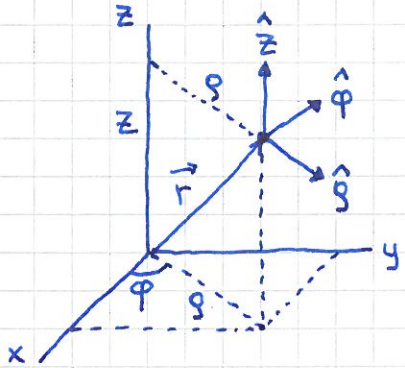
Sirkelbevegelse [YF 9.1–9.3; TM 9.1; LL 1.8; HS 2.1.2]

(Delvis repetisjon, se s. 6–8.)

Anta rotasjon om z-aksen $\Rightarrow$ velger sylinderkoord. (ρ, φ, z)

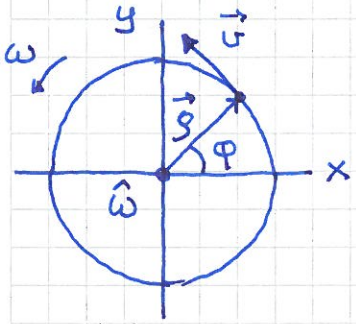


$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \rho\hat{\rho} + z\hat{z}$$

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z$$

$$\rho = \sqrt{x^2+y^2}, \quad \varphi = \arctan(y/x)$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$$



$$\vec{v} = \frac{d\vec{s}}{dt} = \frac{\rho\, d\varphi}{dt}\hat{\varphi} = \rho\omega\hat{\varphi}$$

$$\vec{\omega} = \omega\hat{z} = \omega\hat{\omega}, \quad \vec{\rho} = \rho\hat{\rho}$$

$$\vec{v} = \vec{\omega} \times \vec{\rho}$$

$T = 2\pi/\omega$ = periode, $f = 1/T$ = frekvens

$\alpha = \dot{\omega} = \ddot{\varphi}$ = vinkelakselerasjon

$v = \omega\rho$ = banehastighet

$a_\parallel = \dot{v} = \dot{\omega}\rho = \alpha\rho$ = baneaks. $\quad (\vec{a}_\parallel = \alpha\rho\hat{\varphi})$

$a_\perp = v^2/\rho = \omega v = \omega^2\rho$ = sentripetalaks. $\quad (\vec{a}_\perp = -\omega^2\rho\hat{\rho})$

[Total aks.: $\vec{a} = \dot{\vec{v}} = \dot{\vec{\omega}} \times \vec{\rho} + \vec{\omega} \times \dot{\vec{\rho}}$

$\qquad = \dot{\omega}\rho\hat{\varphi} - \omega v\hat{\rho} = a_\parallel\hat{\varphi} - a_\perp\hat{\rho}$]

Med stivt legeme:

- felles ω og α for hele legemet
- v og a øker med ρ (ρ = avstand fra rot.aksen)

Treghetsmoment [YF 9.4; TM 9.3; LL 6.2, 6.3; HS 4.2]

For ren rotasjon om fast akse:

$$K = K_{rot} = \sum_i \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2}\left(\sum_i m_i \rho_i^2\right)\omega^2 = \tfrac{1}{2} I \omega^2$$

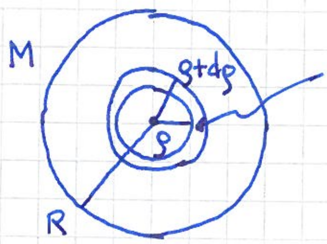
der $I = \sum_i m_i \rho_i^2$ (evt. $I = \int \rho^2 \, dm$) er legemets treghetsmoment mhp rot. aksen.

Notasjon: $I = I_0$ hvis akse gjennom legemets CM.

Eks 1: Ring (og "sylinderskall")

$$I_0 = \int_{ring} \rho^2 \, dm = R^2 \int dm = \underline{\underline{MR^2}}$$

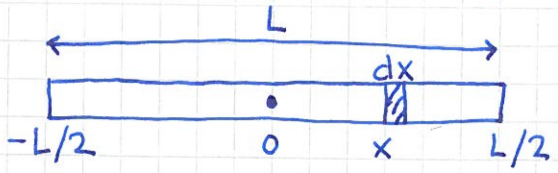
Eks 2: Sirkulær skive (og kompakt sylinder)



$$dm = M \cdot \frac{dA}{A} = M \cdot \frac{2\pi\rho \cdot d\rho}{\pi R^2} = \frac{2M}{R^2}\rho \, d\rho$$

$$I_0 = \int_0^R \rho^2 \frac{2M}{R^2} \rho \, d\rho = \frac{2M}{R^2} \Big|_0^R \tfrac{1}{4}\rho^4 = \underline{\underline{\tfrac{1}{2} MR^2}}$$

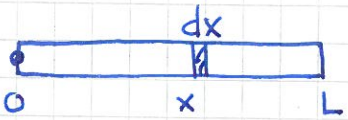
Eks 3: Tynn stang



$$dm = M \cdot \frac{dx}{L}, \quad \rho = x$$

$$\Rightarrow I_0 = \int_{-L/2}^{L/2} x^2 \cdot M \cdot \frac{dx}{L} = \frac{M}{L} \Big|_{-L/2}^{L/2} \tfrac{1}{3} x^3 = \underline{\underline{\tfrac{1}{12} ML^2}}$$

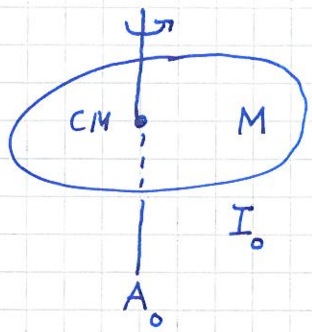
Eks 4: Om stangas ende

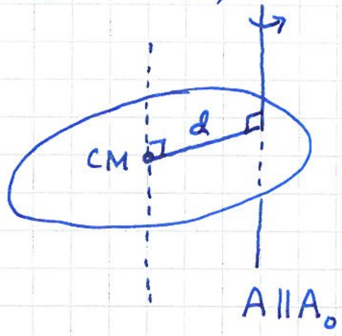


$$I = \int_0^L x^2 \cdot M \cdot \frac{dx}{L} = \underline{\underline{\tfrac{1}{3} ML^2}}$$

Øving 6: Kuleskall: $I_0 = \frac{2}{3} MR^2$ Kompakt kule: $I_0 = \frac{2}{5} MR^2$

Steiners sats [YF 9.5; TM 9.3; LL 6.3; HS 4.3]

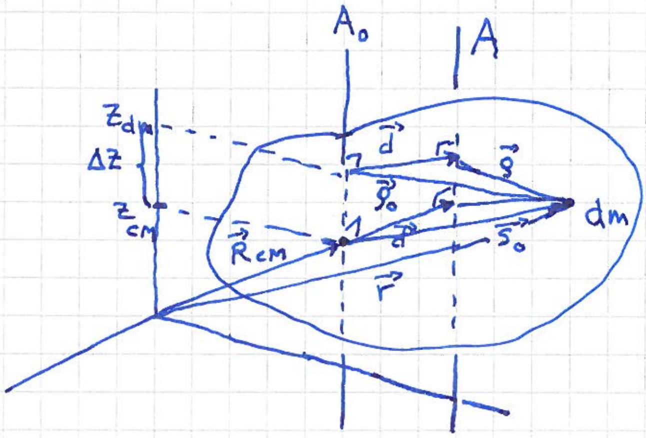




$$\boxed{I = I_0 + Md^2}$$

Bevis:

[Bedre figur på s. 45B]



$$\vec{\varrho}_0 = \vec{d} + \vec{\varrho}$$

$$I_0 = \int \varrho_0^2 \, dm$$

$$\varrho^2 = \varrho_0^2 + d^2 - 2\vec{d} \cdot \vec{\varrho}_0$$

$$\Rightarrow I = \int \varrho^2 \, dm = \underbrace{\int \varrho_0^2 \, dm}_{= I_0} + d^2 \underbrace{\int dm}_{= M} - 2\vec{d} \cdot \int \vec{\varrho}_0 \, dm$$

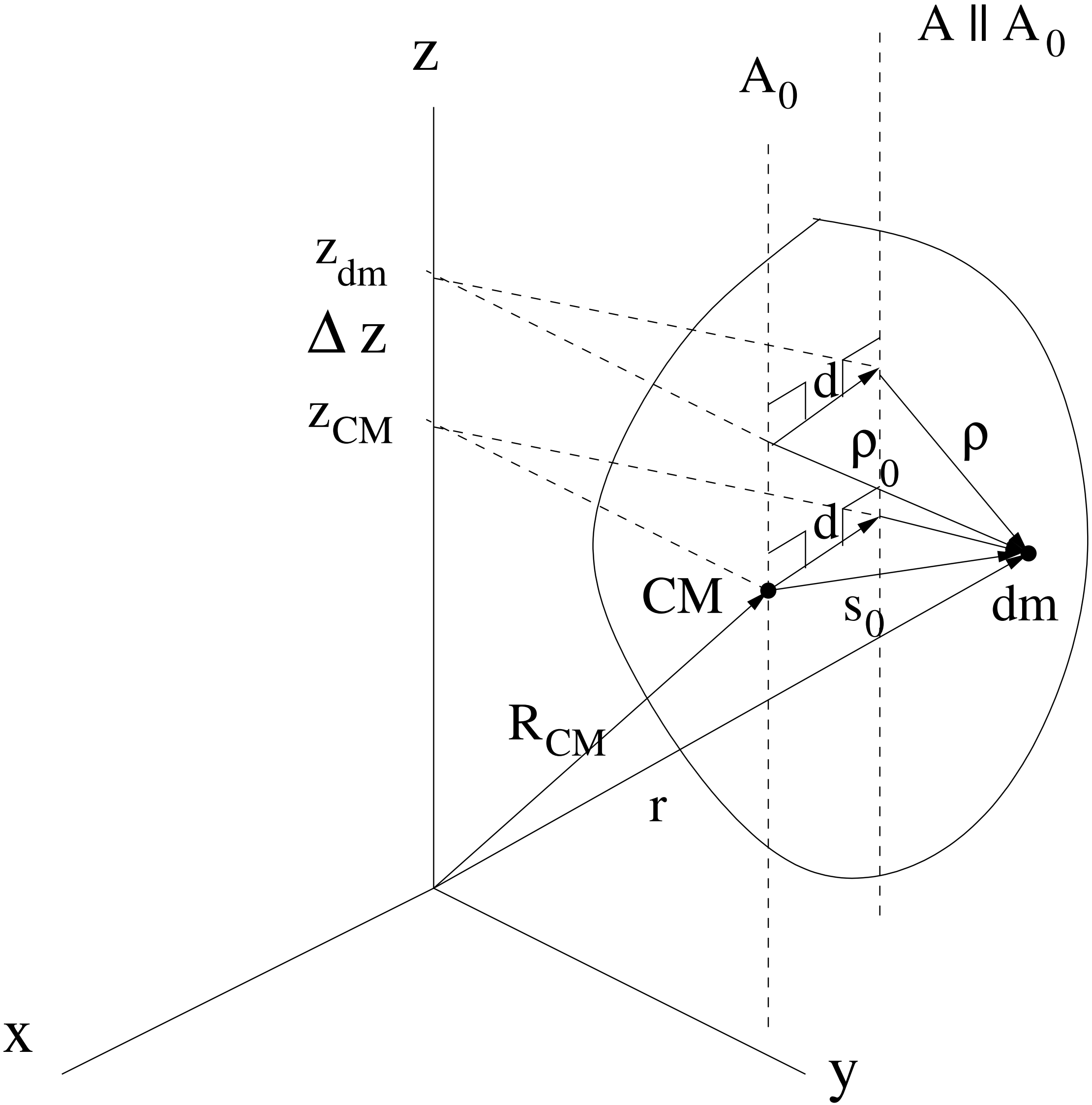
$$\vec{r} = \vec{R}_{CM} + \vec{s}_0 = \vec{R}_{CM} + \Delta z \cdot \hat{z} + \vec{\varrho}_0$$

$$\Rightarrow \vec{d} \cdot \vec{\varrho}_0 = \vec{d} \cdot (\vec{r} - \vec{R}_{CM}) - \Delta z \underbrace{\vec{d} \cdot \hat{z}}_{= 0}$$

$$\Rightarrow \vec{d} \cdot \int \vec{\varrho}_0 \, dm = \vec{d} \cdot \underbrace{\int \vec{r} \, dm}_{= M\vec{R}_{CM}} - \vec{d} \cdot \vec{R}_{CM} \underbrace{\int dm}_{= M} = 0$$

$$\Rightarrow I = I_0 + Md^2 \qquad \text{qed}$$

[Terminologi : Steiners sats = Parallellakseteoremet]



$$\rho_0 = d + \rho$$

$$\begin{aligned} r &= R_{CM} + s_0 \\ &= R_{CM} + \Delta z\, \hat{z} + \rho_0 \end{aligned}$$

Kinetisk energi for stivt legeme

[YF 10.3 ; TM 9.3 ; LL 6.6 ; HS 4.1]

Generell bevegelse for stivt legeme:

Translasjon av CM + Rotasjon om akse A_0 gjennom CM.

Skal vise at: $\boxed{K = K_{trans} + K_{rot} = \frac{1}{2} M V^2 + \frac{1}{2} I_0 \omega^2}$

der

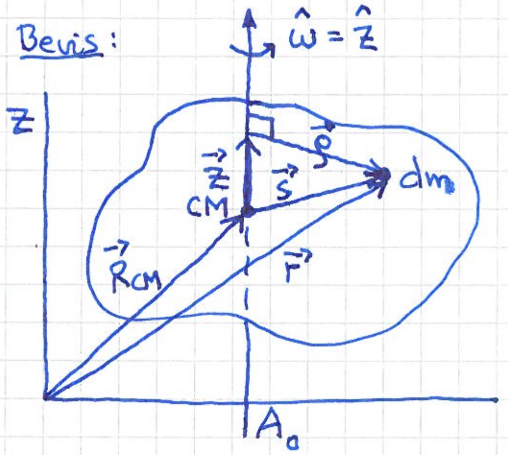
M = legemets masse

$\vec{V} = \dot{\vec{R}}_{CM}$ = hastigheten til CM

I_0 = legemets treghetsmoment om aksen A_0

$\vec{\omega}$ = vinkelhastigheten for rotasjonen om A_0

Bevis:



$\vec{r} = \vec{R}_{CM} + \vec{s} = \vec{R}_{CM} + \vec{z} + \vec{\rho}$

$\vec{v} = \dot{\vec{r}}$ = dm's hastighet

$\vec{V} = \dot{\vec{R}}_{CM}$ = CM's —"—

$\vec{u} = \dot{\vec{s}} = \dot{\vec{\rho}}$ = dm's hastighet relativt CM $\Rightarrow \vec{v} = \vec{V} + \vec{u}$

$dK = \frac{1}{2} dm \cdot v^2$ = dm's kinetiske energi

$\Rightarrow K = \int dK = \int \frac{1}{2} dm \cdot v^2$ = legemets kinetiske energi

$$v^2 = \vec{v} \cdot \vec{v} = (\vec{V} + \vec{u}) \cdot (\vec{V} + \vec{u}) = V^2 + u^2 + 2 \vec{V} \cdot \vec{u}$$

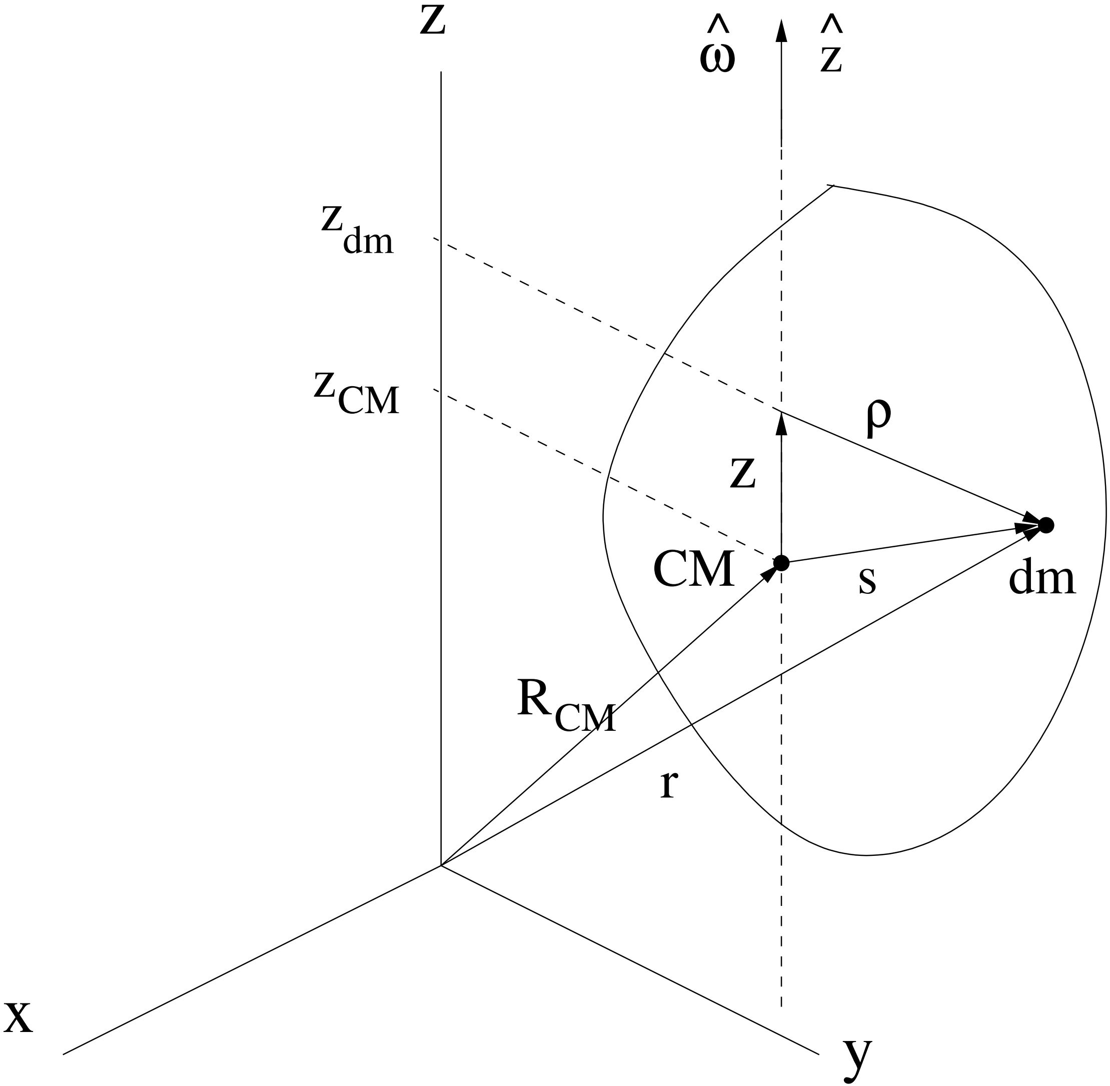
$$\frac{1}{2} \int dm \, V^2 = \underline{\frac{1}{2} M V^2}$$

$$\frac{1}{2} \int dm \, u^2 = \frac{1}{2} \int dm \, (\rho \omega)^2 = \frac{1}{2} \left(\int dm \, \rho^2 \right) \omega^2 = \underline{\frac{1}{2} I_0 \omega^2}$$

$$\int dm \, \vec{V} \cdot \vec{u} = \vec{V} \cdot \frac{d}{dt} \int dm \, \vec{s} = \vec{V} \cdot \frac{d}{dt} \int dm \, (\vec{r} - \vec{R}_{CM})$$

$$= \vec{V} \cdot \frac{d}{dt} \Big\{ \underbrace{\int \vec{r} \, dm}_{= M \cdot \vec{R}_{CM}} - \vec{R}_{CM} \underbrace{\int dm}_{= M} \Big\} = \underline{0}$$

qed!

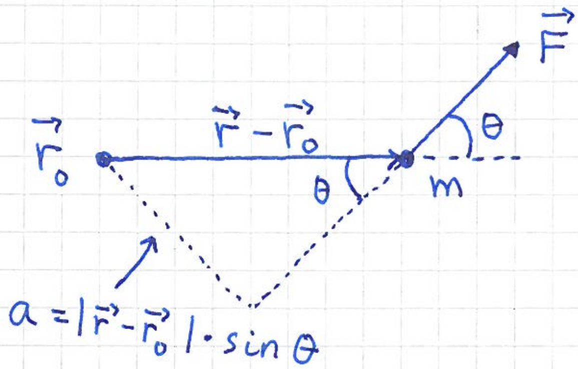


$$r = R_{CM} + s$$

Rotasjonsdynamikk [YF 10; TM 10; LL 6 og 5; HS 5]

(nesten helt) generell beskrivelse

Dreiemoment [YF 10.1; TM 10.2; LL 5.5, 6.4; HS 5.1]



$$\boxed{\vec{\tau} \stackrel{\text{def}}{=} (\vec{r}-\vec{r}_0) \times \vec{F}}$$

= $\vec{F}$'s dreiemoment på m i posisjon $\vec{r}$, relativt (det fritt valgte) referansepunktet $\vec{r}_0$

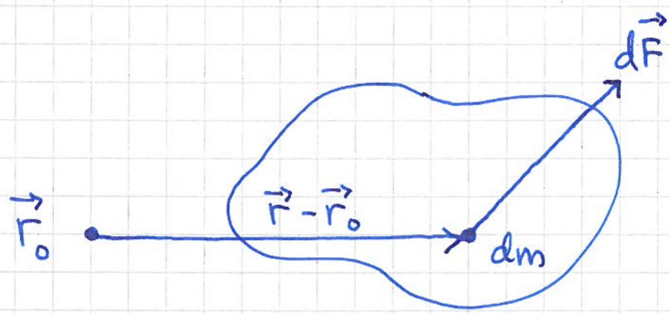
Retning: $\vec{\tau} \perp \vec{F}$ og $\vec{\tau} \perp \vec{r}-\vec{r}_0$

(h.h. regel $\Rightarrow$ $\vec{\tau}$ <u>ut</u> av planet i fig. over)

Absoluttverdi: $|\vec{\tau}| = |\vec{r}-\vec{r}_0| \cdot |\vec{F}| \cdot \sin\theta = a \cdot F$

der a = "armen til $\vec{F}$"

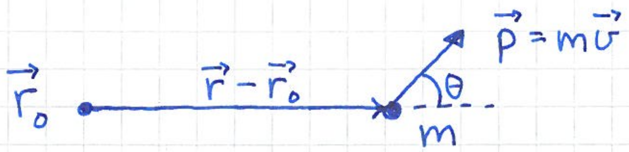
For partikkelsystem (f.eks. stivt legeme):



$$\vec{\tau} = \int d\vec{\tau} = \int_{\text{legemet}} (\vec{r}-\vec{r}_0) \times d\vec{F}$$

Dreieimpuls

[YF 10.5; TM 10.2; LL 6.6; HS 5.3]



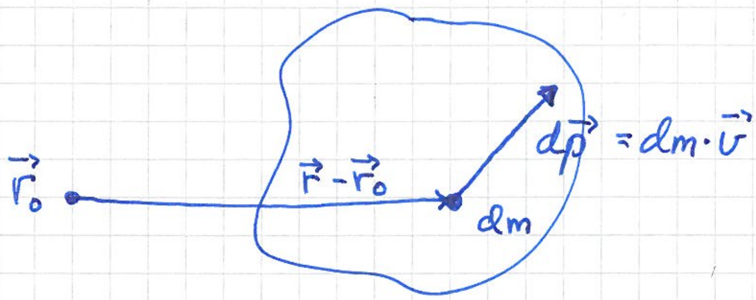
$$\boxed{\vec{L} \overset{\text{def}}{=} (\vec{r} - \vec{r}_0) \times \vec{p} = m(\vec{r} - \vec{r}_0) \times \vec{v}}$$

= m's dreieimpuls relativt $\vec{r}_0$

Retning: $\vec{L} \perp \vec{p}$ og $\vec{L} \perp \vec{r} - \vec{r}_0$

Abs. verdi: $|\vec{L}| = |\vec{r} - \vec{r}_0| \cdot |\vec{p}| \cdot \sin\theta = a \cdot p$ (se s. 47)

For partikkelsystem:



$$\vec{L} = \int d\vec{L} = \int_{\text{legemet}} dm\, (\vec{r} - \vec{r}_0) \times \vec{v}$$

N2, rotasjon [YF 10.5 ; TM 10.3; LL 6.6; HS 5.2]

(="spinnsatsen")

$$\dot{\vec{L}} = \frac{d}{dt}\left\{ m(\vec{r}-\vec{r}_0)\times\vec{v} \right\} = m(\dot{\vec{r}}-\dot{\vec{r}}_0)\times\vec{v} + m(\vec{r}-\vec{r}_0)\times\dot{\vec{v}}$$

<u>Anta</u> $\dot{\vec{r}}_0 = 0$ (<u>fast</u> $\vec{r}_0$) eller $\dot{\vec{r}}_0 \parallel \vec{v}$

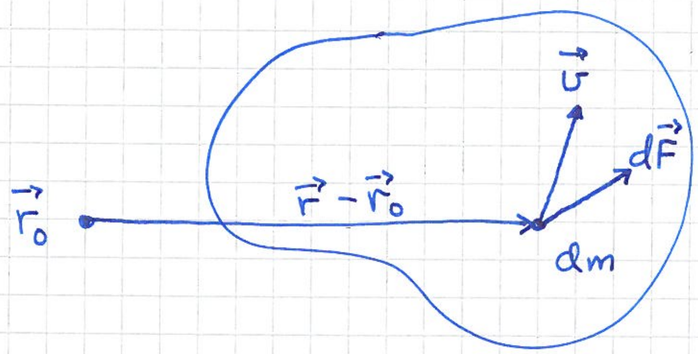
slik at $\dot{\vec{r}}_0 \times \vec{v} = 0$

Har (som s. 41): $\dot{\vec{r}} \times \vec{v} = \vec{v}\times\vec{v} = 0$

$$m\dot{\vec{v}} = m\vec{a} = \vec{F}$$

Dermed: $\dot{\vec{L}} = (\vec{r}-\vec{r}_0)\times\vec{F}$, dvs $\boxed{\vec{\tau} = \dot{\vec{L}}}$

For partikkelsystem:



$$\dot{\vec{L}} = \int d\dot{\vec{L}} = \int \frac{d}{dt}\left\{ dm\,(\vec{r}-\vec{r}_0)\times\vec{v} \right\}$$

$=$ .... som ovenfor, med $\dot{\vec{r}}_0 \times \vec{v} = 0$ fortsatt ....

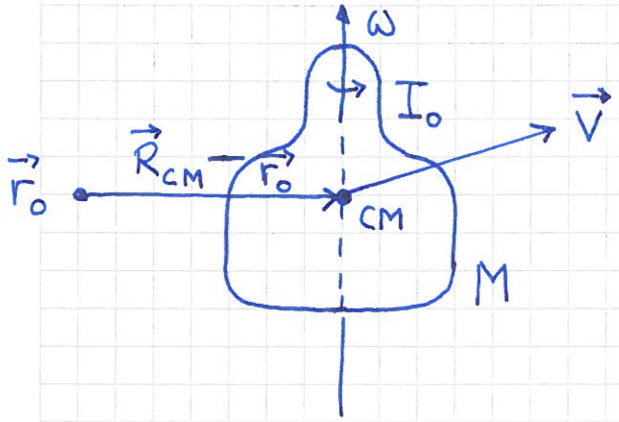
$$= \int (\vec{r}-\vec{r}_0)\times d\vec{F} = \int d\vec{\tau} = \vec{\tau}$$

med $\vec{\tau}$ = totalt dreiemoment på systemet (relativt $\vec{r}_0$)

$\vec{L}$ = total dreieimpuls for —"— (—"—)

Dreieimpuls for stivt legeme [YF 10.5; TM 10.2; LL 6.6; HS 5.3]

Anta at legemet har sylindersymmetri om (den instantane) rotasjonsaksen, "utpekt" med $\hat{\omega}$.



Fra s. 46: $K = K_{trans} + K_{rot} = \frac{1}{2}MV^2 + \frac{1}{2}I_0\omega^2$

Spm: Kan $\vec{L}$ på samme måte skrives som en sum av to ledd, ett assosiert med massesenterets translasjonsbevegelse, og ett assosiert med legemets rotasjonsbevegelse om en akse gjennom CM ?

Svar: Ja! Med antagelsen om sylindersymmetri om $\hat{\omega}$ kan det vises at legemets totale dreieimpuls kan skrives slik:

$$\boxed{\vec{L} = \vec{L}_b + \vec{L}_s = M(\vec{R}_{CM} - \vec{r}_0) \times \vec{V} + I_0\vec{\omega}}$$

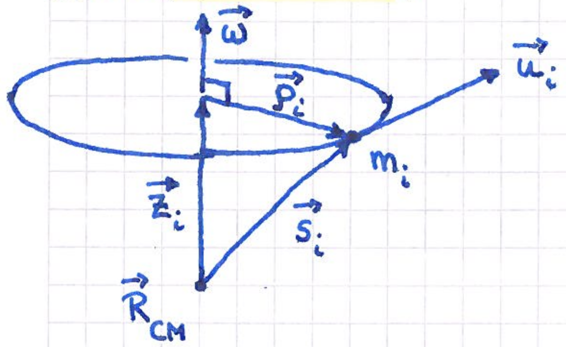
Banedreieimpuls, relativt $\vec{r}_0$: $\vec{L}_b = M(\vec{R}_{CM} - \vec{r}_0) \times \vec{V}$

Indre dreieimpuls ("spinn"; uavh. av $\vec{r}_0$): $\vec{L}_s = I_0\vec{\omega}$

[For bevis, som starter fra definisjonen $\vec{L} = \int dm\,(\vec{r} - \vec{r}_0) \times \vec{v}$, se s. 50 A og 50 B. Litt "kronglete", men ikke særlig vanskeligere enn bevisene s. 45 og 46.]

Bevis :

50 A



$$\vec{r}_i = \vec{R}_{CM} + \vec{s}_i \quad \leftarrow \text{relativkoord.}$$

$$\vec{v}_i = \vec{V} + \vec{u}_i \quad \leftarrow \text{relativhastighet}$$

$$(\dot{\vec{r}}_i = \dot{\vec{R}}_{CM} + \dot{\vec{s}}_i)$$

Fra figur: $\vec{s}_i = \vec{z}_i + \vec{\rho}_i$

Fra før: $\vec{u}_i = \vec{\omega} \times \vec{\rho}_i = \vec{\omega} \times \vec{s}_i$ (siden $\vec{\omega} \times \vec{z}_i = 0$)

$$\vec{L} = \sum_i m_i (\vec{r}_i - \vec{r}_0) \times \vec{v}_i$$

$$= \sum_i m_i (\vec{R}_{CM} - \vec{r}_0 + \vec{s}_i) \times (\vec{V} + \vec{u}_i)$$

$$= \sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times \vec{V} + \sum_i m_i \vec{s}_i \times \vec{V}$$

$$+ \sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times \vec{u}_i + \sum_i m_i \vec{s}_i \times \vec{u}_i$$

1. sum:

$$\sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times \vec{V} = M(\vec{R}_{CM} - \vec{r}_0) \times \vec{V} = \vec{L}_{CM}$$

= banedreieimpulsen relativt $\vec{r}_0$ pga CM's bevegelse (jfr. $\vec{L}$ for punktmasse)

2. sum:

$$\sum_i m_i \vec{s}_i \times \vec{V} = \sum_i m_i (\vec{r}_i - \vec{R}_{CM}) \times \vec{V} = (M\vec{R}_{CM} - M\vec{R}_{CM}) \times \vec{V} = 0$$

3. sum:

$$\sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times \vec{u}_i = \sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times (\vec{\omega} \times \vec{s}_i)$$

$$= (\vec{R}_{CM} - \vec{r}_0) \times (\vec{\omega} \times \underbrace{\sum_i m_i \vec{s}_i}_{=0}) = 0$$

4. sum:

$$\sum_i m_i \vec{s}_i \times \vec{u}_i = \vec{L}_{rel.} = \text{dreieimpuls pga masseelementenes bevegelse relativt CM}$$

50 B

Dermed:

$$\vec{L} = \vec{L}_{CM} + \vec{L}_{rel.} = M(\vec{R}_{CM} - \vec{r}_0) \times \vec{V} + \sum_i m_i \vec{s}_i \times \vec{u}_i$$

som gjelder for <u>vilkårlig</u> partikkelsystem (ikke nødv. vis stivt legeme); alternativt $\int_M dm(\vec{s} \times \vec{u})$ for $\vec{L}_{rel}$ hvis kontinuerlig massefordeling.

Hvis stivt legeme: $\vec{u}_i = \vec{\omega} \times \vec{s}_i$

$$\Rightarrow \sum_i m_i \vec{s}_i \times \vec{u}_i = \sum_i m_i \vec{s}_i \times (\vec{\omega} \times \vec{s}_i)$$

Identitet: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

(kjedelig, men ikke vanskelig å bevise!)

Dermed:

$$\sum_i m_i \vec{s}_i \times (\vec{\omega} \times s_i) = \sum_i m_i \left\{ \vec{\omega} s_i^2 - \vec{s}_i (\vec{s}_i \cdot \vec{\omega}) \right\}$$

$$= \sum_i m_i \left\{ \vec{\omega} (z_i^2 + \rho_i^2) - (\vec{z}_i + \vec{\rho}_i) z_i \omega \right\}$$

$$= \sum_i m_i \left\{ \vec{\omega} (z_i^2 + \rho_i^2) - z_i^2 \vec{\omega} - z_i \omega \vec{\rho}_i \right\}$$

$$= \sum_i m_i \rho_i^2 \vec{\omega} - \omega \sum_i m_i z_i \vec{\rho}_i$$

$$= I_0 \vec{\omega} - \omega \sum_i m_i z_i \vec{\rho}_i$$

<u>Hvis</u> sylindersymmetri om $\hat{\omega}$, dvs om $\hat{z}$:

$$\sum_i m_i z_i \vec{\rho}_i = \sum_i m_i z_i (x_i \hat{x} + y_i \hat{y}) = 0$$

fordi bidragene fra <u>like store</u> masseelementer i (x_i, y_i, z_i) og $(-x_i, -y_i, z_i)$ <u>kansellerer</u>.

Dette er <u>ofte</u> tilfelle, men ikke alltid.

Men <u>hvis</u> sylindersymmetri om $\hat{\omega}$: $\vec{L}_{rel} = I_0 \vec{\omega}$

$$\Rightarrow \vec{L} = \vec{L}_{CM} + \vec{L}_{rel} = M(\vec{R}_{CM} - \vec{r}_0) \times \vec{V} + I_0 \vec{\omega}$$

<u>qed</u>

Bevaringslover

For isolert system (dvs: system som ikke påvirkes av ytre krefter) er energi, impuls og dreieimpuls bevarte størrelser.

$$W = 0 \quad \left(\text{dvs } P = \frac{dW}{dt} = 0\right) \quad \Rightarrow \quad E = \text{konst.}$$

$$\vec{F} = 0 \quad \Rightarrow \quad \vec{p} = \text{konst.}$$

$$\vec{\tau} = 0 \quad \Rightarrow \quad \vec{L} = \text{konst.}$$

Mekanisk likevekt (Statikk)

[YF 11.1–11.3; TM 12.1–12.3; LL 7.1 ; HS 4.6]

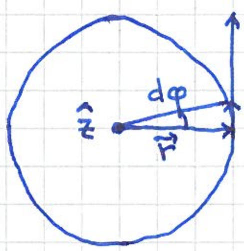
Et stivt legeme er i ro,

$$\vec{p} = 0 \quad \text{og} \quad \vec{L} = 0$$

bare dersom $\sum_i \vec{F}_i = 0$ og $\sum_i \vec{\tau}_i = 0$

Arbeid utført ved rotasjon

[YF 10.4; TM 9.5; LL 6.4; HS 4.4.1]



$\vec{F} = F\,\hat{\varphi}$

$d\vec{s} = r\,d\varphi\,\hat{\varphi}$

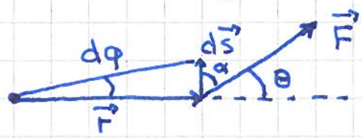
$\vec{r} = r\,\hat{r}$

$$dW = \vec{F}\cdot d\vec{s} = F\cdot r\,d\varphi$$

$$\vec{\tau} = \vec{r}\times\vec{F} = r\cdot F\ \hat{z}$$

$$\Rightarrow \boxed{dW = \tau\,d\varphi} \Rightarrow P = \frac{dW}{dt} = \tau\cdot\omega$$

Enn om $\vec{F}$ og $d\vec{s}$ ikke er parallelle?



$\alpha = \frac{\pi}{2} - \theta$

$$dW = F\cdot ds\cdot\cos\alpha = F\cdot r\,d\varphi\cdot\sin\theta$$

$$\tau = |\vec{\tau}| = |\vec{r}\times\vec{F}| = r\cdot F\cdot\sin\theta$$

$$\Rightarrow \boxed{dW = \tau\,d\varphi}$$