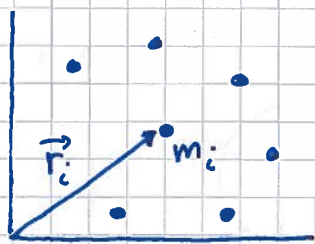


Massesenter

[YF 8.5 og oppg 8.115, 8.116 ; LL 5.6, 5.8, 6.1]

(29)



Massesenter (CM) for N punktmasser

m_1, m_2, \dots, m_N i pos $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$:

$$\vec{R}_{CM} = \frac{1}{M} \sum_i m_i \vec{r}_i$$

$$M = \sum_i m_i = \text{systemets totale masse}$$

Med kontinuerlig massefordeling :

$m_i \rightarrow dm$ (masseelement) og $\sum_i \rightarrow \int$

$$\Rightarrow \vec{R}_{CM} = \frac{1}{M} \int \vec{r} dm \quad ; \quad M = \int dm$$

$$3D: dm = \rho dV \quad 2D: dm = \sigma dA \quad 1D: dm = \lambda dl$$

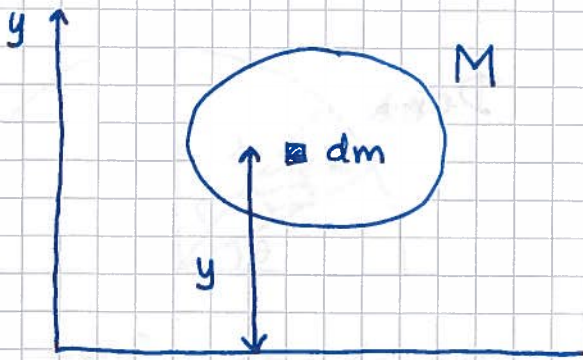
ρ, σ, λ = masse pr hhv volum-, flate-, lengdeenhet

dV, dA, dl = hhv volum-, flate-, lengdeelement

Med uniform massefordeling : $\frac{dm}{M} = \frac{dV}{V}$ osv

Potensiell energi i tyngdefeltet

(30)



Velger $U(0) = 0$

$$U = \int dU = \int g \cdot y \cdot dm$$

Med mindre $y_{\max} - y_{\min} \approx$ jordradien, er $g = \text{konstant}$ for hele systemet.

$$U = g \int y dm = \underline{g \cdot M \cdot Y_{CM}}$$

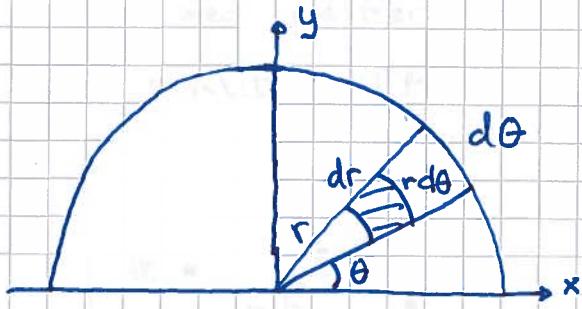
Dvs: Total pot. energi U blir som om hele massen M var samlet i høyden Y_{CM} , f.eks. i massesenteret \vec{R}_{CM}

Tyngdepunktet: Der legemet vil balansere (uten å rotere).

Her er tyngdepunktet samme sted som massesenteret.

Eks 1: \vec{R}_{cm} for halv tynn skive, radius R

(31)



$X_{cm} = 0$ pga symmetri

$$Y_{cm} = \frac{1}{M} \int y dm = \frac{1}{A} \int y dA$$

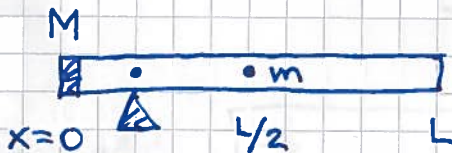
(siden $\frac{dm}{M} = \frac{dA}{A}$)

$$A = \frac{1}{2} \pi R^2, \quad dA = dr \cdot r d\theta, \quad y = r \sin \theta$$

$$\begin{aligned} \Rightarrow Y_{cm} &= \frac{2}{\pi R^2} \int_0^R \int_0^\pi r \sin \theta \, dr \, r d\theta = \frac{2}{\pi R^2} \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta \\ &= \frac{2}{\pi R^2} \cdot \frac{1}{3} R^3 \cdot \int_0^\pi (-\cos \theta) = \frac{4R}{3\pi} \approx \underline{0.42R} \end{aligned}$$

[Halvsirkelformet stang: $Y_{cm} = \frac{2R}{\pi}$; Kompakt halvkule: $Y_{cm} = \frac{3R}{8}$]

Eks 2: Rør med lod i enden



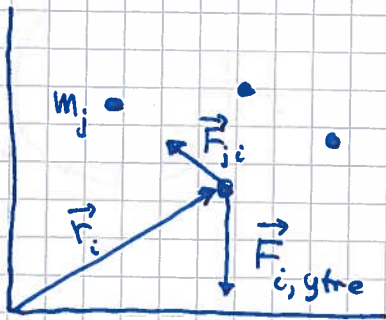
$$m = 165g \quad M = 305g$$

$$X_{cm} = \frac{1}{m+M} \left\{ M \cdot 0 + \underbrace{\int_0^L x \cdot \frac{m dx}{L}}_{= mL/2} \right\} = \frac{mL}{2(m+M)} = \underline{0.18L}$$

Massesenterets bevegelse [YF 8.5 ; LL 5.8]

(32)

Forsøk med plaststrør tyder på at CM bærer seg som om hele massen var samlet i CM. Bevis:



N2 for m_i :

$$m_i \ddot{\vec{r}}_i = \underbrace{\vec{F}_{i,ytre}}_{\text{Total ytre kraft p\aa } m_i} + \underbrace{\sum_{j \neq i} \vec{F}_{ji}}_{\text{Total indre kraft paa } m_i}$$

Legger sammen N2 for alle i :

$$\text{V.S.: } \sum_i m_i \ddot{\vec{r}}_i = \frac{d^2}{dt^2} \left\{ \sum_i m_i \vec{r}_i \right\} = \frac{d^2}{dt^2} \left\{ M \vec{R}_{CM} \right\} = M \ddot{\vec{R}}_{CM}$$

$$\text{H.S.: } \sum_i \vec{F}_{i,ytre} = \vec{F}_{ytre} = \text{netto ytre kraft paa hele systemet}$$

$$\sum_i \sum_{j \neq i} \vec{F}_{ji} = \underbrace{\vec{F}_{21} + \vec{F}_{12}}_{=0 \text{ (N3)}} + \dots + \underbrace{\vec{F}_{N,N-1} + \vec{F}_{N-1,N}}_{=0} = 0$$

Dermed:

$$\boxed{\vec{F}_{ytre} = M \ddot{\vec{R}}_{CM}}$$

Massesenterets (Tyngdepunktets) bevegelse blir som om hele M var samlet i \vec{R}_{CM} og ble utsatt for netto ytre kraft \vec{F}_{ytre}

I tillegg til dette kommer:

For stive legemer: Rotasjon om CM

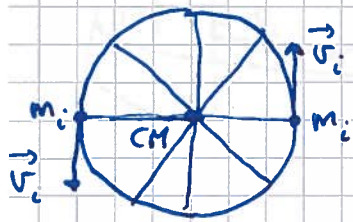
For ikke helt stive legemer: Vibrasjon

Rotasjon

[YF 9,10 ; LL 6 (5)]

33

Ren rotasjon (f. eks. om CM)

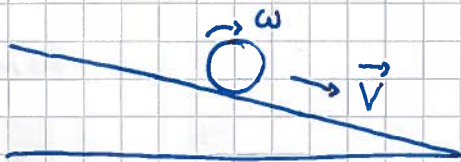


$$K_{\text{trans}} = \frac{1}{2} M \dot{\vec{R}}_{\text{CM}}^2 = 0 ; \quad \vec{p} = \sum_i m_i \vec{v}_i = 0$$

$$K_{\text{rot}} = \sum_i \frac{1}{2} m_i v_i^2 \neq 0$$

$$\vec{L} \neq 0 \quad (\text{Hjulets dreieimpuls})$$

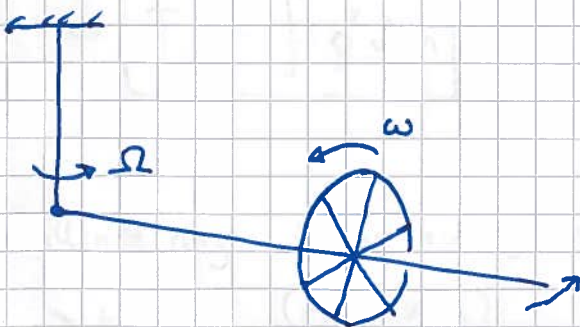
Rulling = Translasjon av CM + Rotasjon om CM



Krefter \parallel skr pnet gir $\dot{V} > 0$

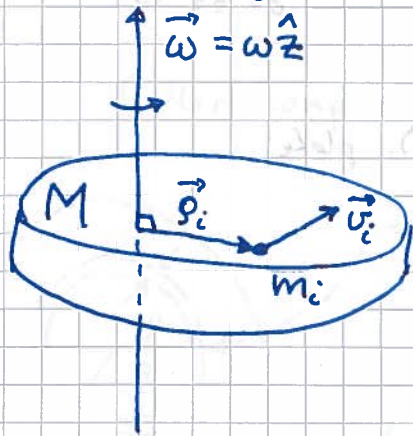
Dreiemoment mhp CM gir $\dot{\omega} > 0$

Komplisert dynamikk



Preesjon. Gyroskop

Rotasjonsenergi og treghetsmoment [YF 9.4; LL 6.4, 6.3] (34)



$\vec{\omega} \parallel$ rotasjonsaksen

$\vec{\rho}_i = \rho_i \hat{\rho}$; $\rho_i =$ avstand fra rot.aksen til m_i

$\vec{v}_i = v_i \hat{\phi}$; $v_i =$ hastigheten til m_i

Fra før : $v_i = \rho_i \omega$

Fra figur : $\vec{v}_i = \vec{\omega} \times \vec{\rho}_i$

(Høyrehåndregel!)

Braker her sylinderkoordinater : ρ, ϕ, z (dvs: polarkoord. + z)

Dermed:

$$K = K_{\text{rot}} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \left\{ \sum_i m_i \rho_i^2 \right\} \omega^2 = \underline{\underline{\frac{1}{2} I \omega^2}}$$

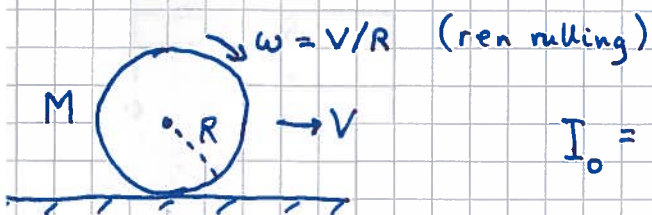
Her er

$I \stackrel{\text{def}}{=} \sum_i m_i \rho_i^2 =$ legemets treghetsmoment mhp den aktuelle (valgte) aksen

Hvis vi har kontinuerlig massefordeling: $m_i \rightarrow dm$ og $\sum_i \rightarrow \int$

$$\Rightarrow \boxed{I = \int \rho^2 dm} \quad \rho = \text{avstand fra aksen til } dm$$

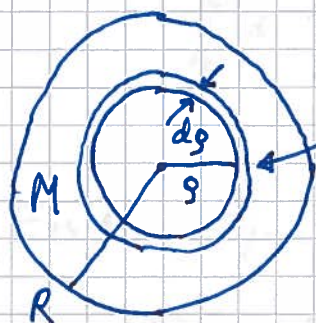
Eks: Rullende ring (eller hul sylinder)



$$I_0 = \int \rho^2 dm = R^2 \int dm = \underline{\underline{MR^2}}$$

$$\Rightarrow K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2} MV^2 + \frac{1}{2} I_0 \omega^2 = \frac{1}{2} MV^2 + \frac{1}{2} MR^2 \cdot \frac{V^2}{R^2} = \underline{\underline{MV^2}}$$

Skive (eller kompakt sylinder)

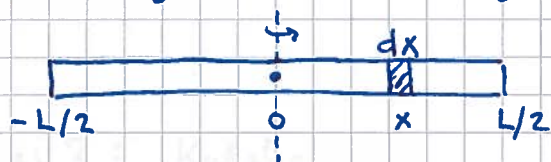


$$dI_0 = g^2 dm = g^2 M \frac{dA}{A}$$

$$= g^2 M \cdot \frac{2\pi g dg}{\pi R^2} = \frac{2M}{R^2} \cdot g^3 dg$$

$$\Rightarrow I_0 = \int dI_0 = \frac{2M}{R^2} \int_0^R g^3 dg = \frac{2M}{R^2} \cdot \frac{R^4}{4} = \underline{\underline{\frac{1}{2} MR^2}}$$

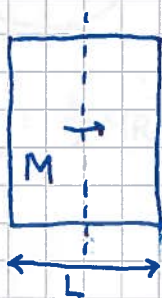
Tynn stang, akse \perp stanga, gjennom CM



$g = x, dm = M \cdot dx / L$

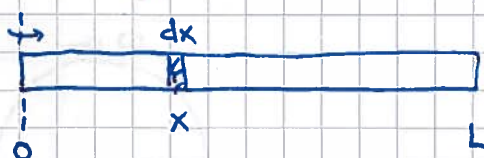
$$\Rightarrow I_0 = \int_{-L/2}^{L/2} x^2 \cdot M \cdot dx / L = \frac{M}{L} \int_{-L/2}^{L/2} \frac{1}{3} x^3 = \frac{M}{L} \cdot \left(\frac{1}{24} + \frac{1}{24} \right) L^3 = \underline{\underline{\frac{1}{12} ML^2}}$$

Tilsvarende for tynn plate (f.eks ei dør):



$I_0 = \frac{1}{12} ML^2$

Tynn stang, akse ved stangas ende:



$$I = \int_0^L x^2 M dx / L = \underline{\underline{\frac{1}{3} ML^2}}$$



Dør:

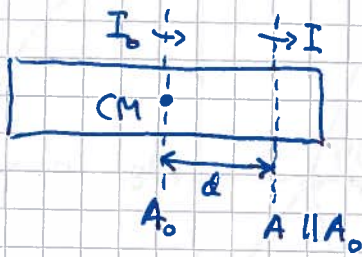
$I = \frac{1}{3} ML^2$

Kuleskall : $I_0 = \frac{2}{3} MR^2$

Kompakt kule : $I_0 = \frac{2}{5} MR^2$

Steiners sats

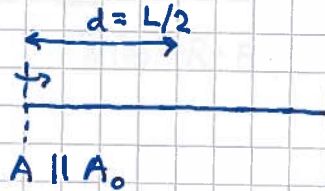
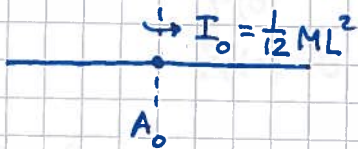
[YF 9.5 ; LL 6.3]



$$I = I_0 + Md^2$$

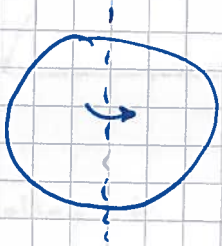
(Se notat for bevis)

Eks 1: Tynn stang

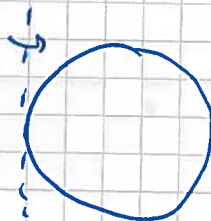


$$\begin{aligned} I &= I_0 + M \cdot (L/2)^2 = \\ &= \frac{1}{12} ML^2 + \frac{1}{4} ML^2 = \underline{\underline{\frac{1}{3} ML^2}} \\ &\text{(som s 35!)} \end{aligned}$$

Eks 2: Kuleskall



$$I_0 = \frac{2}{3} MR^2$$

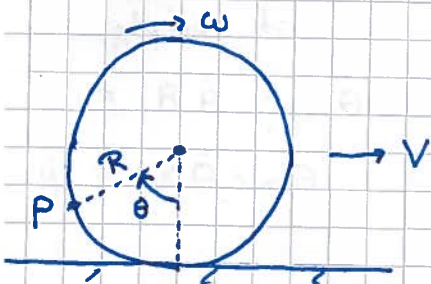


$$d = R$$

$$\underline{\underline{I = \frac{5}{3} MR^2}}$$

Ren rulling

[YF 10.3; LL 6.7]



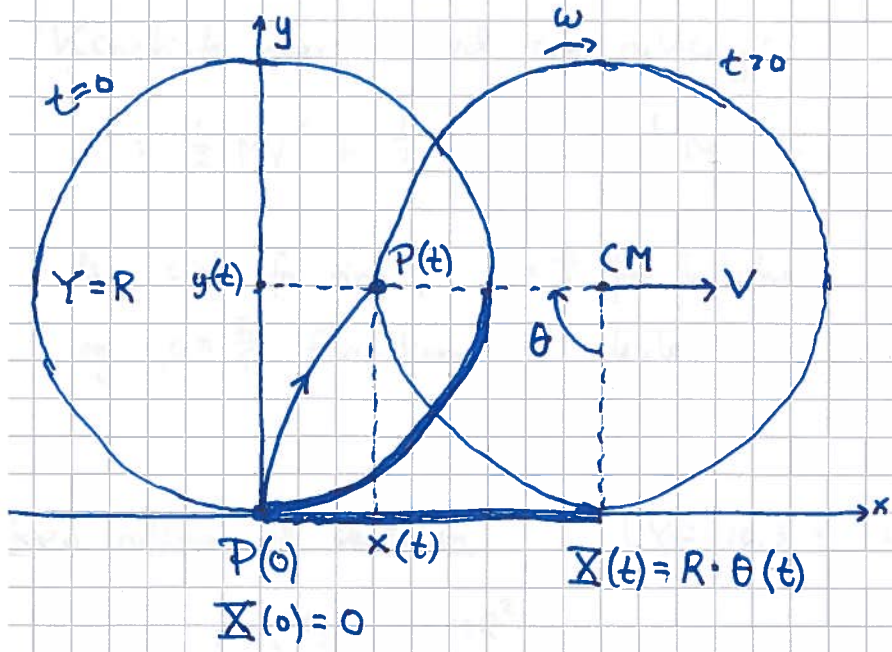
Vi skal vise at $V = \omega R$

ved ren rulling,

samt bestemme hastigheden og

banen til et punkt P

på periferien.



Punkt på periferien, P:
 $x(t), y(t)$ (posisjon)
 $\dot{x}(t), \dot{y}(t)$ (hastighet)

I figuren er $\theta = \pi/2$. Da har CM flyttet seg $1/4$ av omkretsen langs x-aksen, dvs $X = R \cdot \pi/2$.

Fra figur: $x = X - R \sin \theta = R\theta - R \sin \theta$
 $y = Y - R \cos \theta = R - R \cos \theta$ } Dvs, banen til P er en sykloide



Beregelsen til CM:

$$\vec{R}_{CM} = X \hat{x} + Y \hat{y} = R\theta \hat{x} + R \hat{y}$$

$$\vec{V} = R\dot{\theta} \hat{x} = R\omega \hat{x}$$

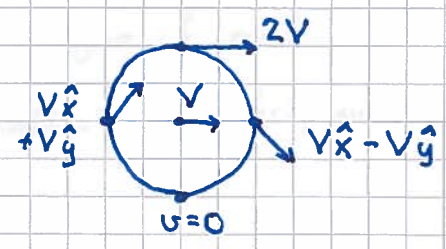
$$\vec{A} = R\ddot{\theta} \hat{x} = R\dot{\omega} \hat{x} = R\alpha \hat{x}$$

Dvs, ulløsbetingelsene er: $V = R\omega, A = R\alpha$

Hastigheten til P:

$$\dot{x} = R\dot{\theta} - R\dot{\theta} \cos \theta = V(1 - \cos \theta)$$

$$\dot{y} = R\dot{\theta} \sin \theta = V \sin \theta$$



Ser at $v=0$ når $\theta = 0, 2\pi, 4\pi, \dots$, dvs når P har kontakt med underlaget. Dermed er $P_f = dW_f/dt = \vec{f} \cdot \vec{v} = 0$, dvs taper ikke mekanisk energi ved ren rulling. (Ideelt sett...!)

Kinetiske energi ved ren rulling:

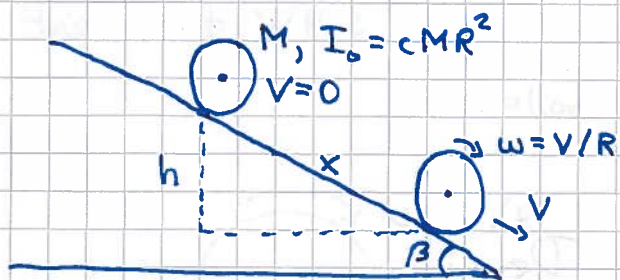
(38)

$$K = \frac{1}{2} MV^2 + \frac{1}{2} I_0 \omega^2 = \frac{1}{2} MV^2 + \frac{1}{2} \cdot cMR^2 \cdot \frac{V^2}{R^2} = \underline{\underline{\frac{1}{2} (1+c) MV^2}}$$

der $c=1$ for ring, $c=\frac{2}{3}$ for kuleskall, $c=\frac{1}{2}$ for kompakt skive,
og $c=\frac{2}{5}$ for kompakt kule.

Ren rulling på skrån

[YF 10.3 ; LL 6.8]



Finne V , \dot{V} , friksjonskraft f ,
og minste μ_s (ert største β)
som gir ren rulling.

Energibevarelse: $Mgh = (1+c) \frac{1}{2} MV^2$; $h = x \sin \beta$

$$\Rightarrow \underline{\underline{V(x) = \left\{ \frac{2gx \sin \beta}{1+c} \right\}^{1/2}}}$$

$\Rightarrow V(\text{kule}) > V(\text{skive}) > V(\text{kuleskall}) > V(\text{hul sylinder})$ som observert

Akselerasjon:

$$\dot{V} = \frac{dV}{dt} = \frac{dV}{dx} \cdot \underbrace{\frac{dx}{dt}}_{=V} = \left\{ \frac{2g \sin \beta}{1+c} \right\}^{1/2} \cdot \frac{1}{2} x^{-1/2} \cdot V = \underline{\underline{\frac{g \sin \beta}{1+c}}}$$

Uten friksjon: $F_{||} = Mg \sin \beta \Rightarrow \dot{V} = g \sin \beta$

Med friksjon er $\dot{V} < g \sin \beta \Rightarrow$ Må være friksjon \vec{f} med
retning oppover langs skrån

N2 || : $Mg \sin \beta - f = M\dot{V}$

$$\Rightarrow \underline{\underline{f = \frac{c}{1+c} Mg \sin \beta}}$$

Ren rulling bare hvis $f \leq f_{\max} = \mu_s N = \mu_s Mg \cos \beta$

(39)

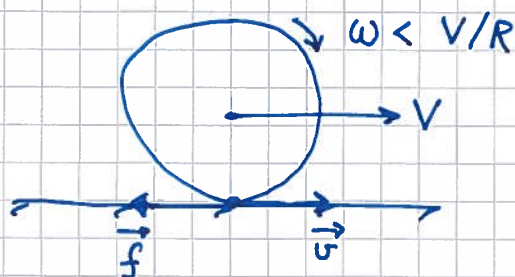
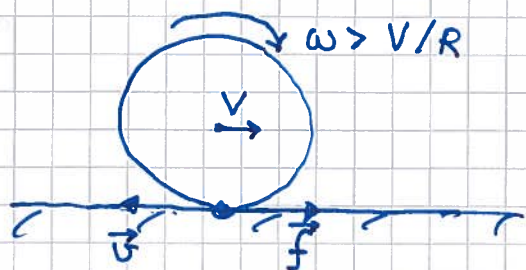
$$\Rightarrow \text{Må ha } \mu_s \geq \frac{c}{1+c} \tan \beta, \text{ ders } \beta \leq \arctan \left\{ \mu_s \frac{1+c}{c} \right\}$$

Eks: Støkkule på plastunderlag. $\mu_s \approx 0.2$, $c = 2/5$

$$\Rightarrow \beta \leq \arctan \left\{ 0.2 \cdot \frac{7/5}{2/5} \right\} = \arctan 0.7 \approx \underline{35^\circ}$$

Sluring [LL 6.7]

Hvis $\omega \neq V/R$: $v = V - \omega R \neq 0$, ders relativ hastighet mellom objekt og underlag i kontaktpunktet



$$\Rightarrow \text{Kinetisk friksjon: } f = \mu_k N$$

$$\Rightarrow \text{Effekttap: } P_f = \vec{f} \cdot \vec{v} < 0$$

Ders, taper mek. energi lik $|P_f|$ pr tidsenhet.