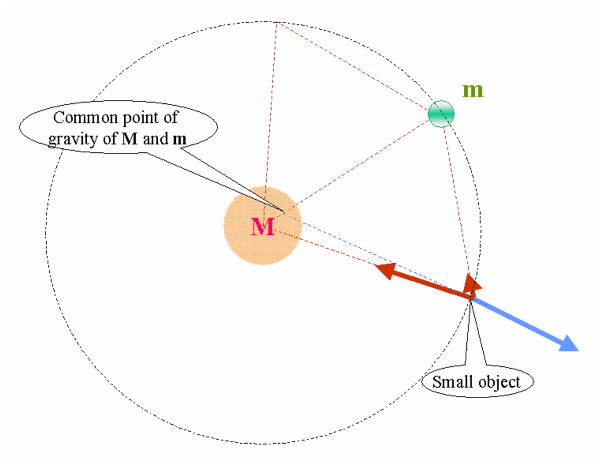


Fig.14a: The stable Lagrange points L4 and L5 are defined through a balance of two gravitational attractions and the centrifugal force, positioned at the angles of two equidistant triangles (where the angles are 60º).

As for any orbiting planet around a star, the earth and the sun create five such “Lagrange points”. Two of them, L1 and L2 close to the earth, L3 in the earth orbit but on the opposite side of the sun, and L4 and L5 in the earth’s orbit but forming an equidistant triangle with the earth and the sun.

Lagrange did not envisage that his discovery could have any practical importance, but in our times L1 and L2 have become favoured positions for space stations. An unseen planet X at Lagrange point L3 was a popular theme in early science fiction stories from where the UFOs and other unexplained phenomenas were supposed to emanate. However, the L1, L2 and L3 points are unstable on a time scale of approximately 23 days. This requires satellites to nudge them back into place from time to time and any planet X would soon drift away from its hidden position behind the sun.

The Lagrange points L4 and L5, however, provide stable balance as long as the mass ration between the two objects is larger than 25 (actually 24.96). In the authoritative literature on celestial mechanics, (see for example p. 66-67 and 74-97 in Murray, C. D. and S.F. Demott, 1999: Solar System Dynamics, Cambridge University Press, 592 pp.), we are told that the mechanism is nothing else than *the Coriolis force.* The mechanism is similar to the one on earth: a balance between gravitation and the centrifugal force, in this case *two* gravitational forces.

However, this Coriolis effect differs from the one we are used to on earth since it is driving moving objects not into inertia circles, but *inertia ellipses.* The Coriolis effect will draw any celestial body back if it tries to “escape”. This is indeed what has happened with a large number of asteroids, which are trapped in two of Jupiter’s Lagrange points (Fig. 14b).

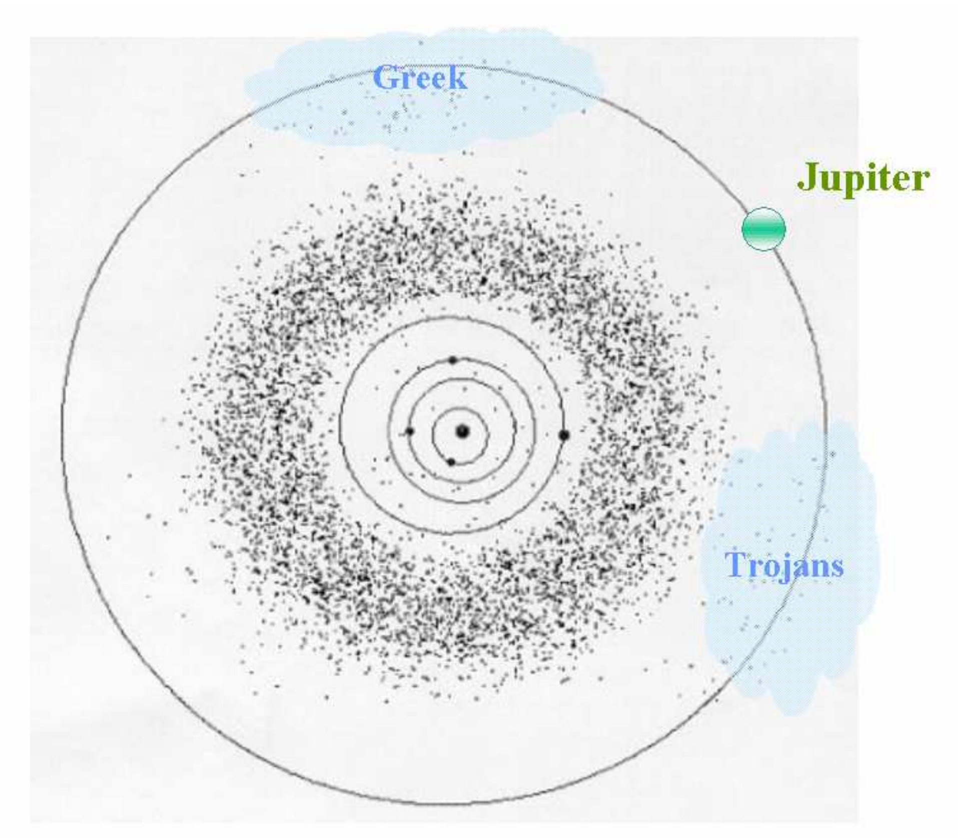


Fig.14b: A schematic view of the inner planets, the asteroid belt and Jupiter. The clusters of asteroids ahead and behind Jupiter (called the “Greek” and the “Trojans” respectively) are trapped around two stable Lagrange points.They form together with the Sun and Jupiter equidistant triangles. It is the Coriolis force that keeps these asteroids in their quasi-stationary positions, very much in the same way as the Coriolis force on earth strives to keep any moving object within a small area.

9. Taylor columns

Geoffrey Ingram Taylor (1886-1975) was one of the great physicists of the twentieth century, among the last masters of both theory and experiment. He started his career as a meteorologist and in 1913 took part in a six-month expedition to the waters off Newfoundland to report on icebergs in the wake of the Titanic disaster. Taylor had a passion for finding agreement between theoretical results and observations, to see how to make the experimental conditions correspond with those assumed in theory. Much of the teaching in hydrodynamics at that time concerned problems which were mathematically solvable, but not necessarily related to reality. One of those hydrodynamical theorems stated:

> In a steady, rapidly rotating flow with angular velocity $\mathbf{\Omega}$, the dominant forces are the pressure gradient $\mathbf{\nabla p}$, and centrifugal forces, and the equation of motion reduce to $\mathbf{2\rho\Omega\times V=-\nabla p}$ where ρ is the fluid density, $\mathbf{V}$ the velocity and p the pressure. Taking the curl yields $\mathbf{\Omega\times\nabla V=0}$ which means that there are no velocity variations along the direction of the axis of rotation, which also means that it is difficult to stretch rapidly rotating bodies.

In other words, for a fluid in uniform rotation, the motion within the fluid does not vary vertically (parallel to the rotation axis). This seemed to Taylor to defy common sense. To test this for himself he designed a series of experiments with a rotating glass tank filled with water.

Taylor set the tank into rotation. The centrifugal force would move water from the inner towards the outer parts of the tank creating a balance between the outward-pointing centrifugal force and the inward-pointing pressure gradient force. The surface of the water would then have a slightly parabolic concave shape and the whole tank-water system rotate with the same angular velocity.

Taylor inserted a drop of ink into the rotating water without stirring it. But, instead of dispersing and colouring the water, it remained for a long time in a vertical column, later known as a “Taylor column”, moving around in the tank as a rigid body (fig.15b). What happened was that when the ink started to spread out in horizontal directions, every ink particle was immediately affected by the Coriolis force acting at right angles to the motion. It forced the particles into curved motions, in circles of surprisingly small radii. If Taylor’s tank rotated with one revolution in two seconds ($\Omega=\pi=$ 3.14 rad s^{-1}), and the ink spread out at 0.5 cm s^{-1}, this would yield inertia circles with a diameter of less than 1 mm.

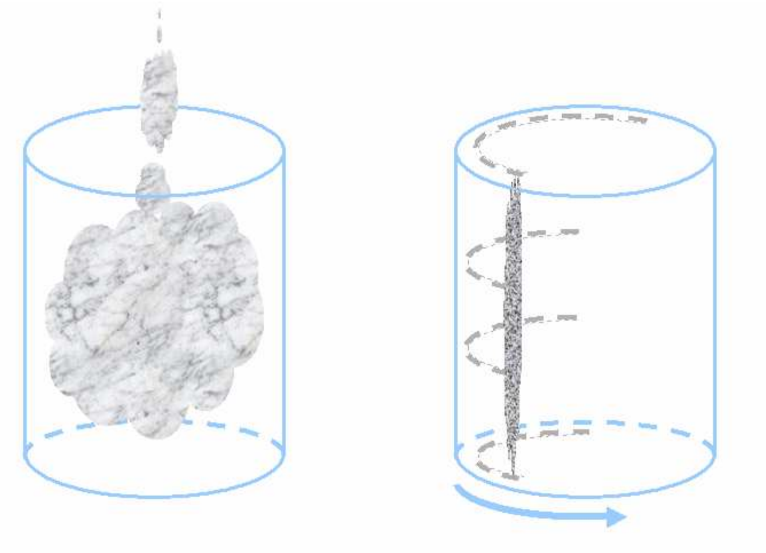


Fig.15: Ink poured down in a non-rotating tank filled with water will disperse in a normal way (a). The same when the tank is rotating leads to the formation of vertical columns of ink, “Taylor columns”(b)

So contrary to all physical intuition, by rotating a fluid we make it change its physical properties, make it "stiff". Taylor's experiment remind us of the fundamental fact that the Coriolis force is not just deflecting' moving bodies, but opposes their displacement by trying to restore them to their initial position (fig.16).

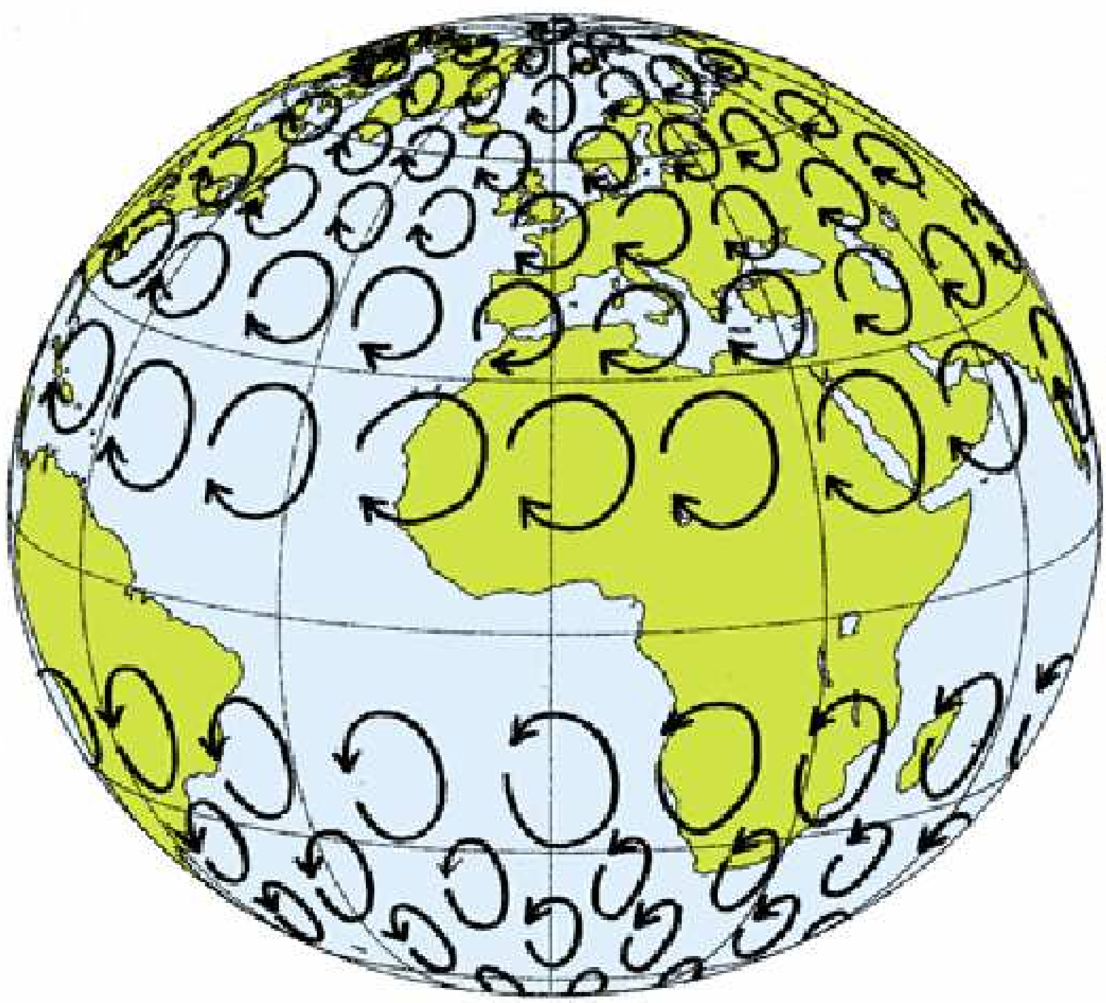


Fig.16: The Coriolis force tends to restore a body to its initial position which hinders the geographical displacement of air masses, "The woollen cap effect".

The vorticies and jet streams are the consequences of two opposing forces, one (the pressure gradient force) trying to equalise large-scale density contrasts, the other (the Coriolis force) trying to restore them.

10. The Coriolis *acceleration* – a simple shortcut?

In 1879, the German meteorologist Adolph Sprung suggested a different mathematical way to deal with the Coriolis effect: *abandon the notion of relative motion and derive the acceleration in a fixed system*. In other words: find the acceleration to *prevent* a relative motion from being deflected! This acceleration, $+2\Omega\times V_r$, achieved by some *real force* is, by some strange convention, called the *Coriolis acceleration*. The derivation is simple and Newton could have done it by the same Euclidean method he used to find the centripetal acceleration in "Principia" (fig.17).

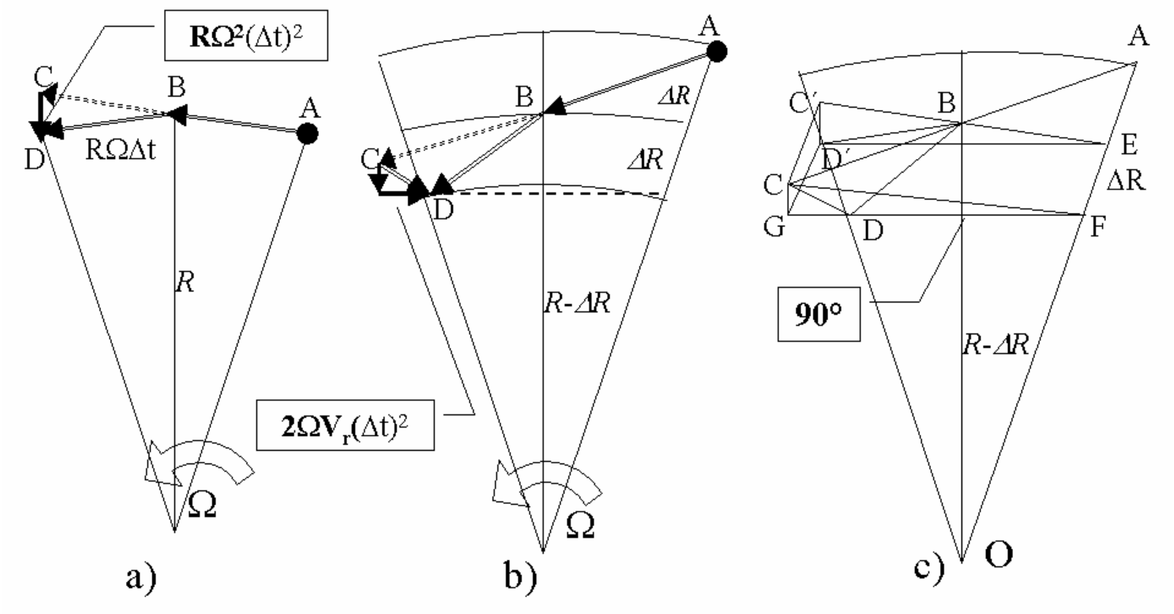
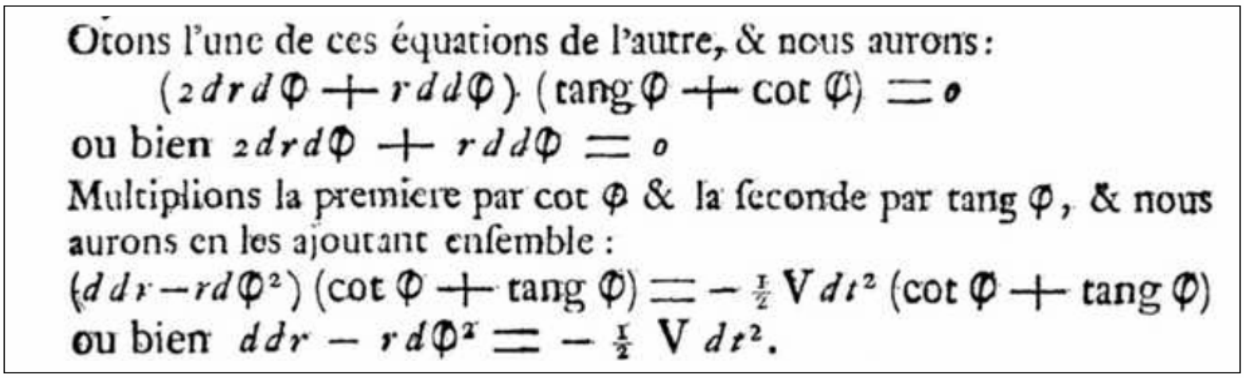


Fig. 17: a) Newton's derivation of the centripetal acceleration: a body is over time Δt by the rotation Ω carried from A to B and would, by pure inertia, in the next time interval have continued to C, had it not been affected by a centripetal impetus which brought it to D. By simple geometry one gets $CD=R\Omega^2(\Delta t)^2$. In case b) the body is also moving radially with relative velocity $V=\Delta R/\Delta t$ and would have continued from B to C, had it not been affected by a centripetal impetus which brought it to D. Since ACF is proportional to ABE and AF=2AE, it follows that $CF=2BE\approx 2\Omega R\Delta t$ and since $DF \approx 2\Omega(R-\Delta R)\Delta t$ it follows that $GD=GF-DF\approx CF-DF=2\Omega\Delta R\Delta t=2\Omega V(\Delta t)^2$. It can easily be shown that D´EFG is a parallelogram with GF perpendicular to OB, so GD ⊥ BO and the Coriolis acceleration is perpendicular to the relative motion – but to the left!

It is not commonly known that Leonard Euler already in 1749 derived analytically what was essentially the Coriolis acceleration (fig. 18).


Otons l'une de ces équations de l'autre, & nous aurons:

$$(2drd\Phi + rdd\Phi)\,(\text{tang}\,\Phi + \cot\Phi) = o$$

ou bien $2drd\Phi + rdd\Phi = o$

Multiplions la premiere par cot Φ & la ſeconde par tang Φ, & nous aurons en les ajoutant enſemble :

$$(ddr - rd\Phi^2)\,(\cot\Phi + \text{tang}\,\Phi) = -\tfrac{1}{2}V dt^2\,(\cot\Phi + \text{tang}\,\Phi)$$

ou bien $ddr - rd\Phi^2 = -\tfrac{1}{2}\,V\,dt^2$.

Fig. 18: Leonard Euler's 1749 derivation of the Coriolis acceleration (2drdφ) and the so-called Euler cceleration (rddφ), which is the acceleration due to variations in the angular velocity. .

11. Popular "common sense" explanations of the Coriolis effect

The common explanation Foucault pendulum is not the only case where "common sense" collides with mathematical or scientific truth. In 1735 George Hadley (1686-1768) suggested from a "common sense" view that, since the surface of the earth at the equator moved faster than the surface at higher latitudes, air that moved towards the equator would gradually lag behind and be observed as a NE wind north of the equator, SE wind south of the equator (fig. 19a).

Hadley's model was a great step forward for its time because it introduced the rotation of the earth for the first time. But his "common sense" explanation is incorrect for three reasons. Bodies moving under frictionless conditions on the surface of a rotating planet will not conserve their absolute velocity. Even if they did, Hadley's scenario will mathematically only explain half the Coriolis force. Finally, Hadley's explanation suggests that the deflection only occurs for meridional motion. In 1843 the American Charles Tracy thought he could explain also the deflection of east-west motion by erroneously invoking great circle motion, an explanation that is still found in many popular books in meteorology (fig.19b).

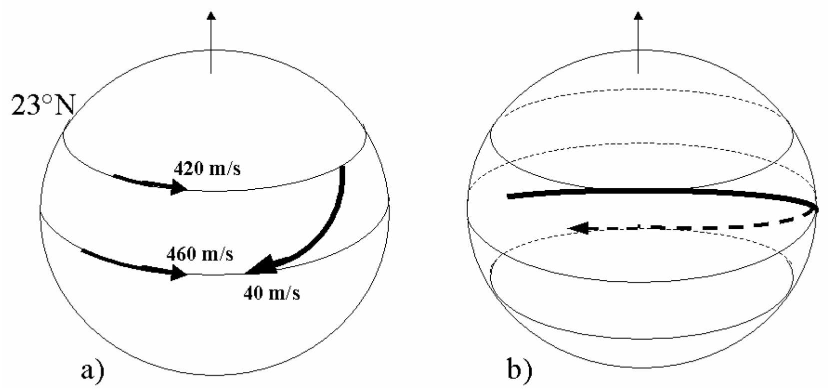


Fig. 19: Two erroneous images of the deflection mechanism: a) conservation of absolute velocity and b) motion along great circles. The latter appears to work for eastward motion, but not for westward.

Coriolis' 1835 paper did not do away with erroneous intuitive explanations. The paper was highly mathematical and not easily accessible. In 1847 the French mathematician Joseph L. F. Bertrand (1822-1900) suggested to the French Academy a "simplified" derivation. He combined two "common sense", but erroneous, assumptions: a) the deflective acceleration is due to conservation of absolute velocity and b) the deflective acceleration on a rotating turntable is constant and only due to the Coriolis effect. The first assumption underestimates the Coriolis effect and the second overestimates it - *so the errors cancel out* (fig.20).

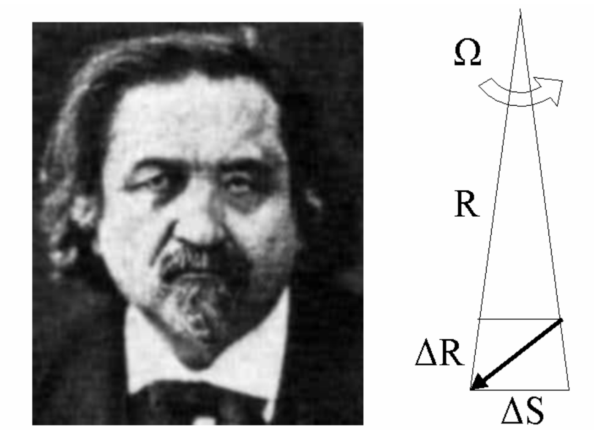


Fig. 20: Joseph Bertrand and his "simplified" derivation. An object on a turntable at a distance R from the centre of rotation is moving radially outwards with a constant speed $V_r = \Delta R/\Delta t$. Due to the rotation Ω the object is subject to a deflective acceleration a, which is assumed constant. The deflected distance ΔS during Δt can be expressed both as $\Delta S = a(\Delta t)^2/2$ and $\Delta S = \Omega\ \Delta R \Delta t$ which yields $a = 2\Omega V_r$.

Bertrand's derivation became popular and entered meteorology in the 1880's. If we today are grappling to understand the Coriolis effect, one source of confusion is this "simple" but deceptive derivation, which appears to justify two frequent misconceptions.

12. Not only the Coriolis effect…

It is often said that what makes dynamic meteorology difficult is its mathematics, which contains non-linear differential equations. But the non-linearity makes *predictions* difficult because of the "Butterfly Effect". The mathematics of the Coriolis effect, a cross-product of two vectors, is not particularly difficult and is *linear*. Euler's equation has been used in celestial mechanics for 250 years without causing any confusions and endless debates. But his equation relates to an *absolute* motion, whereas the Coriolis force relates to *relative* motion, which seems to be difficult to comprehend intuitively.

But frictionless motion is even more out of reach of everyday experience. Correspondents to "American Journal of Physics" have noted that university students nourish naïve, Aristotelian, ideas about how and why things move. The crux of the matter might not lie in the mathematics but in our common senses which are still Aristotelian. For example, according to American university teachers of physics many students believe that forces keep bodies in motion and, conversely, that in the absence of forces bodies are at rest. There are no reasons to assume that students in meteorology are immune to this "Aristotelian physics" as it has been called.

But the confusion in dynamic meteorology does not only depend on misinterpretations of the Coriolis effect. There are some other central principles and conceptual models which needs to be critically re-examined. If we multiply Euler's equation $2\frac{dr}{dt}\frac{d\varphi}{dt}+r\frac{d^2\varphi}{dt^2}=0$ with r and integrate, we will get $r^2\frac{d^2\varphi}{dt^2}=c$ which is today called "angular momentum conservation". This is one of the fundamental laws of both classical and modern physics. It regulates the motion of rotating objects from galaxies to elementary particles. One is the faithful workers in the meteorological garden is the ice skater who regulates her rotation by her arms using the principle of angular momentum conservation.

Fundamental physical laws are of course powerful tools in the scientific workshop. But as with other powerful tools its use is not always trivial and if not properly handled can cause a lot of harm. But this is the start of a new article…

Anders Persson