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Gravitasjon [YF 13, LL 11]

For <u>kort</u> historikk, se notater 2011, s. 106 - 107.

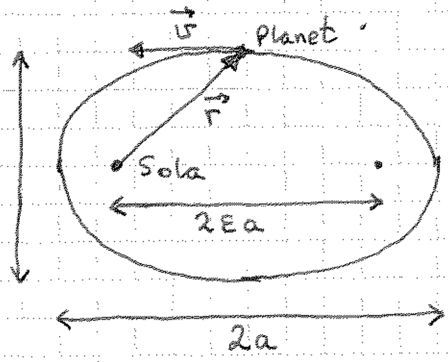
[Copernicus, Brahe, Kepler, Newton, Einstein]
ca 1500 ca 1600 1687 1916

Keplers lover [YF 13.5, LL 11.5]

Analyse av Brahes observasjoner

$\Longrightarrow$

K1: <u>Ellipse</u>formede planetbaner med sola i et av brennpunktene



Eksentrisitet: $\varepsilon = \sqrt{1-(b/a)^2}$

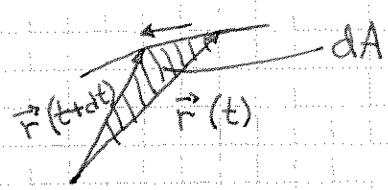
Sirkel: $\varepsilon = 0$,

Ellipse: $0 < \varepsilon < 1$

[$\varepsilon \geq 1$: ikke lukket bane]

K2: $\vec{r}$ sveiper over konstant areal pr tidsenhet,

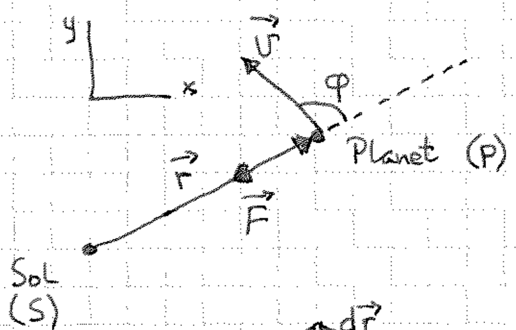
dA/dt = konst.



K3: T^2/a^3 = konst. for alle planetene

T = omløpstid, a = store halvakse

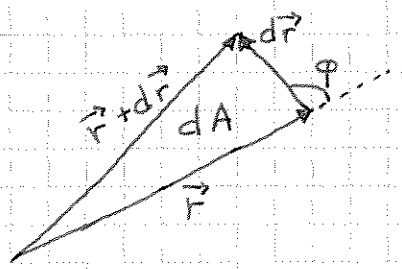
K2 $\Leftrightarrow$ $\vec{L}$ = konstant og $\vec{F} \parallel \vec{r}$:



$$\vec{F} \sim -\hat{r} \Rightarrow \vec{\tau} = \vec{r} \times \vec{F} = 0$$

$$\Rightarrow \vec{L} = L\hat{z} = \vec{r} \times m\vec{v} = \text{konst.}$$

$$L = r m v \sin\varphi$$



$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2} r\, dr \sin\varphi$$

[dA = halvt parallellogram, se R2]

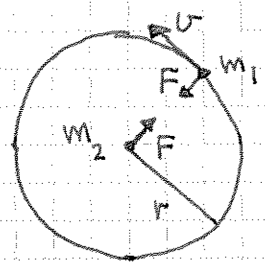
$$\left[\frac{1}{2}|\vec{r} \times (\vec{r}+d\vec{r})| = \frac{1}{2}|\vec{r} \times d\vec{r}|\right]$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r \frac{dr}{dt} \sin\varphi = \frac{1}{2} r v \sin\varphi = \frac{L}{2m} = \text{konst.} \quad \text{(QED)}$$

Newtons gravitasjonslov [YF 13.1; LL 2.5, 11.1]

Anta sirkulær bane, $\varepsilon = 0$, $a = b = r$

(månen : $\varepsilon \approx 0.055$)



m_1 = månens masse

m_2 = jordas masse

- K2 $\Rightarrow \vec{L}$ = konst. $\Rightarrow \vec{F} \sim -\hat{r}$ [dvs. retning på $\vec{F}$]
- sirkel $\Rightarrow F = m_1 \omega^2 r = m_1 \left(\frac{2\pi}{T}\right)^2 r$
- K3 $\Rightarrow T^2 = Ca^3 = Cr^3 \Rightarrow F = m_1 \cdot \frac{4\pi^2}{Cr^3} r = \frac{4\pi}{C} \frac{m_1}{r^2}$
- N3 $\Rightarrow F_{12} = F_{21} \Rightarrow F \sim m_2$ og m_1

$$\Rightarrow \boxed{\vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r}} \quad \text{Gravitasjonsloven}$$

Cavendish 1798 (Du 2012) : $G \approx 6.67 \cdot 10^{-11}$ Nm^2/kg^2

[Nøyaktighet pr i dag : $\Delta G/G \approx 10^{-4}$]

På jordas overflate ($M = 5.974 \cdot 10^{24}$ kg , $R = 6378$ km):

$$F = \frac{GM}{R^2} \cdot m = g \cdot m \quad \text{med } g = \frac{GM}{R^2} \approx 9.81 \, m/s^2$$

Potensiell energi [YF 13.3, LL 11.1]

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r} \quad , \quad \text{med valget } U(\vec{r}_0) = 0$$

Lar $r_0 \to \infty$, dvs $U(\infty) = 0$

$$\Rightarrow U(r) = -\int_{\infty}^{r} \underbrace{(-GmM/r^2)\hat{r}}_{\vec{F}} \cdot \underbrace{dr\,\hat{r}}_{d\vec{r}} = GmM \int_{\infty}^{r} \frac{dr}{r^2}$$

$$= GmM \Big|_{\infty}^{r} \left(-\frac{1}{r}\right) = -\frac{GmM}{r}$$

= pot. energi for to masser m og M

i innbyrdes avstand r

Eks: Masse m i jordas tyngdefelt

z = høyde over bakken

$$U(z) - U(0) = -\frac{GmM}{R+z} + \frac{GmM}{R}$$

$$= GmM \frac{-R + R + z}{(R+z)R} = GmM \frac{z}{(R+z)R}$$

$$\overset{z \ll R}{\approx} GmM \frac{z}{R^2} = m \cdot \frac{GM}{R^2} \cdot z = \underline{mgz} \quad (\text{ok})$$

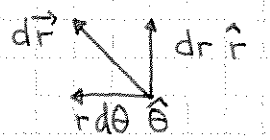
[Oppgave: Vis at feilen vi gjør ved å bruke mgz blir

$$\frac{\Delta U}{U} = \frac{z}{R} \quad]$$

Planetbaner (K1) [LL 11.3+4]

Anta m (planet) $\ll$ M (sola) $\Rightarrow$ M i ro, i origo ($r=0$)

$$\Rightarrow E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{Konst.} \quad \text{(konservativt system)}$$



$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\Rightarrow v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$L = |\vec{r} \times m\vec{v}| = |r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})|$$

$(\hat{r} \times \hat{r} = 0)$

$(\hat{r} \times \hat{\theta} = \hat{z})$

$$mr^2\dot{\theta} = \text{konst.} \quad (\text{fordi } \vec{\tau} = 0)$$

$$\Rightarrow E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\underbrace{\left(\frac{L}{mr^2}\right)}_{\dot{\theta}}^2 - \frac{GMm}{r}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

$$\Rightarrow \frac{dr}{dt} = \sqrt{\frac{2E}{m} + \frac{2GM}{r} - \frac{L^2}{m^2r^2}}$$

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \Rightarrow dt = \frac{mr^2}{L}d\theta$$

$$\Rightarrow d\theta = \frac{L}{mr^2}dt = \frac{L}{mr^2}\frac{dr}{\sqrt{\frac{2E}{m} + \frac{2GM}{r} - \frac{L^2}{m^2r^2}}}$$

$$= \frac{dr}{r^2\sqrt{\frac{2mE}{L^2} + \frac{2GMm^2}{L^2r} - \frac{1}{r^2}}}$$

Løsbart! $\displaystyle\int_{\theta_{min}}^{\theta} d\theta = \int_{r_{min}}^{r} \frac{dr}{r^2\sqrt{\cdots\cdots}}$; substituér $u = \frac{1}{r}$ osv. osv.

Løsning: $\underline{r = \dfrac{r_0}{1 + \varepsilon \cos\theta}}$

med $r_0 = L^2/GMm^2$ og $\varepsilon = \sqrt{1 + 2EL^2/G^2M^2m^3}$

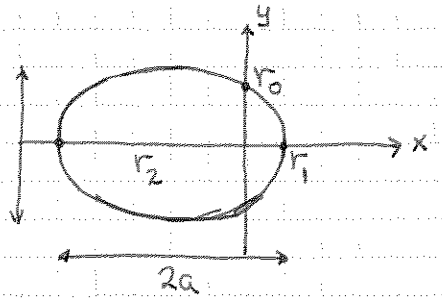
Kjeglesnitt med brennpunkt i origo (der sola er), og valget $\theta_{min} = 0$ der $r = r_{min} = r_0/(1+\varepsilon)$;

$\varepsilon = 0$: sirkel $(E = -G^2M^2m^3/2L^2)$

$0 < \varepsilon < 1$: ellipse $(E < 0)$

$\left.\begin{array}{lll} \varepsilon = 1 : & \text{parabel} & (E = 0) \\ \varepsilon > 1 : & \text{hyperbel} & (E > 0) \end{array}\right\}$ ubundne baner

Ellipse:



a = store halvakse

b = lille —"—

$r_1 \leq r \leq r_2$

$(\theta_1 = 0, \quad \theta_2 = \pi)$

$$2a = r_1 + r_2 = \frac{r_0}{1+\varepsilon} + \frac{r_0}{1-\varepsilon} = \frac{2r_0}{1-\varepsilon^2} \Rightarrow a = -\frac{GMm}{2E} = \frac{GMm}{2|E|}$$

2c = avstand mellom brennpunktene $= 2\varepsilon a \Rightarrow \varepsilon = c/a$

$a^2 = b^2 + c^2$

$$\Rightarrow b = \sqrt{a^2 - c^2} = \sqrt{a^2 - a^2\varepsilon^2} = \ldots\ldots = \frac{L}{\sqrt{-2mE}} = \frac{L}{\sqrt{2m|E|}}$$

Omløpstid T:

Ellipse: $A = \pi ab$ (arealet)

Fra s. 93: $dA/dt = L/2m$ (K2)

$$\Rightarrow A = \oint dA = \int_0^T \frac{L}{2m}\, dt = \frac{L}{2m} T = \pi ab$$

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$$\Rightarrow T = 2m\pi a \frac{b}{L} = 2m\pi a \sqrt{\frac{1}{2m|E|}} = \frac{2m\pi a}{\sqrt{m}} \sqrt{\frac{a}{GMm}} = \underline{\underline{\frac{2\pi}{\sqrt{GM}} a^{3/2}}} \quad (K3)$$

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Satellitter [YF 13.4]

Som regel i sirkulære baner

N2: $F = ma$ med $F = GMm/r^2$ og $a = v^2/r$

$$\Rightarrow \frac{GMm}{r^2} = m\frac{v^2}{r} \Rightarrow \underline{v = \sqrt{\frac{GM}{r}}}$$

Geostasjonær bane har $T = 24$ h, med satellitten over samme sted på ekvator hele tiden $\underset{\text{øv. 13}}{\Longrightarrow}$ $r = 42246$ km

Total energi i sirkulær bane:

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\frac{GM}{r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

Dvs $K = -\frac{1}{2}U = -E > 0$

[Oppgave: Vis at E for sirkel s. 96 stemmer med dette.]

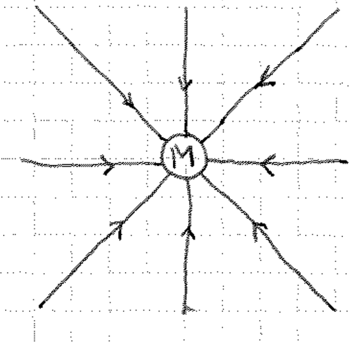
Potensial og felt [LL 11.1]

Feltstyrke $\overset{\text{def}}{=}$ Kraft pr masseenhet:

$$\vec{g} = \lim_{m\to 0} \vec{F}/m = -(MG/r^2)\,\hat{r}$$

Tolkning: M ("referansemassen") omgir seg med feltet $\vec{g}$ (enten "testmassen" m er der eller ei)

Visualisering av $\vec{g}$ med feltlinjer:



- retning tangentielt til $\vec{g}$
- antall feltlinjer pr flateenhet prop. med $|\vec{g}|$

Potensial $\overset{\text{def}}{=}$ Potensiell energi pr. masseenhet:

$$V(r) = \lim_{m \to 0} \frac{U(r)}{m} = -\frac{GM}{r}$$

Tolkning: M omgir seg med det skalare potensialet $V(r)$

Ser at V = konst. på kuleflater med sentrum på (punkt-)massen M.
Slike flater kalles ekvipotensialflater.

Fra før: $U(r) = -\int_{r_0}^{r} \vec{F} \cdot d\vec{r}$ og $\vec{F} = -\nabla U$

Divisjon med m gir direkte sammenhengene

$$V(r) = -\int_{r_0}^{r} \vec{g} \cdot d\vec{r} \quad \text{og} \quad \vec{g} = -\nabla V$$

Dette er generelle sammenhenger for konservative krefter og felt.

Annet eksempel (FY1003/TFY4155):

$$\vec{F} = \frac{q \cdot Q}{4\pi\varepsilon_0 r^2}\hat{r}$$ = kraft mellom ladninger q, Q i innbyrdes avstand r (ε_0 = naturkonstant)

Elektrisk felt: (fra Q) $\vec{E} = \vec{F}/q = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

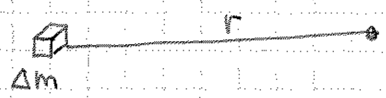
Pot. energi: $U(r) = \frac{qQ}{4\pi\varepsilon_0 r}$

Potensial (omkring Q): $V(r) = \frac{U(r)}{q} = \frac{Q}{4\pi\varepsilon_0 r}$

Sammenhenger: $\vec{E} = -\nabla V$, $V(r) = -\int_{r_0}^{r} \vec{E} \cdot d\vec{r}$

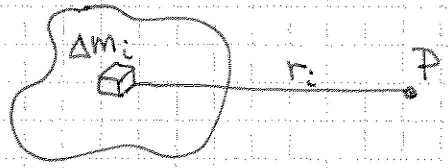
Potensial og felt fra massefordelinger [YF 13.6, LL 11.2]

Potensial fra liten masse Δm:



$$\Delta V(r) = -\frac{G\,\Delta m}{r}$$

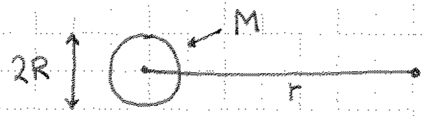
Pot. fra massefordeling:



Potensial i P:

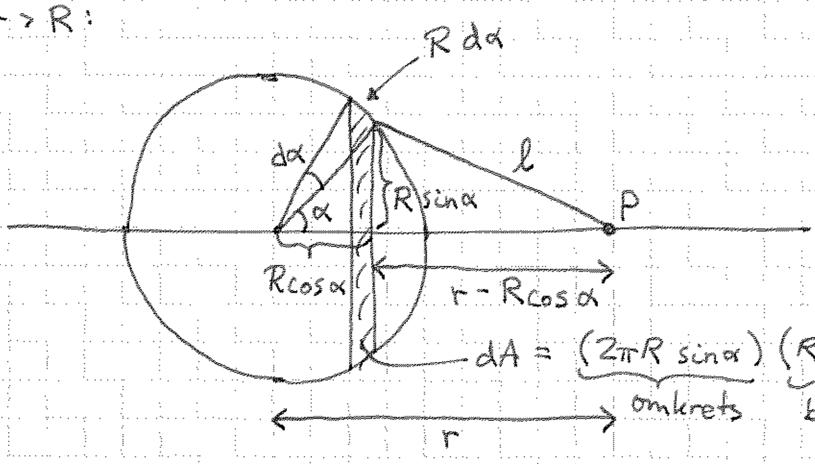
$$V_P = \sum_i V_P^i = -G\sum_i \frac{\Delta m_i}{r_i} \xrightarrow{\Delta m_i \to 0} -G\int \frac{dm}{r}$$

Eks 1: Kuleskall



$V(r) = ?$ $\quad (r > R) \quad$ (evt. $r < R$)

$r > R$:



$dA = \underbrace{(2\pi R \sin\alpha)}_{\text{omkrets}} \underbrace{(R\,d\alpha)}_{\text{bredde}}$ = areal av tynn ring med felles avstand l til punktet P

Fra figur: $l^2 = (R\sin\alpha)^2 + (r - R\cos\alpha)^2$

$$= R^2\sin^2\alpha + r^2 - 2rR\cos\alpha + R^2\cos^2\alpha$$

$$= R^2 + r^2 - 2rR\cos\alpha$$

Bidrag til V i P fra tynn ring:

$dV = -G\,dm/l\,;\quad dm = M\cdot dA/A = M\cdot 2\pi R^2 \sin\alpha\, d\alpha / 4\pi R^2$

$$= \tfrac{1}{2} M \sin\alpha\, d\alpha$$

$\Rightarrow$ Totalt pot. i P:

$$V(r) = \int dV = -G\int dm/\ell = -\tfrac{1}{2}MG \int_0^\pi \frac{\sin\alpha \cdot d\alpha}{\sqrt{R^2+r^2-2rR\cos\alpha}}$$

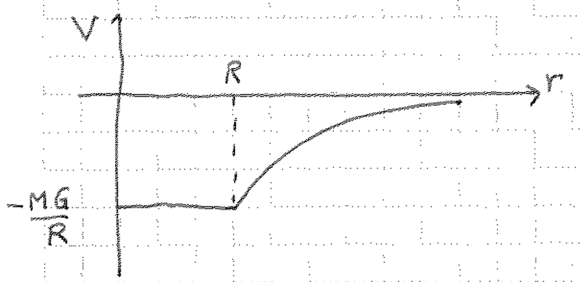
$$= -\tfrac{1}{2}MG \Big| \frac{1}{rR}\sqrt{R^2+r^2-2rR\cos\alpha}\,\Big|_0^\pi$$

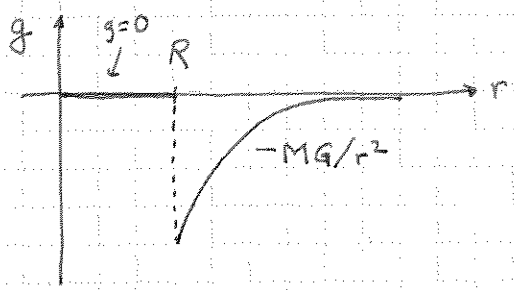
$$= -\frac{MG}{2rR}\Big\{\underbrace{\sqrt{R^2+r^2+2rR}}_{=R+r} - \underbrace{\sqrt{R^2+r^2-2rR}}_{=r-R}\Big\} = \underline{-\frac{MG}{r}}$$

For $\underline{r<R}$ blir alt likt, unntatt at $\sqrt{R^2+r^2-2rR} = R-r$, slik at

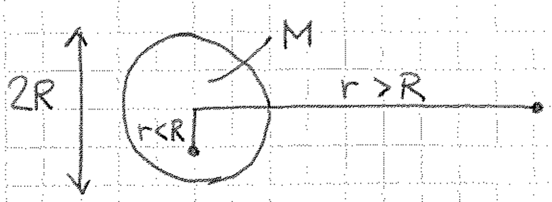
$$V(r) = -\frac{MG}{2rR}\{R+r-(R-r)\} = \underline{-\frac{MG}{R}} \qquad (= \text{konst.})$$

Feltet blir $\vec{g}(r) = -\nabla V = -\hat{r}\frac{dV}{dr} = \begin{cases} 0 & ; \; r<R \\ -\hat{r}\frac{MG}{r^2} & ; \; r>R \end{cases}$





Eks 2: Kompakt kule (f.eks. jorda)

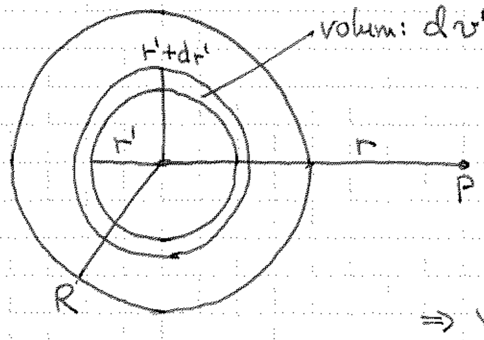


$V(r) = ?$ $\quad \vec{g}(r) = ?$

Strategi: Kompakt kule = sum av mange tynne kuleskall

$\Rightarrow$ Bruk eks. 1 og integrer.

$r > R$:



$$dm' = M \cdot \frac{dv'}{v'} = M \cdot \frac{4\pi r'^2\, dr'}{\frac{4}{3}\pi R^3} = \frac{3M}{R^3}\, r'^2 dr'$$

$$dV = -G\, dm'/r = -\frac{3MG}{rR^3}\, r'^2 dr'$$

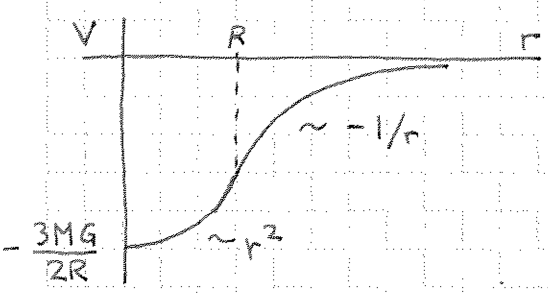
$$\Rightarrow V(r) = \int dV = -\frac{3MG}{rR^3} \underbrace{\int_0^R r'^2 dr'}_{= R^3/3} = \underline{-\frac{MG}{r}}$$

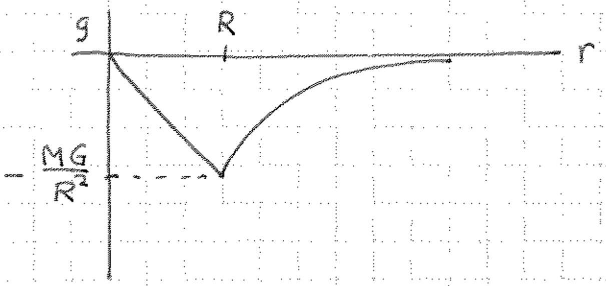
$r < R$:

Kuleskall med radius $r' < r$ gir bidrag $dV = -G\, dm'/r$ } jf. s. 100
—— " —————— $r' > r$ —— " —— $dV = -G\, dm'/r'$ }

$$\Rightarrow V(r) = -\frac{3MG}{R^3}\left\{ \underbrace{\int_0^r \frac{r'^2 dr'}{r}}_{= r^2/3} + \underbrace{\int_r^R \frac{r'^2 dr'}{r'}}_{= R^2/2 - r^2/2} \right\} = \underline{-\frac{3MG}{2R}\left\{1 - \frac{r^2}{3R^2}\right\}}$$

Feltet: $\vec{g}(r) = -\nabla V = -\hat{r}\, dV/dr = \begin{cases} -\hat{r}\, MGr/R^3 & ; \quad r < R \\ -\hat{r}\, MG/r^2 & ; \quad r > R \end{cases}$





Kraft på masse m ved $r = R$: $\vec{F}(R) = m\vec{g}(R) = -\hat{r}\, \frac{MmG}{R^2}$

dvs som om hele massen M var samlet i sentrum av kula!

(Noe vi har antatt hele tiden til nå, for tyngdekraften på masse m ved jordoverflaten; $\vec{F} = m\vec{g}$ med $g = MG/R^2 = 9.81\ \mathrm{m/s^2}$.)

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