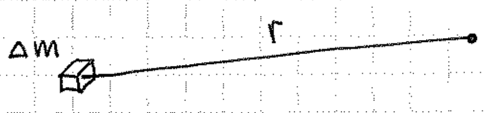
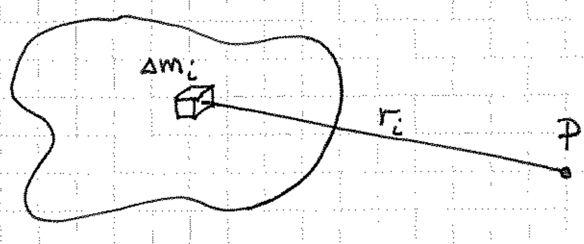
Potensial og felt fra massefordelinger [YF 13.6, LL 11.2]

Potensial fra liten masse Δm:



$$\Delta V(r) = -\frac{G \cdot \Delta m}{r}$$

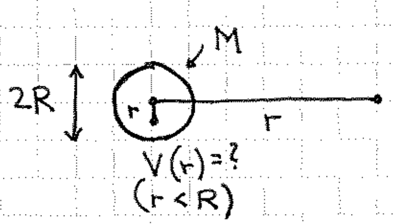
Potensial fra massefordeling:



V_P = potensial i punkt P

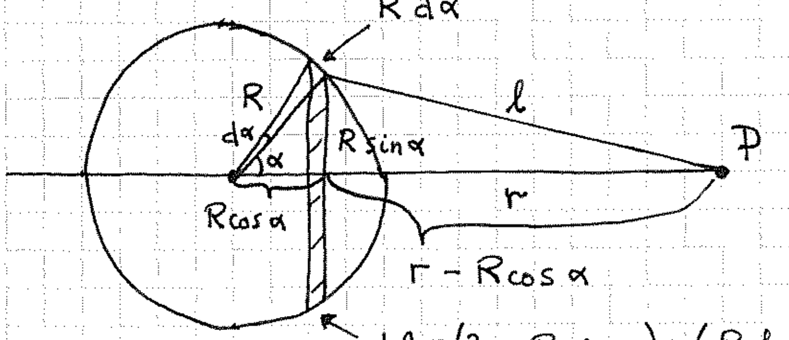
$$V_P = \sum_i V_P^i = -G \sum_i \frac{\Delta m_i}{r_i} \xrightarrow{\Delta m_i \to 0} -G \int \frac{dm}{r}$$

Eks 1: Kuleskall



$V(r) = ?$ $(r > R)$

<u>$r > R$:</u>



$$dA = \underbrace{(2\pi \cdot R \sin\alpha)}_{\text{omkrets}} \cdot \underbrace{(R\,d\alpha)}_{\text{bredde}}$$ = area av tynn ring der alle punkter er i samme avstand ℓ fra P

Pythagoras $\Rightarrow$

$$\ell^2 = (R\sin\alpha)^2 + (r - R\cos\alpha)^2$$
$$= R^2\sin^2\alpha + r^2 - 2rR\cos\alpha + R^2\cos^2\alpha$$
$$= R^2 + r^2 - 2rR\cos\alpha$$

Bidrag til V i punktet P fra tynn ring:

$$dV = -G\,\frac{dm}{\ell} \quad ; \quad dm/M = dA/A = 2\pi R\sin\alpha \cdot R\,d\alpha \,/\, 4\pi R^2$$
$$= \tfrac{1}{2}\sin\alpha\, d\alpha$$

Totalt potensial i P:

$$V(r) = \int dV = -G\int \frac{dm}{\ell} = -\frac{MG}{2}\int_0^{\pi} \frac{\sin\alpha\, d\alpha}{\sqrt{R^2+r^2-2rR\cos\alpha}}$$

$$= -\frac{MG}{2}\Bigg|_0^{\pi} \frac{\sqrt{R^2+r^2-2rR\cos\alpha}}{rR}$$

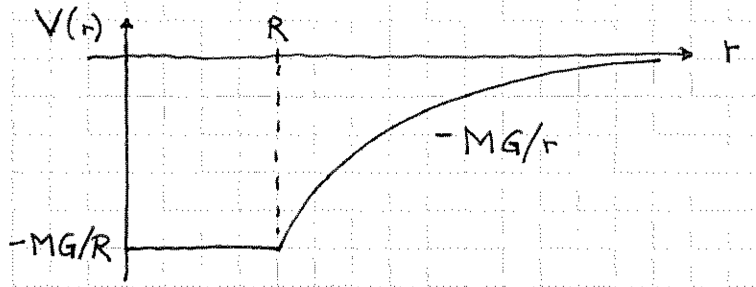
$$= -\frac{MG}{2rR}\left\{\sqrt{R^2+r^2+2rR} - \sqrt{R^2+r^2-2rR}\right\}$$

$$= -\frac{MG}{2rR}\left\{(R+r) - (r-R)\right\} = \underline{-\frac{MG}{r}}$$

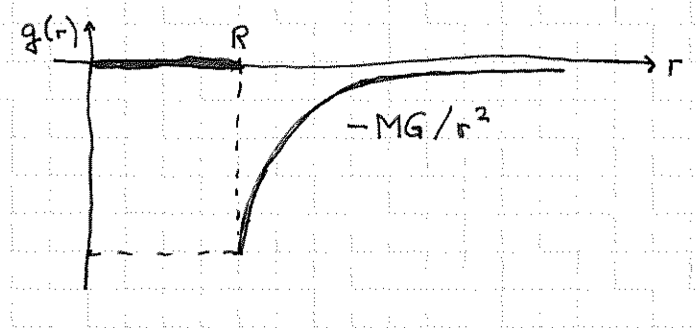
$\underline{r < R}$:

Alt blir som for $r > R$, bortsett fra at $\sqrt{R^2+r^2-2rR} = R-r$, slik at

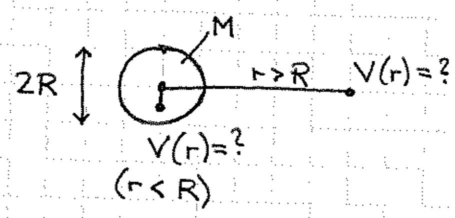
$$V(r) = \ldots\ldots = -\frac{MG}{2rR}\left\{(R+r) - (R-r)\right\} = \underline{-\frac{MG}{R}}$$



Feltet: $\vec{g}(r) = -\nabla V(r) = -\hat{r}\,\frac{dV}{dr} = \begin{cases} 0 & ; \; r < R \\ -\hat{r}\,\frac{MG}{r^2} & ; \; r > R \end{cases}$

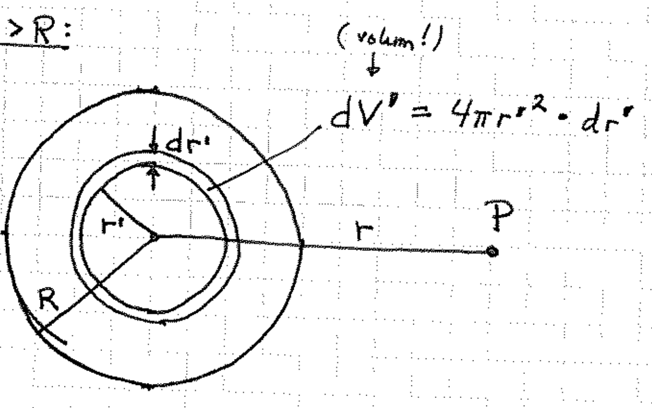


Eks 2: Kompakt kule



Strategi: Del opp kula i tynne kuleskall, bruk resultat fra Eks 1, og integrer.

r > R:



$$\frac{dm'}{M} = \frac{dV'}{V'} = \frac{4\pi r'^2 dr'}{\frac{4}{3}\pi R^3} = \frac{3}{R^3} r'^2 dr'$$

Bidrag fra tynt kuleskall til potensialet i P:

$$dV = -G\,\frac{dm'}{r} = -\frac{3MG}{rR^3}\, r'^2\, dr'$$

Totalt potensial i P:

$$V(r) = \int dV = -\frac{3MG}{rR^3}\int_0^R r'^2 dr' = -\frac{3MG}{rR^3}\cdot\frac{1}{3}R^3 = \underline{-\frac{MG}{r}}$$

r < R:

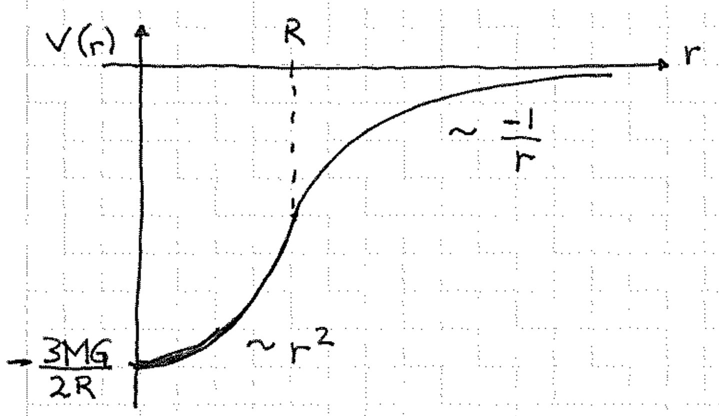
For tynne kuleskall med radius $r' < r$: $dV = -\dfrac{G\cdot dm'}{r}$

——— " ——— $r' > r$: $dV = -\dfrac{G\cdot dm'}{r'}$

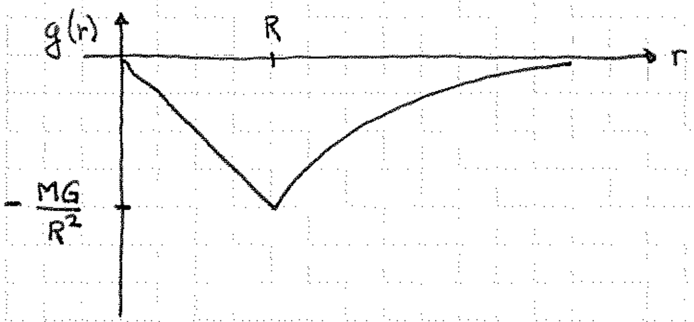
$$\Rightarrow V(r) = -\frac{3MG}{R^3}\left\{\int_0^r \frac{r'^2 dr'}{r} + \int_r^R \frac{r'^2 dr'}{r'}\right\}$$

$$= -\frac{3MG}{R^3}\left\{\frac{1}{3}r^2 + \frac{1}{2}R^2 - \frac{1}{2}r^2\right\}$$

$$= \underline{-\frac{3MG}{2R}\left\{1 - \frac{r^2}{3R^2}\right\}}$$



Feltet: $\vec{g}(r) = -\nabla V(r) = -\hat{r}\,\frac{dV}{dr} = \begin{cases} -\hat{r}\,\frac{MGr}{R^3} & ; \quad r < R \\ -\hat{r}\,\frac{MG}{r^2} & ; \quad r > R \end{cases}$



$\Rightarrow$ Kraft på masse m ved $r = R$:

$\vec{F}(r=R) = m \cdot \vec{g}(R) = -\hat{r}\;MmG/R^2$,

som om hele massen M var plassert i sentrum av kula!

[Denne antagelsen har vi da også gjort "hele tiden" for tyngdekraften på en masse m på jordoverflaten, $\vec{F} = m\vec{g}$ med $g = 9.81\ m/s^2$.]

Hit 28.11.11