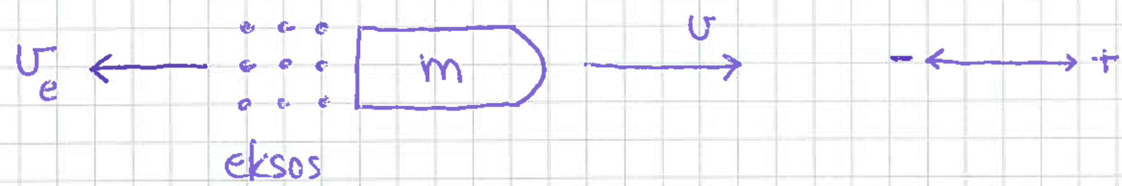
Rakettprinsipp [YF 8.6 ; LL 5.4]



v = rakettens hastighet (målt i fast referansesystem)

v_e = eksosens —"— (—"—)

u = —"— (målt relativt raketten) (= konst.)

$\Rightarrow v_e = u + v \quad ; \quad v > 0, \quad u < 0$

$\dfrac{dm}{dt}$ = rakettens masseendring pr tidsenhet < 0

Antar $F_{ytre} = 0$ (til slutt: $F_{ytre} = -mg$)

$\Rightarrow dp/dt = 0$

$\boxed{m(t)} \longrightarrow v(t) \qquad p(t) = m(t)v(t)$

$\longleftarrow \; \vdots \; \boxed{m(t+dt)} \longrightarrow v(t+dt)$

$v_e(t+dt) \quad dm_e = -dm \quad (>0)$

$$p(t+dt) = m(t+dt)\cdot v(t+dt) + dm_e \cdot v_e(t+dt)$$

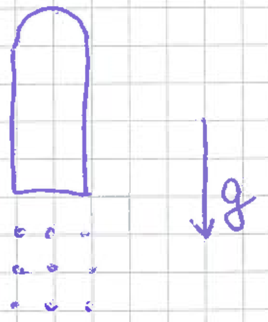
$$= [m(t) + dm]\cdot[v(t)+dv] - dm\cdot[u + v(t) + dv]$$

$$= \underbrace{m(t)v(t)}_{p(t)} + \underbrace{m(t)\,dv - u\,dm}_{=0}$$

$\Rightarrow m \cdot \dfrac{dv}{dt} = u \cdot \dfrac{dm}{dt} = u\dot{m}$, som er på "N2-form",

$m \cdot a = F_{skyv}$, med $F_{skyv} = u\dot{m} > 0$

I tyngdefeltet kommer tyngdekraften i tillegg:



$$F_{ytre} = -mg$$

$\Rightarrow$ N2 for "rest"-raketten blir

$$F = ma$$

med total kraft

$$F = F_{ytre} + F_{skyv} = -mg + u\dot{m}$$

Øving 4: $m\, dv/dt = -mg + u\, dm/dt$

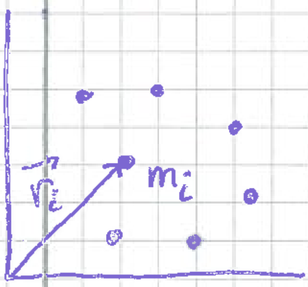
$$\Rightarrow \int dv = -\int g\, dt + u \int dm/m \qquad \text{osv.}$$

Så langt: Punktmasser, evt. ren translasjon av stive legemer.

Nå: Partikkelsystemer. Stive legemer inkl. rotasjon.

Massesenter [YF 8.5 + oppg 8.115/116 ; LL 5.6, 5.8, 6.1]

= tyngdepunkt når g er konstant i hele systemet



Massesenter (CM) for N punktmasser $m_1\ m_2 \cdots m_N$ i posisjon $\vec{r}_1\ \vec{r}_2 \cdots \vec{r}_N$:

$$\vec{R}_{CM} = \frac{1}{M}\sum_i m_i \vec{r}_i \ ; \quad \text{Total masse } M = \sum_i m_i$$

Med kontinuerlig massefordeling : $m_i \to \Delta m_i \xrightarrow{\Delta m_i \to 0} dm$

$$\Rightarrow \vec{R}_{CM} = \frac{1}{M}\int \vec{r}\, dm \ ; \quad M = \int dm$$

Masseelementet dm :

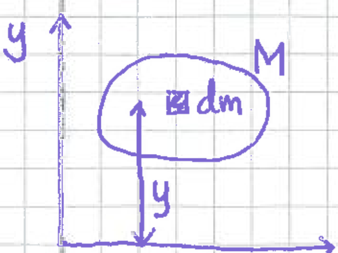
$$dm = \rho\, dV \ (3D) \ ; \quad dm = \sigma\, dA \ (2D) \ ; \quad dm = \lambda\, d\ell \ (1D)$$

ρ, σ, λ = masse pr hhv volum-, flate-, lengdeenhet

$dV, dA, d\ell$ = hhv volum-, flate-, lengdeelement

Med uniform massefordeling er $\frac{dm}{M} = \frac{dV}{V}$ etc

Eks 1: Pot. energi i tyngdefeltet (anta konstant g)



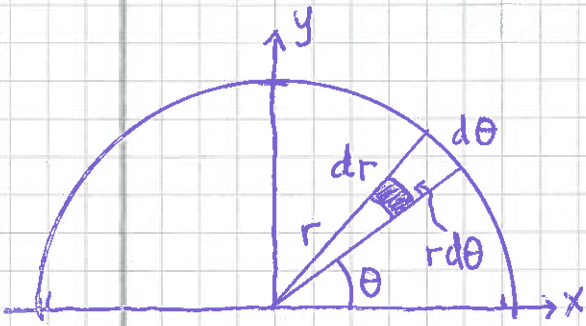
Velger $U(0) = 0$

$$\Rightarrow U = \int dU = \int g y\, dm = g\int y\, dm$$

$$= \underline{\underline{g M Y_{CM}}}$$

Dvs: Legemet har total U som om hele massen $M = \int dm$ var samlet i høyden $Y_{CM} = \frac{1}{M}\int y\, dm$ (og da f.eks. i $\vec{R}_{CM}$)

Eks 2: Halvsirkulær tynn skive (radius R)



$X_{CM} = 0$ pga symmetri $\Rightarrow \vec{R}_{CM} = Y_{CM}\,\hat{y}$

med $Y_{CM} = \frac{1}{M}\int y\,dm = \frac{1}{A}\int y\,dA$

(siden $dm/M = dA/A$)

Vet at $A = \pi R^2/2$; ser at $dA = dr \cdot r\,d\theta$ og $y = r\sin\theta$.

Ser også at hele skiva regnes med når $0<r<R$ og $0<\theta<\pi$ brukes som integrasjonsgrenser.

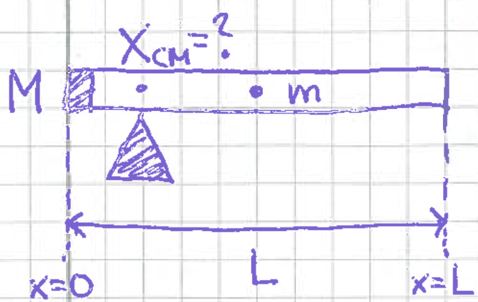
$$\Rightarrow Y_{CM} = \frac{2}{\pi R^2}\int_{r=0}^{R}\int_{\theta=0}^{\pi} r\sin\theta \cdot dr \cdot r\,d\theta$$

$$= \frac{2}{\pi R^2}\int_0^R r^2\,dr \int_0^\pi \sin\theta\,d\theta = \frac{2}{\pi R^2}\cdot\frac{1}{3}R^3\cdot\underbrace{\Big|_0^\pi(-\cos\theta)}_{=2}$$

$$= \frac{4}{3\pi}R \approx \underline{\underline{0.42\,R}} \qquad \text{(Rimelig svar!)}$$

[Vis selv: $Y_{CM} = \frac{2R}{\pi}$ for "bøyle" (1D); $Y_{CM} = \frac{3R}{8}$ for halvkule (3D)]

Eks 3: Rør med lodd i enden



$m = 165\,g$, $M = 305\,g$

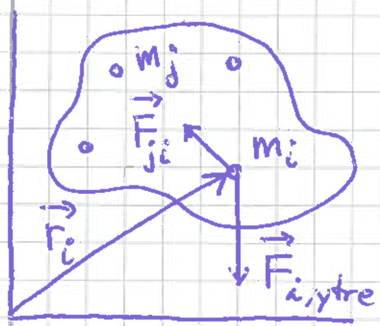
$$X_{CM} = \frac{M\cdot 0 + m\cdot L/2}{M+m} = \frac{165}{940}L \approx \underline{\underline{0.18L}}$$

$$\left[\text{Evt: } X_{CM} = \frac{1}{M+m}\left\{M\cdot 0 + \underbrace{\int_0^L x\cdot \overbrace{m\frac{dx}{L}}^{=dm}}_{=\frac{1}{2}mL}\right\} = \frac{mL}{2(M+m)}\right]$$

Tyngdepunktbevegelsen [YF 8.5 ; LL 5.8]

Forsøk (kast) viser at CM beveger seg som om hele M var samlet i CM ! [Se s. 4]

Bevis:



$$\vec{R}_{CM} = \frac{1}{M}\sum_i m_i \vec{r}_i \quad ; \quad M = \sum_i m_i$$

N2 for m_i:

$$m_i \ddot{\vec{r}}_i = \vec{F}_{i,ytre} + \underbrace{\sum_{j\neq i} \vec{F}_{ji}}_{\text{total indre kraft på } m_i}$$

total ytre kraft på m_i

Adderer N2 for alle $i = 1, 2, \ldots, N$:

$$\sum_i m_i \ddot{\vec{r}}_i = \frac{d^2}{dt^2}\left\{\sum_i m_i \vec{r}_i\right\} = \frac{d^2}{dt^2}\left\{M\vec{R}_{CM}\right\} = M\ddot{\vec{R}}_{CM}$$

$$\sum_i \vec{F}_{i,ytre} = \vec{F}_{ytre} = \text{total ytre kraft på systemet}$$

$$\sum_i \sum_{j\neq i} \vec{F}_{ji} = \vec{F}_{21} + \vec{F}_{12} + \ldots + \vec{F}_{N,N-1} + \vec{F}_{N-1,N} = 0 \quad (\text{pga N3})$$

$$\Rightarrow \quad \boxed{\vec{F}_{ytre} = M\ddot{\vec{R}}_{CM}}$$

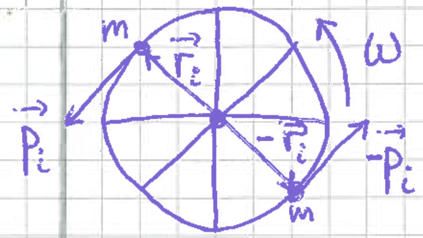
som er N2 for punktmasse M i posisjon $\vec{R}_{CM}$, utsatt for total ytre kraft $\vec{F}_{ytre}$!

I tillegg: Rotasjon om CM. } Neste ca 3 uker.
Vibrasjon om CM. }

Rotasjon [YF 9, 10 ; LL 6 (5)]

Noen innledende observasjoner:

- Ren rotasjon



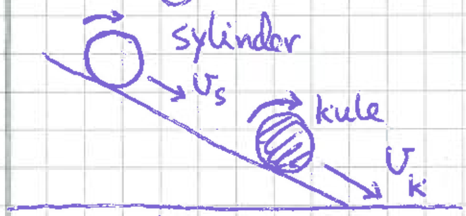
CM i ro $\Rightarrow K_{trans} = \frac{1}{2} M \dot{\vec{R}}_{CM}^2 = 0$

Men $\underline{K_{rot} \neq 0}$

Total impuls $\vec{p} = \sum_i \vec{p}_i = 0$

Men $\underline{\text{dreieimpuls} \neq 0}$

- Rulling



Hvorfor oppstår rotasjon?

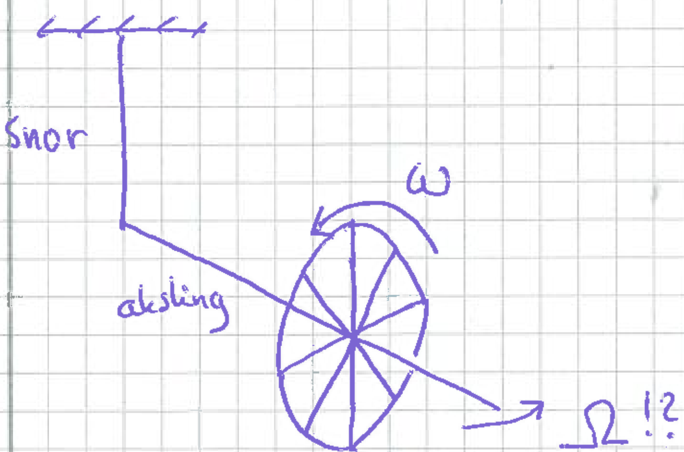
Hvor angriper kreftene?

Må ha et <u>dreiemoment</u>.

Hvorfor $v_k > v_s$?

Hva med friksjon?

- Kompleks dynamikk



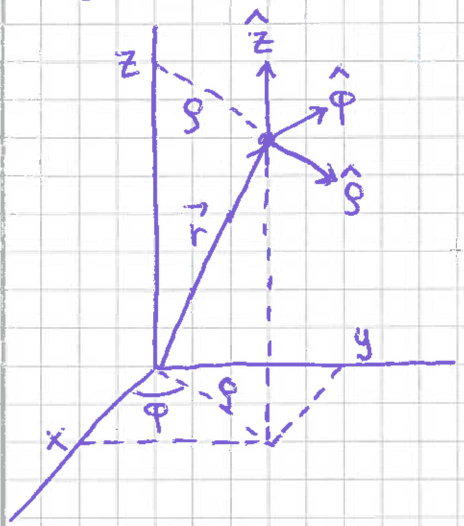
Presesjon!

(Gyroskop)

Sirkelbevegelse [YF 9.1–9.3; LL 1.8]

Anta rotasjon om z-aksen.

Sylinderkoordinater: Polarkoord. (ρ, φ) og z



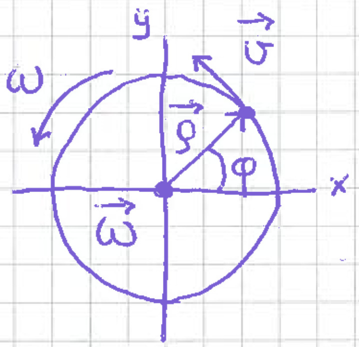
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \rho\hat{\rho} + z\hat{z}$$

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z$$

$$\rho = \sqrt{x^2+y^2}, \quad \tan\varphi = y/x$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$$

Vinkelhastigheten $\vec{\omega}$ peker langs rotasjonsaksen:



$$\vec{\omega} = \omega\hat{z} \quad (= \omega\hat{\omega})$$

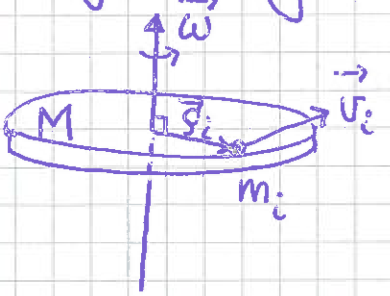
$$\vec{\rho} = \rho\hat{\rho}$$

$$\vec{v} = d\vec{s}/dt = \rho\frac{d\varphi}{dt}\hat{\varphi} = \rho\omega\hat{\varphi}$$

Ser at $\hat{\varphi} = \hat{z} \times \hat{\rho}$ (høyrehåndsregel)

$$\Rightarrow \quad \boxed{\vec{v} = \vec{\omega} \times \vec{\rho}}$$

Rotasjonsenergi. Treghetsmoment [YF 9.4; LL 6.4, 6.3]



$$K = K_{rot} = \sum_i \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2}\Big(\sum_i m_i \rho_i^2\Big)\omega^2$$

$I \overset{def}{=} \sum_i m_i \rho_i^2$ = legemets <u>treghetsmoment</u> mhp valgt akse.

Kontinuerlig massefordeling: $m_i \to \Delta m_i \xrightarrow{\Delta m_i \to 0} dm$

$$\Rightarrow \boxed{I = \int \rho^2\, dm} \qquad \rho = \text{avstand fra aksen til } dm$$

Dermed: $\boxed{K_{rot} = \tfrac{1}{2} I \omega^2}$

Kin. energi for stivt legeme [YF 10.3 ; LL 6.6]

$K = \sum_i \frac{1}{2} m_i v_i^2 = \ldots\ldots$ se notat for bevis/utledning (ikke pensum)

$$\Rightarrow \boxed{K = K_{trans} + K_{rot} = \tfrac{1}{2} M V^2 + \tfrac{1}{2} I_0 \omega^2}$$

der

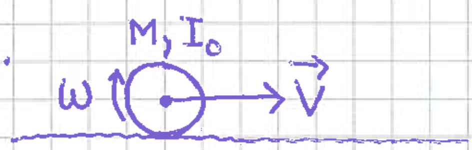
M = legemets masse

$V = \dot{\vec{R}}_{CM}$ = hastigheten til CM

I_0 = treghetsmoment mhp rotasjonsaksen gjennom CM

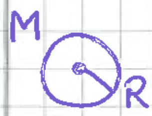
$\vec{\omega}$ = vinkelhastigheten om $\parallel$

Eks: Rullende hjul/ring.



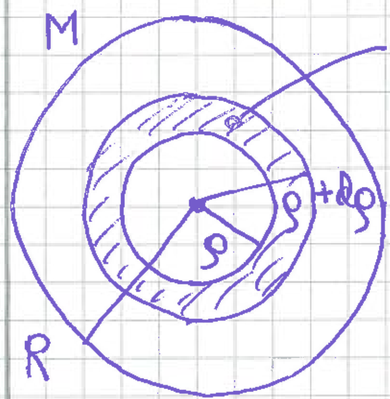
Treghetsmoment, eksempler [YF 9.6 ; LL 6.3]

Eks 1: Ring / Hul sylinder



$$I_0 = \int \rho^2 \, dm = R^2 \int dm = \underline{\underline{MR^2}}$$

Eks 2: Skive / Kompakt sylinder

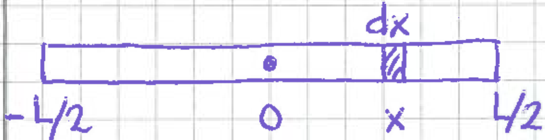


$$dI_0 = \rho^2 \, dm = \rho^2 M \frac{dA}{A}$$

$$= \rho^2 M \frac{2\pi\rho \, d\rho}{\pi R^2} = \frac{2M}{R^2} \rho^3 \, d\rho$$

$$\Rightarrow I_0 = \int dI_0 = \frac{2M}{R^2} \underbrace{\int_0^R \rho^3 \, d\rho}_{= \frac{1}{4}R^4} = \underline{\underline{\tfrac{1}{2} MR^2}}$$

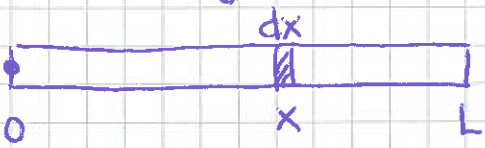
Eks 3: Tynn stang, akse $\perp$ stang, gjennom CM



$$\rho = x, \quad dm = M \, dx / L$$

$$\Rightarrow I_0 = \int_{-L/2}^{L/2} x^2 M \, dx/L = \frac{M}{L} \Big|_{-L/2}^{L/2} x^3/3 = \underline{\underline{\tfrac{1}{12} ML^2}}$$

Eks 4: Tynn stang, akse gjennom stangas ende



$$I = \int_0^L x^2 M \, dx / L = \underline{\underline{\tfrac{1}{3} ML^2}}$$

Eks 5: Kuleskall $I_0 = \frac{2}{3} MR^2$

Eks 6: Kule (kompakt) $I_0 = \frac{2}{5} MR^2$

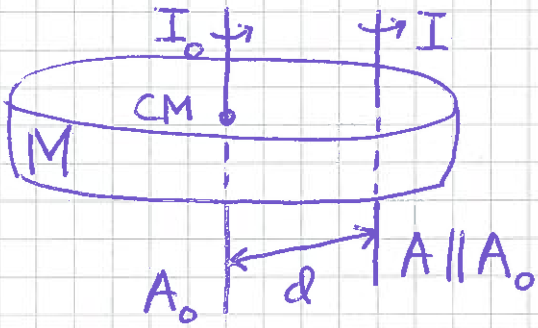
} akse gjennom CM

[Eks 2; 3, 5, 6 oppgis til eksamen]

Steiners sats

(parallellakseteoremet)

[YF 9.5 ; LL 6.3]



$$\boxed{I = I_0 + Md^2}$$

(se notat for utledning)

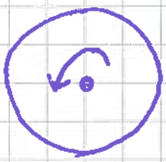
Eks 1: Tynn stang

$I_0 = \frac{1}{12}ML^2$

$\xrightarrow{d=\frac{L}{2}}$

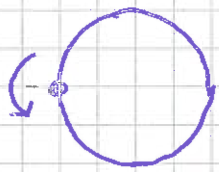
$I = I_0 + M(L/2)^2 = \frac{1}{3}ML^2$

Eks 2: Kompakt kule



$I_0 = \frac{2}{5}MR^2$

$\xrightarrow{d=R}$



$I = I_0 + MR^2 = \frac{7}{5}MR^2$